# UNIFORM CONTROLLABILITY OF THE $1-d$ SEMI-DISCRETE HEAT EQUATION

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Abstract. In this job we present a new proof of the uniform boundary controllability of the $1-d$ semi-discrete heat equation. The proof is based on the transmutation formula, which transmutes the partial controllability of the $1-d$ semi-discrete wave equation to the partial controllability of the corresponding semi-discrete heat equation. We use the three step time iteration method. Firstly, the projection of the solution of the heat system can be controlled. Secondly, we let the heat equation evolve, to make the initial data of the third step to be small enough. In the third step we deduce a global nonuniform control with respect to the small initial data, which obtained from the end of the second step. However, the control still uniform bounded with respect to the initial data at time $t=0$, due to the fact that small initial data compensate the effect of the blow up of the control. Combining the control in this three step we finish to prove our result.

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# Contents

## 1. Introduction

In this paper, we investigate the boundary controllability of the semi-discrete $1-d$ heat equation. The continuous case is by now well understood (see, in [Li],[LZ],[R],[R1]). For instance, the following result is well known: *Given $T>0$ and $u_0 \in L^2(0,1)$ there exists a control function $f \in L^2(0,T)$ such that the solution of the equation*

$$\left\{\begin{array}{ll} \partial_t u - \partial_x^2 u = 0, & (x,t) \in (0,1)\times(0,T) \\ u(0,t)=0, \;\; u(1,t)=f(t), & t \in (0,T) \\ u(x,0)=u_0, & x\in(0,1), \end{array}\right. \tag{1.1}$$

*satisfies*

$$u(x,T)=0. \qquad \forall x \in (0,1). \tag{1.2}$$

*Moreover, there exists a constant $C>0$, depending on the control time $T$, such that*

$$\|f\|_{L^2(0,T)} \le C\,\|u_0\|_{L^2(0,1)}\,. \tag{1.3}$$

Recently, in [L], L. Miller proved the following explicit bound for the control of the system (1.1):

$$\|f\|_{L^2(0,T)} \le Ce^{4\alpha/T}\,\|u_0\|_{L^2(0,1)}\,, \qquad \forall\ \alpha > 2(\frac{36}{37})^2, \tag{1.4}$$

where $C$ is a constant independent of time $T$.

We will analyze the analogue of these control results for the corresponding semi-discrete systems obtained by discretizing the space variable $x$. More precisely, we will show that (1.4) holds uniformly for the boundary control of the semi-discrete heat equation.

Let us introduce the semi-discrete schemes. Given $N\in\mathbb{N}$ we set $h=1/(N+1)$ and consider the following semi-discretization of the system (1.1):

$$\left\{\begin{array}{ll} \partial_t u_j(t) - \Big[\dfrac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2}\Big] = 0, & 0<t<T,\ j=1,\cdots,N \\ u_0(t)=0,\ u_{N+1}(t)=f_h(t), & 0<t<T, \\ u_j(0)=u_j^0,\ j=1,\cdots,N. & \end{array}\right. \tag{1.5}$$

System (1.5) is a finite-difference space semi-discretization of the heat equation (1.1) with control on the extreme $x=1$ (which corresponds to the value $N+1$ of the index $j$) Here and thereafter, $u_j$ is an approximation of the value of the solution $u$ of the continuous heat equation at the node $x_j=jh$.

To simplify the notations we denote by $\Delta_h$ the discrete second derivative with respect to $x$, i.e., for any $\varphi_j$, $\Delta_h\varphi_j$ is as following

$$\Delta_h\varphi_j = \frac{1}{h^2}(\varphi_{j+1}-2\varphi_j+\varphi_{j-1}).$$

Observe that $\{\sin(k\pi x)\}_{k=1}^{\infty} = \{\Phi_k(x)\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(0,1)$. Hence, for any $u_0 \in L^2(0,1)$, there exists an unique sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ such that

$$u_0 = \sum_{k=1}^{\infty} a_k \sin(\pi k x). \tag{1.6}$$

Here and henceforth, we choose

$$u_j^0 = \sum_{k=1}^{N} a_k \Phi_j^k, \qquad j = 1, \cdots, N \tag{1.7}$$

with $\Phi_j^k = \sin(k\pi jh)$ as the initial datum for the semi-discrete system on the mesh of size $h$. This is an approximation of the values of $u_0$ at the nodes $x = jh$.

Recall that, when $h > 0$ is fixed, system (1.5) is null controllable, as a consequence of the fact that the Kalman rank condition is fulfilled. However, this technique does not show whether the control $(f_h)_{h>0}$ is uniformly bounded with respect to $h > 0$.

As A. López and E. Zuazua have shown ([LZ1]), system (1.5) is uniformly controllable, i.e., for any $T > 0$, there exists a $f_h \in L^2(0,T)$, which is uniformly bounded in $L^2(0,T)$ as $h \to 0$, such that the solution of (1.5) satisfies

$$u_j(T) = 0, \qquad j = 1, \cdots, N, \tag{1.8}$$

for any $h > 0$.

The proof in [LZ1] is based on the classical results on series of real exponentials, due to the fact that the spectrum of (1.5) can be computed explicitly.

In this paper we give another proof of this result.

According to the classical theorem of Russell in [R], the controllability of the wave equation implies the null-controllability of the heat equation. The main idea in our new proof is to use the *control transmutation method* relating the null-controllability of the semi-discrete heat equation to the exact controllability of the semi-discrete wave equation. This is the Fourier version of the transmutation which can be written in the physical space by Kannai's formula. The advantage of using this method is that it gives a representation of the control in the physical space, and that the proof of the result is simpler.

However, we could not directly deduce the uniform estimate of the control for the system (1.5), due to the fact that the lack of the uniform controllability of the corresponding $1-d$ semi-discrete wave equation. In despite of this, we attain the uniform estimate of the control for the system (1.5), by means of the *three step time iteration method* (see, for instance, [LZ],[LZZ],[GL]).

The following theorem shows that, system (1.5) is uniformly controllable.

**Theorem 1.1.** *Let $T>0$, $\varepsilon_0\in(0,1)$ and $0<T_1<T$. There exists a control $\tilde f_h\in L^2(0,T)$ such that the solution of system (1.5) satisfies*

$$u_j(T)=0, \qquad j=1,\cdots,N,$$

*for any $h>0$.*

*Moreover, fix $\varepsilon_0\in(0,1)$, for any $\alpha>\alpha_*=2(\frac{36}{37})^2$, there exist two constants $C(\varepsilon_0)>0$ and $L(\varepsilon_0)=\frac{2}{\cos(\pi\varepsilon_0/2)}$ such that*

$$\left\|\tilde f_h\right\|_{L^2(0,T)}\le C(\varepsilon_0)e^{\alpha L(\varepsilon_0)^2/T_1}\Big(h\sum_{j=1}^{N}|u_j^0|^2\Big)^{1/2}$$

*for any $h>0$, where $C(\varepsilon_0)$ and $L(\varepsilon_0)$ depend on $T_1$ but not on $h$.*

*Finally, the control $\tilde f_h$ of the system (1.5) may be built such that*

$$\tilde f_h\to\begin{cases} f, & t\in(0,T_1)\\ 0, & t\in(T_1,T)\end{cases}\quad as\quad h\to 0, \tag{1.9}$$

*where $f$ is a null control of the continuous equation*

$$\begin{cases}\partial_t u-\partial_x^2 u=0, & (x,t)\in(0,1)\times(0,T_1),\\ u(0,t)=0,\;\; u(1,t)=f(t), & t\in(0,T_1),\\ u(x,0)=u_0, & x\in(0,1),\\ u(x,T_1)=0, & x\in(0,1).\end{cases}\tag{1.10}$$

**Remark 1.1.** *Note that the two constants $C(\varepsilon_0)$ and $L(\varepsilon_0)$ are obtained from the partial controllability of (1.1) in the time interval $(0,T_1)$. Both of them are independent on $h$.*

**Remark 1.2.** *Moreover, from the analysis in* [IZ], *it is easy to obtain the following relationship between $C(\varepsilon_0)$, $L(\varepsilon_0)$ and $\varepsilon_0$:*

- *$L(\varepsilon_0)\longrightarrow\infty$ as $\varepsilon_0\longrightarrow 1$ and $L(\varepsilon_0)\longrightarrow 2$ as $\varepsilon_0\longrightarrow 0$.*
- *$C(\varepsilon_0)\longrightarrow\infty$ as $\varepsilon_0\longrightarrow 1$ and $C(\varepsilon_0)\longrightarrow\frac{1}{2(L-2)}$ as $\varepsilon_0\longrightarrow 0$. Here $L$ is the control time of the corresponding semi-discrete $1-d$ wave equation. We will see, from the definition of $L$ in Lemma 3.2, $L$ is strictly larger than $L(\varepsilon_0)$.*

Before the brief explanation of the proof, we first introduce the filtered space of the solution. For each $h$ and $\varepsilon_0\in(0,1)$, we define

$$\mathcal{C}_{\varepsilon_0}(h)=\operatorname{span}\{\Phi^k,k=1,\cdots,[\varepsilon_0/h]\},\;\;\Phi^k=(\Phi_0^k,\cdots,\Phi_{N+1}^k).\tag{1.11}$$

This is a space in which only the low frequencies of the system (1.5) are involved. We denote by $\pi_{\varepsilon_0}$ the projection of the solution over the subspace of the eigenfrequencies involved in the filtered space $\mathcal{C}_{\varepsilon_0}(h)$. It is important to emphasize that the flow associated with the equation (1.5) is invariant on this subspace.

The proof of Theorem 1.1 is based on a control strategy in three steps. Roughly speaking, the proof is as follows. We divide the time interval $[0, T]$ in three subintervals: $I_1 = [0, T_1]$, $I_2 = [T_1, T_2]$ and $I_3 = [T_2, T]$.

In the first time interval, $I_1$, we control to zero the projection of the solution over a suitable subspace containing only sufficiently low frequencies. Note that we fix the parameter $\varepsilon_0 \in (0, 1)$, for filtering the high frequencies of the solution.

In the second time interval, $I_2$, we let the system evolve freely without control. In this way the projection of the solution of (1.5) over the low frequencies remains at rest and, due to the strong dissipativity of system (1.5) in its high frequencies, the size of the solution at the end of the time $t = T_2$ becomes exponentially small, i.e. of the order of $e^{-c(T_2 - T_1)}$ with $c = \frac{4\sin^2(\pi\varepsilon_0/2)}{h^2}$. It is easy to understand that $\{u_j\}_{j=1}^N$ degenerate to zero of the order of $e^{-\frac{CT}{h^2}}$, with $C > 0$ only depending on $\varepsilon_0$.

Finally, in the interval $I_3$ we apply a control driving the whole solution to zero. We denote by $f_h^2$ the control occurred in $I_3$. As we will show in the appendix, there exists a positive constant $C(\beta)$, independent to $h$, such that $f_h^2$ satisfies

$$\left\| f_h^2 \right\|_{L^2(T-T_2,T)} \leq C(\beta) e^{1/h^\beta} \Big( h \sum_{j=1}^{N} |u_j(T_2)|^2 \Big)^{1/2}, \qquad \text{for any} \quad \beta > 1.$$

On the other hand, in view of the analysis made in the interval $I_2$, the $\ell^2$ norm of $u_j(T_2)$ decays by a multiplicative factor of the order of $e^{-C/h^2}$. This phenomena compensates the blow up of the control as $h \to 0$ of the order of $e^{C/h^\beta}$ where $\beta \in (1, 2)$. Hence, the control is uniformly bounded in the total time interval $t \in (0, T)$ as $h \to 0$.

The rest of this paper is organized as follows: In section 2 we will introduce the Kannai's formula, which plays a key role in our new proof. In section 3 we give out the partial controllability of the semi-discrete system (1.5), by using the Kannai's formula and filtering method to the corresponding semi-discrete wave equation. This provided the accuracy of the first step of the iteration of the proof of Theorem 1.1. In section 4 Theorem 1.1 is proved, by using the iteration method. Some open problems are listed in section 5. The final part of the paper is the appendix, in which we put some technical lemmas and a first attempt to Carleman inequality for semi-discrete $1-d$ heat equation. However, we are failed to deduce the uniform controllability property.

## 2. Preliminary: Kannai's formula

In this section we will introduce the Kannai's formula (see, section 6.2 in the book [T]), which relates the null-controllability of the heat equation to the exact controllability of the wave equation.

Let $M$ be a compact Riemannian manifold. The homogenous heat equation is defined as:

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & (t,x) \in \mathbb{R}^+ \times M \\ u(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial M \\ u(0,x) = u_0, & x \in M. \end{cases} \tag{2.1}$$

Here and thereafter, $\Delta$ denote the Laplacian in the space variable $x \in M$.

For any $u_0 \in L^2(M)$ there exists a unique solution $u \in C^\infty((0,\infty)\times\overline{M})$. The homogenous heat semigroup $t \to e^{t\Delta}$ is defined by $e^{t\Delta}u_0 = u(t)$.

The homogenous wave equation is defined as:

$$\begin{cases} \partial_t^2 \omega - \Delta \omega = 0, & (t,x) \in \mathbb{R}^+ \times M \\ \omega(t,x) = 0, & (t,x) \in \mathbb{R}^+ \times \partial M \\ \omega(0,x) = u_0, \ \ \omega_t(0,x) = 0, & x \in M. \end{cases} \tag{2.2}$$

For any $u_0 \in H_0^1(M)$ there exists a unique solution $\omega \in C([0,\infty); H_0^1(M)) \cap C^1([0,\infty); L^2(M))$. The even homogenous wave group $t \to W(t)$ is defined by $W(t)u_0 = \omega(t)$.

It is well known that the geometry of small time asymptotic for the homogeneous heat semigroup $t \mapsto e^{t\Delta}$ on $L^2(M)$ can be understood from the even homogeneous wave group $t \mapsto W(t)$ through Kannai's formula:

$$e^{t\Delta} = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/(4t)} W(s) ds. \tag{2.3}$$

The key point is how to use Kannai's formula to relate the null-controllability of the heat equation to the exact controllability of the wave equation. The main idea is to replace the fundamental solution of the heat equation on the line $e^{-s^2/(4t)}/\sqrt{4\pi t}$ appearing in Kannai's formula by some fundamental controlled solution of the heat equation in $M$ controlled on the boundary $\partial M$.

## 3. Uniform partial controllability

In this section we will prove the uniform partial controllability of system (1.5), in the sense that only the projection $\pi_{\varepsilon_0}$ of the solution is controlled:

**Theorem 3.1.** *Let $T > 0$ and $\varepsilon_0 \in (0,1)$. The projection $\pi_{\varepsilon_0}$ of the solution of (1.5) is uniformly null controllable. More precisely, for any $h > 0$ there exists a control $f_h \in L^2(0,T)$ such that the solution of system (1.5) satisfies*

$$\pi_{\varepsilon_0}\vec{u}(T) = 0, \qquad \vec{u}(T) = (u_1, \cdots, u_N) \tag{3.1}$$

*for any $T > 0$.*

*Moreover, for any $\alpha > \alpha_* = 2(\frac{36}{37})^2$, there exist two constants $C(\varepsilon_0) > 0$ and $L(\varepsilon_0) = \frac{2}{\cos(\pi\varepsilon_0/2)}$, independent of $h$, such that*

$$\|f_h\|_{L^2(0,T)} \leq C(\varepsilon_0) e^{\alpha L(\varepsilon_0)^2/T} (h \sum_{j=1}^{N} |u_j^0|^2)^{1/2} \tag{3.2}$$

*holds for any $h > 0$.*

*Finally, the controls $f_h$ of system (1.5) may be built such that*

$$f_h \to f \ \text{ in } \ L^2(0,T) \ \text{ as } \ h \to 0 \tag{3.3}$$

*where $f$ is a null control for the corresponding continuous heat equation (1.1).*

**Remark 3.1.** *In Theorem 3.1 we control the projection of the solution over the subspace of $\mathcal{C}_{\varepsilon_0}(h)$, which will play a key role in the proof of the Theorem 1.1.*

**Remark 3.2.** *The argument we have presented here is rather general and interesting by itself. One may recover the controllability property of the continuous heat equation as the limit of this partial controllability results since, as $h \to 0$, the projections $\pi_{\varepsilon_0}$ end up covering the whole range of frequencies.*

We follow the idea of Miller ([L]) who uses Kannai's formula to deduce the null-controllability of the heat equation by the exact-controllability of the wave equation.

First we recall two lemmas: the fundamental control solution of the heat equation (in [L]); the partial controllability of the semi-discrete wave equation (in [Z1]).

Here and thereafter $\mathcal{D}'(\mathcal{O})$ denotes the space of distributions on the open set $\mathcal{O}$ endowed with the weak topology and $\mathcal{M}(\mathcal{O})$ denotes the subspace of Random measures on $\mathcal{O}$.

**Lemma 3.1.** *Given $\varepsilon_0 \in (0,1)$. For any $\alpha > \alpha_* = 2(\frac{36}{37})^2$, there exists $C > 0$ such that for all $L > 0$ and $T \in (0, \inf(\frac{\pi}{2}, L)^2)$, there is a $v \in C^0([0,T], \mathcal{M}((-L,L)))$ satisfying*

$$\partial_t v - \partial_s^2 v = 0, \quad \text{in } \mathcal{D}'((0,T)\times(-L,L)) \tag{3.4}$$

$$v|_{t=0} = \delta, \quad v|_{t=T} = 0, \tag{3.5}$$

$$\|v\|_{L^2((0,T)\times(-L,L))} \le Ce^{\alpha L^2/T}. \tag{3.6}$$

**Remark 3.3.** *This lemma shows the estimate of the control of the fundamental solution of the heat equation on $(0,T)\times(-L,L)$. It was proved by Miller in* [L].

**Lemma 3.2.** *Given $0 < \varepsilon_0 < 1$. For $L > L(\varepsilon_0) = \frac{2}{\cos(\pi\varepsilon_0/2)}$, there exists a control $g_h \in L^2(0,L)$ such that the projection $\pi_{\varepsilon_0}$ of the solution of system(the time variable is denoted by s here)*

$$\begin{cases} \partial_s^2\omega_j(s) - \left[\frac{\omega_{j+1}(s)-2\omega_j(s)+\omega_{j-1}(s)}{h^2}\right] = 0, & s\in(0,L),\ j=1,\cdots,N \\ \omega_0(s) = 0, \ \ \omega_{N+1}(s) = g_h(s), & s\in(0,L) \\ \omega_j(0) = u_j^0, \ \ \partial_s\omega_j(0) = 0, & j = 1,\cdots,N \end{cases} \tag{3.7}$$

*satisfies*

$$\pi_{\varepsilon_0}\vec{\omega}(L) = \pi_{\varepsilon_0}\partial_s\vec{\omega}(L) = 0, \qquad \vec{\omega}(L) = (\omega_1,\cdots,\omega_N)$$

*for any $h > 0$.*

*Moreover, there exists a constant $C(\varepsilon_0) > 0$, independent of $h$, such that*

$$\|g_h\|_{L^2(0,L)} \le C(\varepsilon_0)\Big(h\sum_{j=1}^N |u_j^0|^2\Big)^{1/2} \tag{3.8}$$

*holds for any $h > 0$.*

**Remark 3.4.** *This is the uniform boundary controllability of the semi-discrete wave equation with Cauchy data $(\omega_j(0), \partial_s\omega_j(0)) = (u_j^0, 0), j = 1,\cdots,N$. It can be deduced directly from the uniform observability of the corresponding adjoint system (in* [IZ], [Z1]).

Now we begin to prove the Theorem 3.1.

*Proof of Theorem 3.1.* We assume that Lemma 3.1 holds. Let $\alpha > \alpha_*$, $T \in (0, \inf(1, L(\varepsilon_0)^2))$ and $L > L(\varepsilon_0)$ be fixed from now on. Let $C > 0$ and $v \in L^2((0,T)\times(-L,L))$ be the corresponding constant and fundamental controlled solution given by Lemma 3.1.

Let $\{\underline{\omega}_j\}_{j=0}^N$ and $\underline{g}_h(s)$ be the extensions of $\{\omega_j\}_{j=0}^N$ and $g_h(s)$ by even reflection with respect to $s = 0$, i.e.,

$$\underline{\omega}_j(s) = \begin{cases} \omega_j(s), & s\in(0,L) \\ \omega_j(-s), & s\in(-L,0), \end{cases} \qquad j = 0,\cdots,N, \tag{3.9}$$

and

$$g_h(s) = \begin{cases} g_h(s), & s \in (0, L) \\ g_h(-s), & s \in (-L, 0). \end{cases} \tag{3.10}$$

Equation (3.7) implies the projection $\pi_{\varepsilon_0}$ of the solution of

$$\begin{cases} \partial_s^2 \underline{\omega}_j(s) - \Delta_h \underline{\omega}_j(s) = 0, & s \in (-L, L), \ j = 1, \cdots, N \\ \underline{\omega}_0(s) = 0, \ \ \underline{\omega}_{N+1}(s) = \underline{g}_h(s), & s \in (-L, L) \\ \underline{\omega}_j(0) = u_j^0, \ \ \partial_s \underline{\omega}_j(0) = 0, & j = 1, \cdots, N \end{cases} \tag{3.11}$$

satisfies $\pi_{\varepsilon_0} \underline{\omega}_j(s) = \partial_s \pi_\varepsilon \underline{\omega}_j(s) = 0$ at $s = \pm L$. Moreover, from (3.8) we know that

$$\left\| \underline{g}_h \right\|_{L^2(-L,L)} \le 2C(\varepsilon_0) \Big( h \sum_{j=1}^N |u_j^0|^2 \Big)^{1/2}. \tag{3.12}$$

Let $\{\varphi_j\}_{j=1}^N$ be the solution of the equation

$$\begin{cases} \partial_t \varphi_j - \Delta_h \varphi_j = \psi_j, & t \in (0, T), \ \ j = 1, \cdots, N \\ \varphi_0(t) = \varphi_{N+1}(t) = 0, & t \in (0, T) \\ \varphi_j(0) = 0, & 1, \cdots, N, \end{cases} \tag{3.13}$$

with

$$\psi_j = v(t, s) \partial_s \underline{\omega}_j(s) \Big|_{-L}^{L} - \partial_s v(t, s) \underline{\omega}_j(s) \Big|_{-L}^{L}. \tag{3.14}$$

The transmutation formulas

(3.15)

$$u_j(t) = \int_{-L}^{L} v(t, s) \underline{\omega}_j(s) ds + \varphi_j(t), \qquad j = 0, \cdots, N, \ \text{ and } \quad f_h(t) = \int_{-L}^{L} v(t, s) \underline{g}_h(s) ds$$

define $\{u_j\}_{j=1}^N$ and a control $f_h$ on the time interval $(0, T)$.

Now we check that $\{u_j\}_{j=1}^N$ is the solution of (1.5) with the control $f_h(t)$. Moreover, the projection $\pi_{\varepsilon_0}$ of $\{u_j\}_{j=1}^N$ is zero at time $t = T$ with the control $f_h$.

- The definition of $u_j$ in (3.15) yields

$$u_j(0) = \int_{-L}^{L} v(0, s) \underline{\omega}_j(s) ds + \varphi_j(0) = \underline{\omega}_j(0) = u_j^0, \ \ j = 1, \cdots, N.$$

  Moreover, from the definition of $\psi_j$ in (3.14) and the properties of $v(t, s)$ in Lemma A.1 in the Appendix, we have $\psi_j(t) \in L^2(0, T)$. Consequently, from equation (3.13) we know that $\varphi_j \in C^0(0, T)$, and this yields that

$$u_j \in C^0(0, T), \qquad j = 1, ..., N.$$

- We claim that

$$\partial_t u_j(t) - \Delta_h u_j(t) = 0, \qquad t \in (0,T), \;\; j = 1, \cdots, N. \tag{3.16}$$

(3.16) is a consequence of the following computation:

$$\begin{aligned}
\partial_t u_j &= \int_{-L}^{L} \partial_s^2 v(t,s)\underline{\omega}_j(s)ds + \partial_t \varphi_j(t) \\
&= \partial_s v(t,s)\underline{\omega}_j(s)\Big|_{-L}^{L} - v(t,s)\partial_s\underline{\omega}_j(s)\Big|_{-L}^{L} + \int_{-L}^{L} v(t,s)\partial_s^2\underline{\omega}_j(s)ds + \Delta_h\varphi_j(t) + \psi_j(t) \\
&= \int_{-L}^{L} v(t,s)\Delta_h\underline{\omega}_j(s)ds + \Delta_h\varphi_j(t) = \Delta_h\Big(\int_{-L}^{L} v(t,s)\underline{\omega}_j(s)ds + \varphi_j(t)\Big) \\
&= \Delta_h u_j.
\end{aligned}$$

- For the boundary point $u_0(t)$, we have

$$u_0(t) = \int_{-L}^{L} v(t,s)\underline{\omega}_0(s)ds = 0, \qquad \forall t \in (0,T).$$

- Now we check that

$$\pi_{\varepsilon_0}\vec{u}(T) = 0, \qquad \vec{u} = (u_1, \cdots, u_N).$$

First we claim that $\pi_{\varepsilon_0}\vec{\varphi}(T) = 0$, $\vec{\varphi} = (\varphi_1, \cdots, \varphi_N)$. Taking into account that $\pi_{\varepsilon_0}\underline{w}_j(s) = \pi_{\varepsilon_0}\partial_s\underline{w}_j(s) = 0$ at $s = \pm L$, we obtain that for any $t \in (0,T)$,

$$\begin{aligned}
\pi_{\varepsilon_0}\psi_j(t) &= \pi_{\varepsilon_0}\Big(v(t,s)\partial_s\underline{\omega}_j(s)\Big|_{-L}^{L} - \partial_s v(t,s)\underline{\omega}_j(s)\Big|_{-L}^{L}\Big) \\
&= v(t,s)\pi_{\varepsilon_0}\partial_s\underline{\omega}_j(s)\Big|_{-L}^{L} - \partial_s v(t,s)\pi_{\varepsilon_0}\underline{\omega}_j(s)\Big|_{-L}^{L} \\
&= 0.
\end{aligned}$$

This means that $\pi_{\varepsilon_0}\varphi_j(t)$ satisfies the equation

$$\begin{cases}
\partial_t\pi_{\varepsilon_0}\varphi_j - \Delta_h\pi_{\varepsilon_0}\varphi_j = \pi_{\varepsilon_0}\psi_j = 0, & t \in (0,T), \;\; j = 1, \cdots, N \\
\pi_{\varepsilon_0}\varphi_0(t) = \pi_{\varepsilon_0}\varphi_{N+1}(t) = 0, & t \in (0,T) \\
\pi_{\varepsilon_0}\varphi_j(0) = 0, & 1, \cdots, N.
\end{cases} \tag{3.17}$$

Hence

$$\pi_{\varepsilon_0}\varphi_j(T) = 0, \qquad j = 1, \cdots, N.$$

Taking into account that $v(T,s) = 0$ for any $s \in (-L, L)$, we get

$$\pi_{\varepsilon_0}u_j|_{t=T} = \pi_{\varepsilon_0}\int_{-L}^{L} v(T,s)\underline{\omega}_j(s)ds + \pi_{\varepsilon_0}\varphi_j(T) = 0, \qquad j = 1, \cdots, N.$$

- Taking (3.6) and (3.12) into account, we obtain

$$\|f_h\|_{L^2(0,T)} \leq \|v\|_{L^2((0,T)\times(-L,L))} \left\|\underline{g}_h\right\|_{L^2(-L,L)} \leq C(\varepsilon_0)e^{\alpha L^2/T}\Big(h\sum_{j=1}^{N}|u_j^0|^2\Big)^{1/2}. \tag{3.18}$$

We have proved that for all $\alpha > \alpha_*$ and $h > 0$ there is a $C(\varepsilon_0) > 0$ such that for all $\Big(h\sum_{j=1}^{N}|u_j^0|^2\Big)^{1/2} \leq \infty$, $T \in (0, \min(1, L)^2)$ and $L > L(\varepsilon_0)$, there is a control $f_h$ which uniformly solves the null-controllability problem (1.5) with the projection $\pi_{\varepsilon_0}$ at a cost so estimated in (3.18). Let $\alpha$ and $L$ tend respectively to $\alpha_*$ and $L(\varepsilon_0)$ completes the proof of theorem 3.1.

**Remark 3.5.** *Note that we could not construct $u_j(t)$ by $\int_{-L}^{L} v(t,s)\underline{\omega}_j(s)ds$, which is the directly approximation of the continuous case in* [L]. *This is due to the fact that the transmutation formulas make sense if and only if the semi-discrete wave equation is exactly controllable. Since we use the partial controllability results (Lemma 3.2) of the semi-discrete wave equation, an extra "error function ", $\varphi_j$, must be put in $u_j$ and balance the equation (3.16), as we have defined in (3.15).*

## 4. Proof of the main result

*Proof.* Given $T>0$, $\varepsilon_0\in(0,1)$. We divide the time interval in three subintervals

$$[0,T]=I_1\cup I_2\cup I_3, \tag{4.1}$$

with $I_1=[0,T_1]$, $I_2=[T_1,T_2]$ and $I_3=[T_2,T]$.

Given a initial datum $\{u_j^0\}_{j=0}^N$ to be controlled we proceed as follows:

• *First step.* In the first time interval $I_1$ we control the *low frequencies* of the solutions. In other words, we build a control $f_h^1\subset L^2(0,T_1)$ such that the solution of

$$\begin{cases} \partial_t u_j(t)-\Delta_h u_j(t)=0, & 0<t<T_1,\ j=1,\cdots,N\\ u_0(t)=0,\ u_{N+1}(t)=f_h^1(t), & 0<t<T_1,\\ u_j(0)=u_j^0,\ j=1,\cdots,N, & \end{cases} \tag{4.2}$$

satisfies

$$\pi_{\varepsilon_0}\vec{u}(T_1)=0. \tag{4.3}$$

By theorem 3.1, this can be done uniformly for any $\varepsilon_0\in(0,1)$. More precisely, we deduce directly from theorem 3.1 the existence of two positive constants $C(\varepsilon_0)>0$ and $L(\varepsilon_0)>0$, independent of $h$, such that

$$\left\|f_h^1\right\|_{L^2(0,T_1)}\le C(\varepsilon_0)e^{\alpha L(\varepsilon_0)^2/T_1}\Big(h\sum_{j=1}^N|u_j^0|^2\Big)^{1/2} \tag{4.4}$$

holds for all $h>0$.

In view of the uniform bound of the control (4.4), there exists a constant $C>0$, independent of $h$, such that

$$\Big(h\sum_{j=1}^N|u_j(T_1)|^2\Big)^{1/2}\le C\Big(h\sum_{j=1}^N|u_j^0|^2\Big)^{1/2}. \tag{4.5}$$

We denote by $\vec{v}^{T_1}$ the solution obtained at the end of the first time interval $I_1$, i.e.

$$\vec{v}^{T_1}=(v_1^{T_1},\cdots,v_N^{T_1})=(u_1(T_1),\cdots,u_N(T_1)) \tag{4.6}$$

.

• *Second step.* In the second time interval $I_2$ we let the equation evolve freely. In other words, we solve

$$\begin{cases} \partial_t u_j(t)-\Delta_h u_j(t)=0, & T_1<t<T_2,\ j=1,\cdots,N\\ u_0(t)=0,\ u_{N+1}(t)=0, & T_1<t<T_2,\\ u_j(T_1)=v_j^{T_1}, & j=1,\cdots,N, \end{cases} \tag{4.7}$$

Taking into account that $\pi_{\varepsilon_0} u_j = 0, \forall j = 1, \cdots, N$, and using the development of solutions of (4.7) in Fourier series we deduce that

$$\pi_{\varepsilon_0}\vec{u}(t) = 0, \qquad \text{for all } \ t \in (T_1, T_2). \tag{4.8}$$

Furthermore, we know that there exists a positive constant $C_0$ such that

$$\Big(h\sum_{j=1}^{N}|u_j(t)|^2\Big)^{1/2} \le C_0 e^{-\lambda^h_{[\varepsilon_0/h]+1}(t-T_1)}\Big(h\sum_{j=1}^{N}|v_j^{T_1}|^2\Big)^{1/2} \tag{4.9}$$

for all $t \in [T_1, T_2]$. In particular, for $t = T_2$, taking into account that

$$\lambda_k^h = \frac{4}{h^2}\sin^2(\frac{\pi k h}{2}) \qquad \text{and} \qquad ([\varepsilon_0/h]+1)h > \varepsilon_0,$$

we get

$$\lambda^h_{[\varepsilon_0/h]+1} \ge \frac{4}{h^2}\sin^2(\frac{\pi\varepsilon_0}{2}).$$

Consequently,

$$\Big(h\sum_{j=1}^{N}|u_j(T_2)|^2\Big)^{1/2} \le C_0 e^{-\frac{C_2(T_2-T_1)}{h^2}}\Big(h\sum_{j=1}^{N}|v_j^{T_1}|^2\Big)^{1/2}, \tag{4.10}$$

where the constants $C_0, C_2$ are independent of $h$.

Therefore, at the end of the second step, i.e. at time $t = T_2$, we obtain a state $\{u_j(T_2)\}_{j=1}^N$ with a exponential small norm(as $h \to 0$). We denote by $\vec{v}^{T_2}$ the solution at time $t = T_2$, i.e.

$$\vec{v}^{T_2} = (v_1^{T_2}, \cdots, v_N^{T_2}) = (u_1(T_2), \cdots, u_N(T_2)). \tag{4.11}$$

• *Third step.* In the third step we control the whole solution to zero.

As a consequence of Lemma 3.1, we claim that: For $T \in (0, T_2 + \inf(\frac{\pi}{2}, L)^2)$, there exists a $v \in C^0([T_2, T], \mathcal{M}((-L, L)))$ and a constant $C > 0$, such that

$$\begin{cases} \partial_t v - \partial_s^2 v = 0, & \text{in } \ \mathcal{D}'((T_2, T)\times(-L, L)) \\ v|_{t=T_2} = \delta, \ \ v|_{t=T} = 0, & \\ \|v\|_{L^2((T_2,T)\times(-L,L))} \le Ce^{\alpha L^2/(T-T_2)} & \end{cases} \tag{4.12}$$

holds for any $\alpha > \alpha_* = 2(\frac{36}{37})^2$.

Now we combine the estimate of $v$ in 4.12 and Lemma A.2 in the appendix $A$. We know that there exists $f_h^2$ such that the solution of

$$\begin{cases} \partial_t u_j(t) - \Delta_h u_j(t) = 0, & T_2 < t < T, \ j = 1, \cdots, N \\ u_0(t) = 0, \ u_{N+1}(t) = f_h^2(t), & T_2 < t < T, \\ u_j(T_2) = v_j^{T_2}, \ j = 1, \cdots, N, & \end{cases} \tag{4.13}$$

satisfies

$$u_j(T)=0, \qquad j=1,\cdots,N. \tag{4.14}$$

Moreover, for $\alpha>\alpha_*$,$\beta_0\in(0,1)$ and $L>0$,there exist positive constants $A,B$ such that

$$\left\|f_h^2\right\|_{L^2(T_2,T)} \le Ae^{B/h^{\beta_0}}e^{\alpha L^2/(T-T_2)}\Big(h\sum_{j=1}^N|v_j^{T_2}|^2\Big)^{1/2} \tag{4.15}$$

This can be done similarly by using the Kannai's formula to combine the solutions and control of the two systems (4.12) and (A.5), i.e. by using the same technique in the proof of Theorem 3.1. Note that due to the exact controllability of the semi-discret wave equation (A.5), the error function $\varphi_j$(recall system (3.13)) vanishes everywhere.

Combining (4.5),(4.10) and (4.15) and taking (4.6) and (4.11) account we get

$$\left\|f_h^2\right\|_{L^2(T_2,T)} \le C_0Ae^{\frac{\alpha L^2}{T-T_2}+\frac{B}{h^{\beta_0}}-\frac{C_2(T_2-T_1)}{h^2}}\Big(h\sum_{j=1}^N|u_j^0|^2\Big)^{1/2}. \tag{4.16}$$

Since $\beta_0\in(1,2)$, $h^{\beta_0}>h^2$ holds for $h$ small enough, and we get

$$\frac{\alpha L^2}{T-T_2}+\frac{B}{h^{b_0}}-\frac{C_2(T_2-T_1)}{h^2}\to-\infty \qquad \text{as } \ h\to 0.$$

Hence $\|f_h^2\|_{L^2(T_2,T)}$ is uniformly bounded.

*Conclusion:* Putting all these results together we conclude that the control

$$\tilde{f}_h=\begin{cases} f_h^1 & \text{in} \quad [0,T_1],\\ 0 & \text{in} \quad [T_1,T_2],\\ f_h^2 & \text{in} \quad [T_2,T]\end{cases} \tag{4.17}$$

is such that

$$\left\|\tilde{f}_h\right\|_{L^2(0,T)} \le C(\varepsilon_0)e^{\alpha L(\varepsilon_0)^2/T_1}\Big(h\sum_{j=1}^N|u_j^0|^2\Big)^{1/2}, \tag{4.18}$$

with a constant $C>0$ independent of $h$. Moveover, the solution of (1.5) satisfies

$$u_j(T)=0, \qquad j=1,\cdots,N.$$

This is the conclusion we wanted to prove.

Note also that the control $\tilde{f}_h$ given by (4.17) is concentrated on the intervals $[0,T_1]$ and $[T_2,T]$ and that its restriction to the last interval $[T_2,T]$ is exponentially small with respect to $h\to 0$. $\square$

## 5. Open problems

**Problem 1. Complete discretization.** In this paper we present a result relating to the semi-discrete schemes with respect to the space variable. It obvious to see that the time derivative remains to be continuous. However, engineers always interested in the *complete* approximation schemes, i.e., the discrete approximation both in time and on space. This is due to the fact that the computer can not understand what the "derivative" means. It is still an open problem whether the series of controls, which is obtained from the complete discrete systems, remains to converge to the continuous one.

**Problem 2. Carleman Inequalities.** There is another possibility to obtain the controllability by means of the Carleman inequality. In the context of continuous models there are a number of fixed point techniques that allow one to extend the results of controllability of linear waves and heat equations to semilinear equations with moderate nonlinearities (globally Lipschitz ones, for instance)[Z]. These techniques need to be combined with Carleman or multiplier inequalities to estimate the dependence of the observability constants on the potential of the linearized equation. It is still an open problem whether there exist some kind of numerical approximation schemes allowing to obtain the uniform Carleman estimates making it possible to extend the results of the linear discrete systems to semilinear ones. We attempt to deduce a corresponding discrete Carleman inequalities, in terms of some special numerical approximation schemes for the $1-d$ continuous constant coefficient heat equation in the appendix. However, we failed to obtain the result on that way.

**Problem 3. Multi-dimensional heat equation.** The controllability for the $1-d$ semi-discrete heat equation has been developed very well. However, in the author's knowledge, there still not any reference relating to the multi-dimensional semi-discrete heat equation. However, as E. Zuazua has shown in [Z2], some uniform control properties exist for the $2-d$ space semi-discrete wave equation, by means of the filtering method. It is a very interesting problem that whether the transmutation method could be applied to attain some similar results of the heat equation by means of the wave one.

## Appendix A. Some technical proofs

A.1. **Boundary regularity of the heat kernel.** We first give out the boundary regularity property of the heat kernel.

**Lemma A.1.** *Let $v(t,s)$ is the solution of system (3.4)-(3.6). It holds*

$$v(t,\pm L)\in C^0(0,T),\qquad \partial_s v(t,\pm L)\in L^2(0,T). \tag{A.1}$$

*Proof.* We first reduce the problem to the case $L=\pi/2$ using the rescaling $(t,s)\mapsto(\sigma^2 t,\sigma s),\sigma>0$ with $\sigma=\pi/(2L)$. Since $\sigma$ is finite, it is enough to prove Lemma A.1 in the particular case $L=\pi/2$.

For any fixed $\varepsilon\in(0,1)$ the fundamental control $v\in C^0([0,T],\mathcal{M}((-L,L)))$ is defined as the solution of

$$\partial_t v-\partial_s^2 v=0\ \text{ in}(0,T)\times(-L,L),\ \ (v|_{s=L},v|_{s=-L})=b,\ \ v|_{t=0}=\delta,$$

where the control $b$ is defined by $b(t)=0$ for $t\le\varepsilon T$ and $\partial_s b\in L^2(\varepsilon T,T)^2$ for $t\in(\varepsilon T,T)$. This is due to the Proposition 5.2 in [L]. Hence we need to prove that $\partial_s b\in L^2(0,\varepsilon T)$.

Assume that $v$ is the solution of the equation

$$\begin{cases}\partial_t v-\partial_s^2 v=0, & t\in(0,\varepsilon T),\ \ s\in(-L,L)\\ v(t,L)=v(t,-L)=0, & t\in(0,\varepsilon T)\\ v|_{t=0}=\delta. & \end{cases} \tag{A.2}$$

Setting $e_j(s)=\sin(j(s+\pi/2))/\sqrt{\pi}$ defines an orthonormal basis $(e_j)_{j\in\mathbb{Z}}$ of $L^2((-L,L))$ such that $e_j$ is an eigenvector of $-\Delta_s$ with eigenvalue $j^2$. In the weak topology, the Dirac mass can be decomposed in this basis as $\delta_s=\sum_j e_j(0)e_j(s)$. Note that $e_j(0)=\sin(j\pi/2)\in\{0,1\}$, so that the sequence $(e_j(0))_{j\in\mathbb{Z}}$ is bounded. For $t\in(0,\varepsilon T)$, we introduce the coordinates $(v_j(t))_{j\in\mathbb{Z}}$ of $v(t,\cdot)\in L^2((-L,L))$ in the Hilbert basis $(e_j)_{j\in\mathbb{Z}}$. Now the function $v$ can be written as

$$v(t,s)=\sum_j v_j(t)e_j(s). \tag{A.3}$$

As in [FR], these coordinates can be computed by $v_j(0)=e_j(0)$ and

$$v_j(t)=e^{-j^2t}v_j(0). \tag{A.4}$$

Taking into account that $L=\pi/2$ and the definition of $e_j$, we have

$$\partial_s(t,\pm L)=\pm\sum_j(-1)^j je^{-j^2t}\qquad\text{for}\qquad t\in(0,\varepsilon T).$$

By directly computation we know that $\partial_s(t,\pm L)$ is bounded, hence $v\in L^2(0,\varepsilon T)$.

Combining $\partial_s v(t,\pm L)\in L^2(0,\varepsilon T)$ and $\partial_s v(t,\pm L)\in L^2(\varepsilon T,T)$ we get the desired regularity of $v$ in (A.1). □

A.2. **Global null controllability of the $1-d$ semi-discrete wave equation.** Let us consider the $1-d$ semi-discrete wave equation:

$$\left\{\begin{array}{ll} \partial_s^2\omega_j - \Delta_h\omega_j = 0, & s\in(0,L),\ \ j=1,\cdots,N, \\ \omega_0(s)=0,\ \ \omega_{N+1}(s)=g_h(s), & s\in(0,L), \\ \omega_j(0)=\omega_j^0,\ \ \partial_s\omega_j(0)=0 & j=1,\cdots,N. \end{array}\right. \tag{A.5}$$

System (A.5) is a finite-difference space semi-discretization of the wave equation with control on the extreme $x=1$.

In the following Lemma we state a result guaranteeing the null controllability of solutions of system (A.5) with a control that may grow exponentially as $h\to 0$.

**Lemma A.2.** *Let $L>0$. Then there exist positive constant $A,B>0$ such that for every $h>0$ there exists a control $g_h\in L^2(0,L)$ such that the solution of (A.5) satisfies*

$$u_j(L)=\partial_s u_j(L)=0,\qquad j=1,\cdots,N \tag{A.6}$$

*with the estimates*

$$\|g_h\|_{L^2(0,L)}\le Ae^{B/h^{\beta_0}}\Big(h\sum_{j=1}^N|\omega_j^0|^2\Big)^{1/2} \tag{A.7}$$

*for any $\beta_0\in(1,2)$.*

This result may be proved using HUM as a direct consequence of the following observability estimate for the adjoint system:

$$\left\{\begin{array}{ll} \partial_s^2\varphi_j - \Delta_h\varphi_j = 0, & j=1,\cdots,N,\ \ s\in(0,L) \\ \varphi_0(s)=\varphi_{N+1}(s)=0, & s\in(0,L), \\ \varphi_j(L)=\varphi_j^0,\ \ \partial_s\varphi_j(0)=0, & j=1,\cdots,N. \end{array}\right. \tag{A.8}$$

**Proposition A.1.** *Let $L>0$. Then there exist positive constant $A,B>0$ such that*

$$h\sum_{j=1}^N|\varphi_j^0|^2\le Ae^{B/h^{\beta_0}}\int_0^L\Big|\frac{\varphi_N(s)}{h}\Big|^2ds \tag{A.9}$$

*for every solution of (A.8) and $\beta_0\in(1,2)$.*

Before to prove Proposition A.1 we introduce a extended Ingham inequality which was proved by Micu and Zuazua in [MZ]:

**Lemma A.3.** *Let $f=f(s)$ be of the form $f(s)=\sum_{k\in\mathbb{Z}}a_ke^{i\mu_ks}$, where $\mu_k$ is a sequence of real numbers. We assume that there exist $k\in\mathbb{Z}$, $\gamma>0$, and $\gamma_\infty>0$ such that*

$$\mu_{k+1}-\mu_k\ge\gamma_\infty>0 \text{ if } |k|>N, \tag{A.10}$$

$$\mu_{k+1}-\mu_k\ge\gamma>0 \text{ for any } k\in\mathbb{Z}. \tag{A.11}$$

*Let $J=[0,L]\subset\mathbb{R}$ be a finite interval with $L>\frac{2\pi}{\gamma_\infty}$. Then, there exist two positive constants $C_1, C_2>0$ such that*

$$C_1\sum_{k\in\,\mathbb{Z}}|a_k|^2\le\int_J|f(s)|^2ds\le C_2\sum_{k\in\,\mathbb{Z}}|a_k|^2 \tag{A.12}$$

*for all $(a_k)_{k\in\mathbb{Z}}\in\ell^2$.*

*More precisely, $C_1=C_1(2N+1)$ and $C_2=C_2(2N+1)$, where $C_i(j), i=1,2,$ are given by the following recurrent formulas:*

$$\begin{cases} C_1(j+1)=\Big[\Big(\dfrac{2C_2(j)}{|J|}+1\Big)\dfrac{4}{C_1(j)(|J|\gamma_\infty-2\pi)^2\gamma^2}+\dfrac{2}{|J|}\Big]^{-1}, \\ C_2(j+1)=2(|J|(j+1)+C_2(0)), j=0,1,\cdots, \end{cases} \tag{A.13}$$

*and $C_1(0)$, $C_2(0)$ are such that (A.12) holds in the particular case in which $\gamma_\infty=\gamma>0$.*

**Remark A.1.** *Lemma A.3 allows us to deduce that (A.12) holds when the length of the interval $J$ is smaller. Indeed, it suffices that $|J|>2\pi/\gamma_\infty$, $\gamma_\infty$ being the "asymptotic gap" of the sequence $\{\mu_k\}$, which is general larger than $\gamma$. As we will show later, this extended Ingham inequality allows us to deduce the boundary observability estimate (A.9) for any $L>0$. Lemma A.3 was proved by S.Micu and E.Zuazua in* [MZ].

*Proof of Proposition A.1.* It was shown in [IZ] that the energy of (A.8) is conserved. More precisely, taking into account that $\partial_s\varphi_j(0)=0, j=1,\cdots,N$, the energy $E_h(s)$ of (A.8) is given by

$$E_h(s)=E_h(0)=\frac{h}{2}\sum_{j=0}^{N}\Big|\frac{\varphi_{j+1}^0-\varphi_j^0}{h}\Big|^2. \tag{A.14}$$

Let us recall the system that eigenfunction $\Phi=(\Phi_1,\cdots,\Phi_N)$ and eigenvalues $\lambda$ of system (A.8) satisfy:

$$\begin{cases} -\Delta_h\Phi_j=\lambda\Phi_j, \qquad j=1,\cdots,N, \\ \Phi_0=\Phi_{N+1}. \end{cases} \tag{A.15}$$

The eigenvalues and eigenvectors of system (A.15) can be computed explicitly. We have (see [IK], p. 456):

$$\lambda_k(h)=\frac{4}{h^2}\sin^2(\frac{\pi kh}{2}), \qquad j=1,\cdots,N$$

and

$$\Phi^k=(\Phi_j^k)=\sin(\frac{\pi jkh}{2}), \qquad k=1,\cdots,N, j=1,\cdots,N.$$

Solutions of (A.8) admit a Fourier development on the basis of eigenvectors of (A.15). More precisely, every solution $\varphi = (\varphi_1, \cdots, \varphi_N)$ of (A.8) can be written as

$$\varphi(s) = \sum_{k=1}^{N} \Big[ a_k (\cos \sqrt{\lambda_k(h)} s) + b_k (\sin \sqrt{\lambda_k(h)} s) \Big] \Phi^k, \tag{A.16}$$

for suitable $a_k, b_k \in \mathbb{R}, k = 1, \cdots, N$, that can be computed explicitly in terms of the initial data in (A.8).

Moreover, due to the fact that $\partial_s \varphi_j(0) = 0$, we have $b_k = 0, k = 1, \cdots, N$. This allows us to rewrite (A.16) as the form:

$$\varphi(s) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \tilde{a}_k e^{i\mu_k s} \Phi^k, \tag{A.17}$$

where $\mu_k, \tilde{a}_k, k \in \mathbb{Z}$ satisfy:

$$\mu_k = \begin{cases} \sqrt{\lambda_k(h)}, & k = 1, \cdots, N, \\ \sqrt{\lambda_{-k}(h)}, & k = -N, \cdots, -1, \\ k\gamma_\infty, & |k| \geq N \text{and } k = 0, \end{cases} \tag{A.18}$$

$$\tilde{a}_k = \begin{cases} a_k, & k = 1, \cdots, N, \\ a_{-k}, & k = -N, \cdots, -1, \\ 0, & |k| \geq N \text{and } k = 0, \end{cases} \tag{A.19}$$

It is easy to check that $\mu_k, k \in \mathbb{Z}$ satisfy (A.10) and (A.11), by choosing $\gamma = \pi \cos(\frac{\pi h}{2})$. Consequently we obtain that: For any $L > \frac{2\pi}{\gamma_\infty}$, there exist two positive constants $C_1, C_2$ such that

$$C_1 \sum_{k \in \mathbb{Z}} |\tilde{a}_k|^2 \leq \int_0^L |\tilde{a}_k e^{i\mu_k s}|^2 ds \leq C_2 \sum_{k \in \mathbb{Z}} |\tilde{a}_k|^2, \tag{A.20}$$

with $C_1, C_2$ are defined in (A.13).

On the other hand, according to the identity of Lemma 1.1 in [IZ] it follows that

$$h \sum_{j=1}^{N} \Big| \frac{\Phi^k_{j+1} - \Phi^k_j}{h} \Big|^2 = \frac{2}{4 - \lambda_k(h) h^2} \Big| \frac{\Phi^k_N}{h} \Big|^2, \qquad k = 1, \cdots, N. \tag{A.21}$$

Taking into account (A.20) and (A.21), recalling $\lambda_k^h = \frac{4}{h^2}\sin^2(\frac{\pi k h}{2})$, we deduce that for $L > 2\pi/\gamma_\infty$,

$$
\begin{aligned}
&\int_0^L \Big|\frac{u_N(s)}{h}\Big|^2 ds = \int_0^T \Big|\sum_{k\in\mathbb{Z}} \tilde{a}_k e^{i\mu_k t}\frac{\Phi_N^k}{h}\Big|^2 dt \\
&\geq C_1 \sum_{k\in\mathbb{Z}} |\tilde{a}_k|^2 \Big|\frac{\Phi_N^k}{h}\Big|^2 \geq C_1 \sum_{k\in\mathbb{Z}} |\tilde{a}_k|^2 h \sum_{j=1}^N \frac{2C_1}{4-\lambda_k(h)h^2}\Big|\frac{\Phi_{j+1}^k - \Phi_j^k}{h}\Big|^2 \\
&\geq \frac{C_1}{2\sin^2(\frac{\pi k h}{2})} \sum_{k\in\mathbb{Z}} |\tilde{a}_k|^2 h \sum_{j=1}^N \Big|\frac{\Phi_{j+1}^k - \Phi_j^k}{h}\Big|^2 \\
&= \frac{C_1}{\sin^2(\frac{\pi k h}{2})} \sum_{k=1}^N |a_k|^2 h \sum_{j=1}^N \Big|\frac{\Phi_{j+1}^k - \Phi_j^k}{h}\Big|^2.
\end{aligned} \tag{A.22}
$$

Moreover,

$$
\begin{aligned}
E_h(0) &= \frac{1}{2}\sum_{k=1}^N \Big[|b_k|^2 h \sum_{j=1}^N |\Phi_j^k|^2\Big] + \frac{1}{2}\sum_{k=1}^N \Big[|a_k|^2 h \sum_{j=1}^N \Big|\frac{\Phi_{j+1}^k - \Phi_j^k}{h}\Big|^2\Big] \\
&= \frac{1}{2}\sum_{k=1}^N \Big[|a_k|^2 h \sum_{j=1}^N \Big|\frac{\Phi_{j+1}^k - \Phi_j^k}{h}\Big|^2\Big].
\end{aligned} \tag{A.23}
$$

Therefore, as a consequence of (A.22) it follows that

$$
E_h(0) \leq \frac{\sin^2(\frac{\pi k h}{2})}{2C_1} \int_0^L \Big|\frac{u_N(s)}{h}\Big|^2 ds \tag{A.24}
$$

holds for any $L > 2\pi/\gamma_\infty$. Here $C_1$ was defined in (A.13).

Now we estimate $C_1$. Taking into account $\gamma = \pi\cos(\frac{\pi h}{2})$, we compute

$$
(C_1)^{-1} = (C_1(2N+1))^{-1} \leq \frac{2C_2(2N+1)}{|J|}\frac{4}{(|J|\gamma_\infty - 2\pi)^2\gamma^2}(C_1(2N))^{-1}
$$

Since $|J| > \frac{2\pi}{\gamma_\infty}$, there exists a small $\delta > 0$, such that $|J|\gamma_\infty > 2\pi + \delta$, hence

$$
\begin{aligned}
(C_1)^{-1} &\leq \Big(\frac{16}{\delta^2 h^2}\Big)^{2N+1}(2N+1)!(C_1(0))^{-1} \\
&= \Big(\frac{16N^2}{\delta^2}\Big)^{2N+1}(2N+1)!(C_1(0))^{-1} \\
&\leq e^{2(2N+1)\log\frac{16N}{\delta^2} + \log(2N+1)!}(C_1(0))^{-1} \\
&\leq e^{BN^\beta}(C_1(0))^{-1}
\end{aligned} \tag{A.25}
$$

holds for a positive constant $B > 0$ and any $\beta > 1$. On the other hand, the classical result of Young ([Y]) shows that $(C_1(0))^{-1}$ tends to infinity as the order of $N$, if $N \to \infty$. This means

$$(C_1)^{-1} \leq C(\beta)e^{BN^\beta} \tag{A.26}$$

holds for any $\beta > 1$.

Combining (A.24),(A.26) and $h = 1/(N+1)$, we get

$$E_h(0) \leq C(\beta)e^{B/h^\beta} \int_0^L \Big|\frac{\varphi_N(s)}{h}\Big|^2 ds \tag{A.27}$$

for any $T > 2\pi/\gamma_\infty$ and $\beta > 1$.

By directly computation we claim that there exists a constant $C$, such that

$$h\sum_{j=1}^{N} |\varphi_j^0|^2 \leq Ch\sum_{j=1}^{N} \Big|\frac{\varphi_{j+1}^0 - \varphi_j^0}{h}\Big|^2 = CE_h(0). \tag{A.28}$$

Combining (A.27) and (A.28), it holds

$$h\sum_{j=1}^{N} |\varphi_j^0|^2 \leq C(\beta)e^{B/h^\beta} \int_0^L \Big|\frac{\varphi_N(s)}{h}\Big|^2 ds \tag{A.29}$$

for any $T > 2\pi/\gamma_\infty$ and $\beta > 1$.

Since $\gamma_\infty$ is a positive parameter, we know that (A.29) holds for any $T > 0$.

Fix $\beta = \beta_0 \in (1, 2)$, we finish to prove the desired inequality (A.9) by setting $A = C(\beta_0)$.

## Appendix B. A first attempt to Carleman inequality for semi-discrete $1-d$ heat equation

In this appendix we will try to establish discrete Carleman-type schemes and corresponding Carleman estimate. A complicate numerical approximation are introduced for obtaining the discrete Carleman estimate.

we establish the space semi-discretizations of the $1-d$ heat equation via the Carleman-type schemes, that we will briefly explain now.

Set $\Omega = (0, L)$. Let $\partial\Omega$ be the boundary of $\Omega$ and $\omega = (a, b) \neq \emptyset$ be a given subdomain of $\Omega$. Let $\omega_0$ be another open subsets of $\Omega$ such that $\overline{\omega_0} \subset \omega$. Put

$$Q \triangleq \Omega \times (0, T), \qquad Q^{\omega} \triangleq \omega \times (0, T), \qquad Q^{\omega_0} \triangleq \omega_0 \times (0, T).$$

The parabolic operator $L$ is defined as

$$Lu \triangleq u_t - u_{xx}. \tag{B.1}$$

We consider the following $1-d$ heat equation:

$$\begin{cases} Lu = g(x,t), & (x,t) \in \Omega \times (0,T) \\ u(0,t) = u(L,t) = 0, & t \in (0,T) \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{B.2}$$

For each $g \in L^2(0, T; H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, there exists a unique weak solution $u \in C(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ of system (B.2).

In the continuous case, we can choose some special weight function $\theta = e^{\lambda\alpha}$ and change the function $u$ to $v = \theta u$.

Let $\psi \in C^\infty(\bar{\Omega})$ so that $\psi > 0$ in $\Omega$ and $\psi = 0$ on $\partial\Omega$ and $|\psi_x| > 0$ for all $x \in \bar{\Omega} \setminus \omega_0$. The existence of such a function $\psi$ is obvious. For example, we should take $\psi(x) = x(L - x)$, which satisfies all of the properties when $\omega$ contains the point $x = L/2$.

For any given parameters $\lambda$ and $\mu$, we set

$$\begin{aligned} \alpha(x,t) &= (t(T-t))^{-1}(e^{\mu\psi(x)} - e^{2\mu\|\psi\|_{C(\bar{\Omega})}}), \\ \varphi(x,t) &= (t(T-t))^{-1}e^{\mu\psi(x)}. \end{aligned} \tag{B.3}$$

By straightforward computation we get

$$u_t = \theta^{-1}(v_t - \lambda\alpha_t v), \qquad u_{xx} = \theta^{-1}(\lambda^2\alpha_x^2 v - \lambda\alpha_{xx}v - 2\lambda\alpha_x v_x + v_{xx}). \tag{B.4}$$

Hence (B.2) is transformed into the new system

$$\text{(B.5)}\quad \begin{cases} Pv \overset{\triangle}{=} v_t - \lambda(\alpha_t + \lambda\alpha_x^2 - \alpha_{xx})v - v_{xx} + 2\lambda\alpha_x v_x = \theta g(x,t), & (x,t) \in \Omega \times (0,T) \\ v(0,t) = v(L,t) = 0, & t \in (0,T) \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases}$$

Obviously, by the change of variable $v = \theta u$ and the well-posedness of system (B.2) we can deduce that system (B.5) is well-posed in the same functional space.

It is well known that the Carleman estimate holds for the parabolic operator $P$. More precisely, there is some $\mu_0 > 0$ such that for all $\mu > \mu_0$, one can find two constants $C = C(\mu) > 0$ and $\lambda_1 = \lambda_1(\mu)$ so that for the solution of system (B.5), the estimate

$$\text{(B.6)}\qquad \lambda^3 \int_Q \varphi^3 v^2 dxdt + \lambda \int_Q \varphi v_x^2 dxdt \le C\Big( \int_Q \theta^2 g^2 + \int_{Q^\omega} (\lambda^3\varphi^3 v^2 + \lambda\varphi v_x^2) dxdt \Big)$$

holds for all $\lambda > \lambda_1$.

We would like to get an analogue of (B.6) for the discrete case in the rest of the paper.

We are trying to obtain the estimate by making an analogue of the process of proving the continuous Carleman estimate. But for providing the convergence of the discrete solution, we only can conclude a discrete Carleman inequality with an extra term, related to the initial data. This extra term is very small and will vanish when mesh-size tends to zero. It is still an open problem whether it is possible to deduce the unique uniform continuation by this discrete Carleman estimate.

B.1. **The Carleman-type semi-discretzation.** We consider the system (B.5) and discretize the $v$, $v_t, v_x$ and $v_{xx}$ as follows:

$$\text{(B.7)}\qquad \begin{aligned} v &\longrightarrow \frac{v_{j+1} + v_{j-1}}{2}, & v_t &\longrightarrow v_j' \\ v_x &\longrightarrow \frac{v_{j+1} - v_{j-1}}{2h}, & v_{xx} &\longrightarrow \frac{v_{j+1} + v_{j-1} - 2v_j}{h}. \end{aligned}$$

For the temporary variables $v_j, j = 0, \cdots, N+1$ we define that $v_j = \theta_j u_j$, $\theta_j = e^{\lambda\alpha_j}$ and $\alpha_j$ is as the form

$$\text{(B.8)}\qquad \alpha_j = \frac{e^{\mu\psi_j} - e^{2\mu\|\psi_j\|}}{(t+s(h))(T-t+s(h))},$$

with $s(h)$ is a positive function such that $s(h) < Ch^{1/5}$.

We then establish the Carleman-type discretization of system (B.2) by means of the structure of (B.5):

$$\begin{cases} P_h v_j = \theta_j(t) g_j(t), & t \in (0,T) \\ v_0(t) = \theta_0(t) u_0(t) = 0, \; v_{N+1}(t) = \theta_{N+1}(t) u_{N+1}(t) = 0, & t \in (0,T) \\ v(x_j, 0) = \theta_j(0) u_j(0), & x \in \Omega, \end{cases} \tag{B.9}$$

with

$$P_h v_j \triangleq v_j' - \lambda(\alpha_j' + \lambda(d_h\alpha_j)^2 - \Delta_h \alpha_j)\frac{v_{j+1}+v_{j-1}}{2} - \Delta_h v_j + \lambda d_h \alpha_j \frac{v_{j+1}-v_{j-1}}{h}. \tag{B.10}$$

Here and thereafter, $'$ denotes the derivation with respect to time $t$.

**Remark B.1.** *Although the new function $v_j, j = 0, \cdots, N+1$ appear in (B.9),it is easy to see that (B.9) is a discrete schemes for $u_j$ by simply replacing $v_j$ by $\theta_j u_j$.*

B.2. **Discrete Carleman inequality.** Our main result for this new type of schemes (B.9) is:

**Theorem B.1. Discrete Carleman inequality**

*Let $s$ be a positive function of $h$ so that $s(h) \le Ch^{1/5}$. Set*

$$\varphi_j = \frac{e^{\mu\psi_j}}{(t+s(h))(T-t+s(h))}, \tag{B.11}$$

*with $\psi_j = \psi(jh)$. Then there is some $\mu_0 > 0$ such that for all $\mu > \mu_0$, one can find two constants $C = C(\mu) > 0$ and $\lambda_1 = \lambda_1(\mu)$ so that for the solution $U = (u_0, \cdots, u_{N+1})$ of the Carleman-type discretization of the heat equation (B.2), the estimate*

$$\begin{aligned} &\int_0^T h\sum_{j=1}^N \Big(\lambda^3\varphi_j^3\theta_j^2 u_j^2 + \lambda\varphi_j\theta_j^2(d_h u_j)^2\Big)dt \\ &\le C\int_0^T h\sum_{j=1}^N \theta_j^2 g_j^2 dt + C\int_0^T h\sum_{a\le x_j\le b}\Big(\lambda^3\varphi_j^3\theta_j^2u_j^2 + \lambda\varphi\theta_j^2(d_hu_j)^2\Big)dt \\ &\quad + C\frac{1}{s^2(h)}\max_j \theta_j^2(0)\Big(\int_0^T h\sum_{j=1}^N g_j^2 dt + h^{-1}\sum_{j=1}^N u_j^2(0)\Big) \end{aligned} \tag{B.12}$$

*holds for all $h > 0$ and all $\lambda > \lambda_1$.*

**Remark B.2.** *The inequality (B.12) has the same structure as the continuous one (B.6), except for some extra terms. However it is obvious that the limit of (B.12) is (B.6) because $\theta_j(0)$ has a exponential decay respect to the mesh size $h$.*

*Proof :* We analyze the discrete operator $P_h v_j$ with the similar process of the continuous case.

We set

$$I_1 = -\lambda d_h \alpha_j \frac{v_{j+1} - v_{j-1}}{h} - v_j', \qquad I_2 = \Delta_h v_j + \lambda \mathcal{A}_j \frac{v_{j+1} + v_{j-1}}{2}, \tag{B.13}$$

with

$$\mathcal{A}_j = \alpha_j' + \lambda (d_h \alpha_j)^2 - \Delta_h \alpha_j. \tag{B.14}$$

For the derivative respect to $t$, we have the following estimates

$$|\alpha_j'| \leq C\varphi_j^2, \qquad |\alpha_j''| \leq C\varphi_j^3, \qquad |\varphi_j'| \leq C\varphi_j^2. \tag{B.15}$$

For $2I_1 I_2$, one sees that
(B.16)

$$\begin{aligned}
2I_1 I_2 &= -2\lambda d_h \alpha_j \frac{v_{j+1} - v_{j-1}}{h} \Delta_h v_j - 2v_j' \Delta_h v_j - \lambda^2 d_h \alpha_j \frac{v_{j+1}^2 - v_{j-1}^2}{h} - \lambda \mathcal{A}_j v_j (v_{j+1} + v_{j-1}) \\
&= -2\lambda \frac{1}{h} (d_h \alpha_j (d_h v_j)^2 - d_h \alpha_{j-1} (d_h v_{j-1})^2) - (2 + \lambda h^2 \mathcal{A}_j) \frac{1}{h} (v_{j+1}' d_h v_j - v_j' d_h v_{j-1}) \\
&\quad -\lambda^2 \frac{1}{h} (d_h \alpha_j \mathcal{A}_j v_{j+1}^2 - d_h \alpha_{j-1} \mathcal{A}_j v_j^2) - \lambda^2 \frac{1}{h} (d_h \alpha_j \mathcal{A}_j v_j^2 - d_h \alpha_{j-1} \mathcal{A}_{j-1} v_{j-1}^2) \\
&\quad + \Big( (1 + \lambda \frac{h^2}{2} \mathcal{A}_j)(d_h v_j)^2 - \lambda \mathcal{A}_j v_j^2 \Big)' + 2\lambda \Delta_h \alpha_j (d_h v_{j-1})^2 - \lambda \frac{h^2}{2} \mathcal{A}_j' (d_h v_j)^2 \\
&\quad + \lambda^2 d_h (d_h \alpha_j \mathcal{A}_j) v_j^2 + \lambda^2 d_h (d_h \alpha_{j-1} \mathcal{A}_{j-1}) v_{j-1}^2 + \lambda \mathcal{A}_j' v_j^2.
\end{aligned}$$

From the definition of $I_1, I_2$ we have $|P_h v_j|^2 \geq 2I_1 I_2$. Sum it from 1 to $N$ respect to $j$ and integrate it from 0 to $T$ respect to $t$, and we obtain

$$\begin{aligned}
&\int_0^T h\sum_{j=1}^N |P_h v_j|^2 dt \\
&\geq \int_0^T h\sum_{j=1}^N \Big( -2\lambda\frac{1}{h}(d_h\alpha_j(d_hv_j)^2 - d_h\alpha_{j-1}(d_hv_{j-1})^2) \\
&\qquad -(2+\lambda h^2\mathcal{A}_j)\frac{1}{h}(v'_{j+1}d_hv_j - v'_jd_hv_{j-1}) \\
&\qquad -\lambda^2\frac{1}{h}(d_h\alpha_j\mathcal{A}_jv_{j+1}^2 - d_h\alpha_{j-1}\mathcal{A}_jv_j^2) \\
&\qquad -\lambda^2\frac{1}{h}(d_h\alpha_j\mathcal{A}_jv_j^2 - d_h\alpha_{j-1}\mathcal{A}_{j-1}v_{j-1}^2)\Big)dt \\
&\quad +\int_0^T h\sum_{j=1}^N \Big(2\lambda\Delta_h\alpha_j(d_hv_{j-1})^2 - \lambda\frac{h^2}{2}\mathcal{A}'_j(d_hv_j)^2 + \lambda\mathcal{A}'_jv_j^2 \\
&\qquad +\lambda^2d_h(d_h\alpha_j\mathcal{A}_j)v_j^2 + \lambda^2d_h(d_h\alpha_{j-1}\mathcal{A}_{j-1})v_{j-1}^2\Big)dt \\
&\quad +\int_0^T h\sum_{j=1}^N \Big((1+\lambda\frac{h^2}{2}\mathcal{A}_j)(d_hv_j)^2 - \lambda\mathcal{A}_jv_j^2\Big)'dt.
\end{aligned} \tag{B.17}$$

We can see that the right hand of (B.17) could be divided into three parts. Next we analysis these three parts one by one.

First, from the definition of $\mathcal{A}_j$ and the properties of $\alpha_j$, we get

$$\begin{aligned}
&0 \leq \\
&\int_0^T h\sum_{j=1}^N \Big\{-2\lambda\frac{1}{h}(d_h\alpha_j(d_hv_j)^2 - d_h\alpha_{j-1}(d_hv_{j-1})^2) \\
&-(2+\lambda h^2\mathcal{A}_j)\frac{1}{h}(v'_{j+1}d_hv_j - v'_jd_hv_{j-1}) \\
&-\lambda^2\frac{1}{h}(d_h\alpha_j\mathcal{A}_jv_{j+1}^2 - d_h\alpha_{j-1}\mathcal{A}_jv_j^2) - \lambda^2\frac{1}{h}(d_h\alpha_j\mathcal{A}_jv_j^2 - d_h\alpha_{j-1}\mathcal{A}_{j-1}v_{j-1}^2)\Big\}dt.
\end{aligned} \tag{B.18}$$

Secondly, from the lemma B.8 which is put in the appendix $C$, we have

$$\begin{aligned}
&\int_0^T h\sum_{j=1}^N \Big((1+\lambda\frac{h^2}{2}\mathcal{A}_j)(d_hv_j)^2 - \lambda\mathcal{A}_jv_j^2\Big)'dt \geq \\
&-C\Big(\max_j \theta_j^2(0)h\sum_{j=0}^N u_j^2(0) + \frac{\lambda}{s^2(h)}\max_j \theta_j^2(0)\int_0^T h\sum_{j=0}^N(\varphi_j^3\theta_j^2u_j^2 + g_j^2)dt\Big).
\end{aligned} \tag{B.19}$$

Finally, we calculate the rest of the terms.

Provided by the lemma B.7, we obtain

$$
\begin{aligned}
&\frac{1}{h}\int_0^T h\sum_{j=1}^N 2\lambda\Delta_h\alpha_j(d_hv_{j-1})^2dt\\
\geq\;& -C\lambda\mu\int_0^T h\sum_{j=1}^N \varphi_j(d_hv_{j-1})^2dt\\
&+\lambda\mu^2\frac{1}{h}\int_0^T h\sum_{j=1}^N \varphi_j((d_h\psi_j)^2+(d_h\psi_{j-1})^2)(d_hv_{j-1})^2dt\\
&+2\lambda o(h)\frac{1}{h}\int_0^T h\sum_{j=1}^N \varphi_j(d_hv_{j-1})^2dt\\
\geq\;& -C\lambda\mu\frac{1}{h}\int_0^T h\sum_{j=1}^N \varphi_j(d_hv_{j-1})^2dt\\
&+\lambda\mu^2\frac{1}{h}\int_0^T h\sum_{j=1}^N \varphi_j((d_h\psi_j)^2+(d_h\psi_{j-1})^2)(d_hv_{j-1})^2dt.
\end{aligned}
\tag{B.20}
$$

Here we use the notation $o(h)$ to denote the lower level of $h$, i.e., when $h$ tends to zero, $o(h)$ also tends to zero.

By some explicit computation we deduce that

$$-\lambda\frac{h^2}{2}\mathcal{A}_j'(d_hv_j)^2\geq -C\lambda^2h^2\varphi_j^3(d_hv_j)^2. \tag{B.21}$$

Further,

$$
\begin{aligned}
d_h(d_h\alpha_j\mathcal{A}_j) &= d_h\Big(d_h\alpha_j(\alpha_j'+\lambda(d_h\alpha_j)^2-\Delta_h\alpha_j)\Big)\\
&= d_h(d_h\alpha_j\alpha_j')+d_h(\lambda d_h\alpha_j^3)-d_h(d_h\alpha_j\Delta_h\alpha_j).
\end{aligned}
\tag{B.22}
$$

We now compute those terms in the right hand of (B.22) carefully.

First we have

$$
\begin{aligned}
d_h(d_h\alpha_j\alpha_j') &= \frac{1}{h}\Big(\frac{\alpha_{j+2}-\alpha_{j+1}}{h}\alpha_{j+1}'-\frac{\alpha_{j+1}-\alpha_j}{h}\alpha_j'\Big)\\
&= \Big(\frac{\alpha_{j+2}-\alpha_{j+1}}{h}-\frac{\alpha_{j+1}-\alpha_j}{h}\Big)\alpha_{j+1}'+\frac{\alpha_{j+1}-\alpha_j}{h}\frac{\alpha_{j+1}-\alpha_j}{h}\\
&= \Delta_h\alpha_{j+1}\alpha_{j+1}'+d_h\alpha_jd_h\alpha_j',
\end{aligned}
\tag{B.23}
$$

and

$$
\begin{aligned}
d_h(\lambda(d_h\alpha_j)^3) &= \lambda\frac{1}{h}\Big[\Big(\frac{\alpha_{j+2}-\alpha_{j+1}}{h}\Big)^3-\Big(\frac{\alpha_{j+1}-\alpha_j}{h}\Big)^3\Big]\\
&= \lambda\Delta_h\alpha_{j+1}((d_h\alpha_{j+1})^2+(d_h\alpha_j)^2+d_h\alpha_{j+1}d_h\alpha_j).
\end{aligned}
\tag{B.24}
$$

Moreover,

(B.25)

$$\begin{aligned} d_h(d_h\alpha_j\Delta_h\alpha_j) &= d_h\Big(\frac{\alpha_{j+1}-\alpha_j}{h}\frac{\alpha_{j+1}+\alpha_{j-1}-2\alpha_j}{h^2}\Big) \\ &= \Big[\Big(\frac{\alpha_{j+2}-\alpha_{j+1}}{h}-\frac{\alpha_{j+1}-\alpha_j}{h}\Big)\frac{\alpha_{j+2}+\alpha_j-2\alpha_{j+1}}{h^2}+\frac{\alpha_{j+1}-\alpha_j}{h}d_h(\Delta_h\alpha_j)\Big] \\ &= \Delta_h\alpha_{j+1}\Delta_h\alpha_j+d_h\alpha_j d_h(\Delta_h\alpha_j). \end{aligned}$$

Combining (B.22)-(B.25) together and with the carefully computation, we find that

$$\begin{aligned} &\frac{1}{h}\int_0^T h\sum_{j=1}^N\Big(\lambda^2 d_h(d_h\alpha_j\mathcal{A}_j)v_j^2+\lambda^2 d_h(d_h\alpha_{j-1}\mathcal{A}_{j-1})v_{j-1}^2+\lambda\mathcal{A}_j'v_j^2\Big)dt \\ &\geq -C\lambda^3\mu^3\frac{1}{h}\int_0^T h\sum_{j=1}^N\varphi_j^3v_j^2dt+2\lambda^3\mu^4\frac{1}{h}\int_0^T h\sum_{j=1}^N\varphi_j^3(d_h\psi_j)^4v_j^2dt. \end{aligned} \tag{B.26}$$

Here the change of the subscript $j$ to $j-1$ does not affect the results since $v_j$ vanishes exponentially at the boundary.

Combining (B.17)-(B.21) and (B.26) we get the designed discrete Carleman inequality (B.12).

B.3. **Comparison with the classical semi-discrete system.** In the previous subsection we introduced a semi-discretization of the equation (B.5) by means of the weight function $\theta=e^{\lambda\alpha}$. However, the schemes is too complicated to deduce the convergence of the solution. For facilitating the proof, we present another discrete format to compare with the classical schemes.

**Definition B.1.** *Let $x_j=jh, j=0,\cdots,N+1$. $\theta_j=e^{\lambda\alpha_j}$, $\alpha_j$ is defined in (B.11). $c_1,c_1',c_2,c_2'$ are given in Lemma (B.9). $f_j=f(x_j), u_j(0)=u(x_j,0)$. A Carleman-type discretization of* 1*-d heat equation is of the following:*

$$\begin{cases} u_j'-\dfrac{u_{j+1}+u_{j-1}-2u_j}{h^2}+hf(u_j)=g_j & t\in(0,T) \\ u_0(t)=u_{N+1}(t)=0, & t\in(0,T) \\ u_j(0)=u(x_j,0) & j=1,\cdots,N. \end{cases} \tag{B.27}$$

*where*

$$f(u_j)=\frac{\lambda^2}{2}\mathcal{K}_1u_{j+1}+\frac{\lambda^2}{6}\mathcal{K}_2u_j+\frac{\lambda^3}{2}\mathcal{K}_3u_{j-1}+\frac{\lambda^2}{2}\mathcal{K}_4d_hu_j+\lambda\mathcal{K}_5d_hu_{j-1}, \tag{B.28}$$

*and*

$$
\begin{aligned}
\mathcal{K}_1 = &\ \lambda(d_h\alpha_j - \mathcal{A}_j)e^{\lambda c_1}(d_h\alpha_j)^2 + 2\Delta_h\alpha_j\mathcal{A}_j,\\
\mathcal{K}_2 = &\ \lambda e^{\lambda c_1'}(d_h\alpha_j)^3 - \lambda e^{\lambda c_2'}(d_h\alpha_{j-1})^3 - 3h(\Delta_h\alpha_j)^2,\\
\mathcal{K}_3 = &\ (\mathcal{A}_j - d_h\alpha_j)e^{\lambda c_2}(d_h\alpha_{j-1})^2,\\
\mathcal{K}_4 = &\ 2(d_h\alpha_j)^2 - e^{\lambda c_1}(d_h\alpha_j)^2 - 2d_h\alpha_{j-1}\mathcal{A}_j,\\
\mathcal{K}_5 = &\ h\Delta_h\alpha_j - \lambda d_h\alpha_j d_h\alpha_{j-1} - \frac{\lambda}{2}e^{\lambda c_2}(d_h\alpha_{j-1})^2 + \lambda d_h\alpha_{j-1}\mathcal{A}_j.
\end{aligned}
\tag{B.29}
$$

*Here $\mathcal{A}_j$ is defined in (B.14) and $e^{\lambda c_1}, e^{\lambda c_2}, e^{\lambda c_1'}, e^{\lambda c_2'}$ are also defined in lemma B.9.*

In the rest of this subsection we will deduce that the two definition of the Carleman-type schemes are equivalent, i.e., by changing the variable $v_j = \theta_j u_j$, these two definitions are the same.

First we compute the forms of $(v_j)', (v_{j+1}+v_{j-1})/2, \Delta_h v_j$ and $(v_{j+1}-v_{j-1})/h$.

We have

$$
(\theta_j u_j)' = \theta_j' u_j + \theta_j u_j' = \theta_j(\alpha_j' u_j + u_j'),
\tag{B.30}
$$

and

$$
\begin{aligned}
\frac{v_{j+1}+v_{j-1}}{2} &= \frac{\theta_{j+1}u_{j+1}+\theta_{j-1}u_{j-1}}{2}\\
&= \theta_j\frac{u_{j+1}+u_{j-1}}{2} + u_{j+1}\frac{\theta_{j+1}-\theta_j}{2} + u_{j-1}\frac{\theta_{j-1}-\theta_j}{2}\\
&= \theta_j\frac{u_{j+1}+u_{j-1}}{2} + \theta_j u_{j+1}\Big(\lambda h d_h\alpha_j + \frac{\lambda^2}{2}h^2 e^{\lambda c_1}(d_h\alpha_j)^2\Big)\\
&+\theta_j u_{j-1}\Big(-\lambda h d_h\alpha_{j-1} + \frac{\lambda^2}{2}h^2 e^{\lambda c_2}(d_h\alpha_{j-1})^2\Big).
\end{aligned}
\tag{B.31}
$$

Consequently,

(B.32)

$$\begin{aligned}
\Delta_h v_j &= \theta_j \Delta_h u_j + \frac{\theta_{j+1}-\theta_j}{h^2} u_{j+1} - \frac{\theta_j - \theta_{j-1}}{h^2} u_{j-1} \\
&= \theta_j \Delta_h u_j + \frac{\theta_{j+1}-\theta_j}{h^2}(u_{j+1}-u_j) + \frac{\theta_{j+1}-2\theta_j+\theta_{j-1}}{h^2} u_j + \frac{\theta_j-\theta_{j-1}}{h^2}(u_j - u_{j-1}) \\
&= \theta_j \Delta_h u_j + \theta_j \Big(\lambda d_h \alpha_j + \frac{\lambda^2}{2} h e^{\lambda c_1} (d_h\alpha_j)^2\Big) d_h u_j \\
&\qquad + \theta_j \Big(\lambda d_h \alpha_{j-1} - \frac{\lambda^2}{2} h e^{\lambda c_2} (d_h\alpha_{j-1})^2\Big) d_h u_{j-1} \\
&\qquad + \theta_j \Big(\lambda \Delta_h \alpha_j + \frac{\lambda^2}{2}\Big[(d_h\alpha_j)^2 + (d_h\alpha_{j-1})^2\Big]\Big) u_j \\
&\qquad + \theta_j \frac{\lambda^3}{6} h \Big(e^{\lambda c_1'} (d_h\alpha_j)^3 - e^{\lambda c_2'} (d_h\alpha_{j-1})^3\Big) u_j
\end{aligned}$$

moreover,

$$\begin{aligned}
\frac{v_{j+1}-v_{j-1}}{h} &= \frac{v_{j+1}-v_j}{h} + \frac{v_j - v_{j-1}}{h} \\
&= \frac{\theta_{j+1}u_{j+1}-\theta_j u_j}{h} + \frac{\theta_j u_j - \theta_{j-1}u_{j-1}}{h} \\
&= +\theta_j \Big(\lambda d_h\alpha_j + \frac{\lambda^2}{2} h e^{\lambda c_1} (d_h\alpha_j)^2\Big) u_{j+1} + \theta_j d_h u_j \\
&\quad +\theta_j \Big(\lambda d_h\alpha_{j-1} - \frac{\lambda^2}{2} h e^{\lambda c_2} (d_h\alpha_{j-1})^2\Big) u_{j-1} + \theta_j d_h u_{j-1}.
\end{aligned} \tag{B.33}$$

Now we put those items (B.30)-(B.33) into (B.10), and we have

$$\begin{aligned}
& v_j' - \lambda \mathcal{A}_j \frac{v_{j+1}+v_{j-1}}{2} \\
= \;& \theta_j \lambda \alpha_j' u_j + \theta_j u_j' - \theta_j \lambda \alpha_j' u_j - \theta_j \lambda^2 (d_h\alpha_j)^2 u_j + \theta_j \lambda \Delta_h \alpha_j u_j - \lambda h^2 \mathcal{A}_j \Delta_h(\theta_j u_j) \\
= \;& \theta_j (u_j' - \lambda^2 (d_h\alpha_j)^2 u_j + \lambda \Delta_h \alpha_j u_j) - \lambda h^2 \mathcal{A}_j \Delta_h (\theta_j u_j).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& v_j' - \lambda \mathcal{A}_j \frac{v_{j+1}+v_{j-1}}{2} - \Delta_h(\theta_j u_j) \\
= \quad & \theta_j(u_j' - \lambda^2(d_h\alpha_j)^2 u_j + \lambda\Delta_h\alpha_j u_j) - \lambda h^2 \mathcal{A}_j \Delta_h(\theta_j u_j) - \theta_j \Delta_h u_j - \theta_j \lambda \Delta_h \alpha_j u_j \\
& -\theta_j \frac{\lambda^2}{2}\Big((d_h\alpha_j)^2 + (d_h\alpha_{j-1})^2\Big) u_j - \theta_j\lambda(d_h\alpha_j d_h u_j + d_h\alpha_{j-1} d_h u_{j-1}) \\
& +\theta_j \frac{\lambda^2 h}{2}\Big(e^{\lambda c_1}(d_h\alpha_j)^2 d_h u_j + e^{\lambda c_2}(d_h\alpha_{j-1})^2 d_h u_{j-1}\Big) - \theta_j \frac{\lambda^3 h}{6}\Big(e^{\lambda c_1'}(d_h\alpha_j)^3 - e^{\lambda c_2'}(d_h\alpha_{j-1})^3\Big) u_j.
\end{aligned}$$

Taking into account that the left hand of above identity is equal to $\theta_j g(x_j,t) + \lambda d_h\alpha_j(v_{j+1} - v_{j-1})/h$, we obtain

$$\begin{aligned}
g_j = \quad & u_j - \Delta_h u_j \\
& + \frac{\lambda^2 h}{2}\Big(\lambda(d_h\alpha_j - \mathcal{A}_j)e^{\lambda c_1}(d_h\alpha_j)^2 + 2\Delta_h\alpha_j\mathcal{A}_j\Big) u_{j+1} \\
& + \frac{\lambda^2 h}{6}\Big(\lambda e^{\lambda c_1'}(d_h\alpha_j)^3 - \lambda e^{\lambda c_2'}(d_h\alpha_{j-1})^3 - 3h(\Delta_h\alpha_j)^2\Big) u_j \\
& + \frac{\lambda^3 h}{2}\Big((\mathcal{A}_j - d_h\alpha_j)e^{\lambda c_2}(d_h\alpha_{j-1})^2\Big) u_{j-1} \\
& + \frac{\lambda^2 h}{2}\Big(2(d_h\alpha_j)^2 - e^{\lambda c_1}(d_h\alpha_j)^2 - 2d_h\alpha_{j-1}\mathcal{A}_j\Big) d_h u_j \\
& + \lambda h\Big(h\Delta_h\alpha_j - \lambda d_h\alpha_j d_h\alpha_{j-1} - \frac{\lambda}{2}e^{\lambda c_2}(d_h\alpha_{j-1})^2 + \lambda d_h\alpha_{j-1}\mathcal{A}_j\Big) d_h u_{j-1}.
\end{aligned} \tag{B.34}$$

This is the desired discretization which is easy to compare with the classical one. Also it is easy to prove the convergence of the schemes with this new schemes.

B.4. **Convergence of the Carleman-type semi-discretization.** In this subsection, we wish to show that (B.27) is well posed. In fact, we only need to show that the discrete solutions $U = (u_0, \cdots, u_{N+1})$ converge to the solution of the 1-d heat equation when the mesh-size $h$ tends to zero.

First we introduce the following identities:

**Lemma B.1. (Discrete Poincaré inequality)**
*There exists a positive constant $C$ such that the inequality*

$$h\sum_j u_j^2 \le C_1 h \sum_j (d_h u_j)^2 \tag{B.35}$$

*holds for all $h > 0$.*

*Proof:* This is the discrete form of Poincaré inequality. To get it, we simply compute $u_j$, and integrate it respect to the net $x_0, \cdots, x_{N+1}$.

For $u_j$, we have

$$u_j = \sum_{i=1}^{j}(u_i - u_{i-1}) = h\sum_{i=1}^{j}\frac{u_i - u_{i-1}}{h}\cdot 1 \le \Big(h\sum_{i=1}^{j}(\frac{u_i - u_{i-1}}{h})^2\Big)^{1/2}\Big(h\sum_{i=1}^{j}1^2\Big)^{1/2}. \tag{B.36}$$

Taking $h\sum_{i=1}^{j}1^2 = jh$ in and add all $u_j$ together, $j = 1, \cdots, N$, we obtain

$$h\sum_{j=1}^{N}u_j^2 \le Nhh\sum_{j=1}^{N}h\sum_{i=1}^{j}(\frac{u_i - u_{i-1}}{h})^2 \le (Nh)^2 h\sum_{j=1}^{N}(\frac{u_j - u_{j-1}}{h})^2. \tag{B.37}$$

In view of that $(N+1)h = L$, we get the desired discrete Poincaré inequality (B.35).

**Lemma B.2.** *For*

$$\mathcal{A}_j = \lambda\alpha_j' + \lambda^2(\alpha_j)^2 - \lambda\Delta_h\alpha_j, \qquad \varphi_j = \frac{e^{\mu\psi_j}}{(t+s(h))(T-t+s(h))}, \tag{B.38}$$

*there exists a positive constant $C$ such that*

$$|\mathcal{A}_j| \le C\varphi_j^2 \tag{B.39}$$

*holds for $j = 1, \cdots, N+1$. Moreover, we have*

$$\max_{j=1,\cdots,N}|\mathcal{A}_j| \le C\frac{1}{s^2(h)}e^{2\mu\|\psi_j\|} \tag{B.40}$$

*holds. Here $\|\cdot\|$ denote the maximum norm of $\psi_j$.*

*Proof:* First we estimate $\alpha_j'$:

$$\alpha_j' = \Big(\frac{e^{\mu\psi_j} - e^{2\mu\|\psi_j\|}}{(t+s(h))(T-t+s(h))}\Big)' = \frac{(T-2t)(e^{\mu\psi_j} - e^{2\mu\|\psi_j\|})}{((t+s(h))(T-t+s(h)))^2} \le \varphi_j^2 \tag{B.41}$$

On the other hand, recalling lemma B.7, we deduce that

$$\begin{aligned} &d_h\alpha_j \le C\mu\,\|d_h\psi_j\|\,\varphi_j;\\ &\Delta_h\alpha_j \le C\mu^3(\|\Delta_h\psi_j\| + \|d_h\psi_j\|^2 + \|d_h\psi_j\|^3)\varphi_j. \end{aligned} \tag{B.42}$$

Taking these three estimates account, we conclude that

$$|\mathcal{A}_j| = |\lambda\alpha_j' + \lambda^2\alpha_j^2 - \lambda\Delta_h\alpha_j| \le C\lambda^2\mu^3|\varphi_j|^2, \tag{B.43}$$

where $C$ is a positive constant depending on the property of $\psi_j$. Since $\lambda, \mu$ are parameters, we obtain (B.39).

Moreover, by the definition of $\varphi_j$ in (B.11), we observe that

$$|\varphi_j| \le \frac{e^{2\mu\psi_j}}{s(h)[T+s(h)]} \le C\frac{1}{s(h)}e^{2\mu\psi_j}, \tag{B.44}$$

and $s(h) \to 0$ when $h \to 0$. Therefore (B.45) holds with some positive constant $C$.

**Lemma B.3.** *For $\mathcal{K}_1, \cdots, \mathcal{K}_5$, the estimate*

$$\max_{i=1,\cdots,5} |\mathcal{K}_i| \le C \frac{1}{s^4(h)} e^{2\mu\|\psi_j\|} \tag{B.45}$$

*holds, with a positive constant $C$ independent to $h$.*

*Proof:* From those inequalities given in the proof of lemma B.2, we know that $d_h\alpha_j$, $\Delta_h\alpha_j$ are restricted by $\varphi_j$, also $\mathcal{A}_j$ is restricted by $\varphi_j^2$. By this we observe that $\mathcal{K}_i, i = 1, \cdots, 5$ are restricted by $\varphi_j^4$, since those coefficients in $\mathcal{K}_i$ are all bounded. Taking (B.44) account we finish the proof.

**Lemma B.4.** *There exists a constant $C_2 > 0$ such that the estimate*

$$|h \sum_j f(u_j) u_j| \le C_2 \frac{h}{s^4(h)} h \sum_j (|u_j|^2 + h(d_h u_j)^2) \tag{B.46}$$

*holds for $\forall\ h > 0$ and $j = 1, \cdots, N$.*

*Proof:* As $f(u_j)$ defined in (B.28), we take lemma B.3 in and deduce that

$$\begin{aligned} |h \textstyle\sum_j f(u_j) u_j| &\le C \frac{1}{s^4(h)} h \sum_j \Big( |u_{j+1} u_j| + u_j^2 + |u_j u_{j-1}| + \Big| d_h u_j u_j \Big| + \Big| d_h u_{j-1} u_j \Big| \Big) \\ &\le C \frac{1}{s^4(h)} h \sum_j \Big( u_j^2 + \Big| d_h u_j u_j \Big| + \Big| d_h u_{j-1} u_j \Big| \Big). \end{aligned} \tag{B.47}$$

On the other hand, taking into account that

$$2 \sum_j (u_{j+1} - u_j) u_j = \sum_j (u_{j+1} - u_j)^2 \tag{B.48}$$

we have

$$h \sum_j \Big( \Big| \frac{u_{j+1} - u_j}{h} u_j \Big| + \Big| \frac{u_j - u_{j-1}}{h} u_j \Big| \le h \sum_j h (d_h u_j)^2. \tag{B.49}$$

Combing (B.47) and (B.49) we deduce (B.46).

**Lemma B.5.** *For $\forall\ h > 0$, there exists a positive constant $C_3$ such that the inequality*

$$\Big| \frac{\lambda^2}{2} h \sum_j \mathcal{A}_j \Delta_h u_j u_j \Big| \le C_3 \frac{h}{s^2(h)} h \sum_j |d_h u_j|^2 \tag{B.50}$$

*holds.*

*Proof:* In view of lemma B.2 we obtain

$$\begin{aligned}\Big|\sum_j \frac{\lambda}{2}h\mathcal{A}_j\Delta_h u_j u_j\Big| &\le h\sum_j |\frac{\lambda}{2}h\mathcal{A}_j||\Delta_h u_j u_j| \le C\frac{h}{s^2(h)}\sum_j |\Delta_h u_j u_j| \\ &= C\frac{h}{s^2(h)}h\sum_j |d_h u_j|^2.\end{aligned} \tag{B.51}$$

We now give a energy estimate to the discrete system (B.27):

**Lemma B.6.** *Let $s(t) = h^{1/5}$. We denote*

$$E_h(t) = h\sum_j u_j^2(t) \tag{B.52}$$

*to the discrete energy of (B.27). Then there exists a constant $C > 0$, which is independent to $h$, such that the estimate*

$$E_h(t) \le e^{-Ct}E_h(0) + C\int_0^t h\sum_j |g_j(s)|^2 ds \tag{B.53}$$

*holds for $\forall\ h > 0$.*

*Proof:* Multiplying $u_j$ to (B.27), integrating it on $\Omega$ in the discrete level and taking lemma (B.4), (B.5) in, we conclude that

$$\begin{aligned}h\sum_j |g_j u_j| \ge\ & \frac{1}{2}\frac{\partial}{\partial t}(h\sum_j u_j^2) + h\sum_j (d_h u_j)^2 \\ &-C_2\frac{h}{s^4(h)}h\sum_j (u_j^2 + h(d_h u_j)^2) - C_3\frac{h}{s^2(h)}h\sum_j (d_h u_j)^2.\end{aligned} \tag{B.54}$$

By the discrete Poincaré inequality in lemma (B.1), we can replace the terms $(d_h u_j)^2$ by $u_j^2$ if $h$ is sufficiently small. We now arrive at (recall that $s(h) = h^{1/5}$)

$$h\sum_j |g_j u_j| \ge \frac{1}{2}\frac{\partial}{\partial t}(h\sum_j u_j^2) + \Big(\frac{1 - C_2h^{1/5} - C_3h^{3/5}}{C_1} - C_2h^{1/5}\Big)h\sum_j u_j^2. \tag{B.55}$$

However, by Cauchy inequality, it holds

$$h\sum_j |g_j u_j| \le \frac{1}{2\varepsilon}h\sum_j g_j^2 + \frac{\varepsilon}{2}h\sum_j u_j^2 \tag{B.56}$$

where $\varepsilon$ is a positive constant.

Taking (B.56) into (B.55) we obtain

$$\frac{\partial}{\partial t}(h\sum_j u_j^2) + Mh\sum_j u_j^2 \le \frac{1}{\varepsilon}h\sum_j g_j^2, \tag{B.57}$$

where

$$M = 2\frac{1 - C_2h^{1/5} - C_3h^{3/5}}{C_1} - 2C_2h^{1/5} - \varepsilon.$$

Integrating (B.57) respect to $t$, we get the estimate of the discrete energy $E_h(t)$(recall (B.52)):

$$E_h(t) \le e^{-Mt}E_h(0) + \frac{1}{\varepsilon}\int_0^t h\sum_j g_j^2(s)ds \tag{B.58}$$

Let $C = \max\{M, 1/\varepsilon\}$, we get the desired results.

Since the discrete energy is uniformly bounded, we now deduce the convergence of the discrete solution.

**Theorem B.2.** *Let $U$ and $u$ be the solutions of (B.27) and (B.2), respectively. Then*

$$\lim_{h\to 0}\max_{t\in[0,T]} h\sum_j |u_j(t) - u(x_j, t)|^2 = 0. \tag{B.59}$$

*Proof:* We observe that (B.2) can be written as

$$u'(x_j) - \frac{u(x_{j+1}) + u(x_{j-1}) - 2u(x_j)}{h^2} = g(x_j) + f_2(u(x_j)). \tag{B.60}$$

It is obvious that $f_2$ converge to zero in $C(0, T; L^2(\Omega))$.

Now we define

$$e_j \triangleq u_j - u(x_j), \qquad j = 1, \cdots, N+1. \tag{B.61}$$

Combing (B.27) and (B.60) we obtain the system of the error function $e_j$:

$$\begin{cases} e_j' - \dfrac{e_{j+1} + e_{j-1} - 2e_j}{h^2} = f_2(u(x_j)) - hf(u_j), & j = 1, \cdots, N \\ e_0(t) = e_{N+1}(t) = 0, & t \in (0, T) \\ e_j(0) = 0, & j = 0, \cdots, N+1. \end{cases} \tag{B.62}$$

We then compute

$$h\sum_j \Big(f_2(u(x_j)) - hf(u_j)\Big)e_j = h\sum_j e_j'e_j - h\sum_j \Delta_h e_j e_j = \frac{1}{2}h\frac{\partial}{\partial t}\sum_j e_j^2 + h\sum_j (d_h e_j)^2. \tag{B.63}$$

For $e_j$, there exists a uniform positive constant $C$ such that

$$h\sum_j e_j^2 \le Ch\sum_j (d_h e_j)^2. \tag{B.64}$$

Also by Cauchy's inequality with $\varepsilon$, we obtain

$$h\sum_j \Big(f_2(u(x_j)) - hf(u_j)\Big)e_j \le \frac{1}{2}\varepsilon h\sum_j |f_2(u(x_j)) - hf(u_j)|^2 + \frac{h}{2\varepsilon}\sum_j e_j^2. \tag{B.65}$$

Taking (B.64) and (B.65) into (B.63), we get out the estimate of the error function

$$h\sum_j (e_j(t))^2 \le e^{(-\frac{1}{C}-\frac{1}{\varepsilon})t} h\sum_j (e_j(0))^2 + 2\varepsilon\int_0^t h\sum_j \Big(|f_2(u(x_j))|^2 + h^2|f(u_j)|^2\Big)ds. \tag{B.66}$$

Since $f_2(u(x_j)$ vanishes in $C([0,T];L^2(\Omega))$ when $h \to 0$ and $f(u_j)$ is also bounded in $C([0,T];L^2(\Omega))$, we observe that the right hand of (B.66) tends to zero if $h \to 0$. This means that the desired identity in the theorem is true.

B.5. **Some technical proofs.** We put some notations and technical proofs which are useful for the proof of the discrete Carleman estimate in this section.

**Notations**

- For $\Omega \in \mathbb{R}^n$ we denote by $\mathcal{A}(\Omega)$ the linear space of functions on $\Omega$, and for bounded continuous functions we define the maximum-norm

$$\|v\|_{\mathcal{A}(\Omega)} = \sup_{x\in\Omega} |v(x)|. \tag{B.67}$$

When $\Omega$ is a bounded and closed set, i.e., a compact set, the supremum in (B.67) is attained and we may write

$$\|v\|_{\mathcal{A}(\Omega)} = \max_{x\in\Omega} |v(x)|. \tag{B.68}$$

- For a bounded domain $\Omega$ and $k$ a non-negative integer we denote by $\mathcal{A}^k(\Omega)$ the set of $k$ times continuously differentiable functions in $\Omega$. We use the norm

$$\|v\|_{\mathcal{A}^k(\Omega)} = \max_{\alpha\le k} \|D^\alpha v\|_{\mathcal{A}(\Omega)}\,. \tag{B.69}$$

**Some lemmas**

**Lemma B.7.** *By using Taylor expansion we obtain*

(B.70)

$$d_h\alpha_j = \mu\varphi_j d_h\psi_j + \frac{h}{2}b_1\mu\varphi_j(d_h\psi_j)^2, \qquad b_1 \in (0, e^{\mu(\psi_{j+1}-\psi_j)}),$$

$$\Delta_h\alpha_j = \mu\varphi_j\Delta_h\psi_j + \frac{\mu^2}{2}\varphi_j((d_h\psi_j)^2 + (d_h\psi_{j-1})^2) + \frac{h}{6}\mu^3\varphi_j(b_2(d_h\psi_j)^3 + b_3(d_h\varphi_{j-1})^3),$$

$$b_2 \in (0, e^{\mu(\psi_{j+1}-\psi_j)}),\ b_3 \in (0, e^{\mu(\psi_j-\psi_{j-1})}).$$

*Proof:* By direct computation we have

(B.71)
$$
\begin{aligned}
d_h\alpha_j &= \frac{\alpha_{j+1}-\alpha_j}{h} = \frac{1}{h}\Big(\frac{e^{\mu\psi_{j+1}}}{(t+s(h))(T-t+s(h))} - \frac{e^{\mu\psi_j}}{(t+s(h))(T-t+s(h))}\Big)\\
&= \frac{e^{\mu\psi_j}}{(t+s(h))(T-t+s(h))}\frac{e^{\mu(\psi_{j+1}-\psi_j)}-1}{h}\\
&= \mu\varphi_j(d_h\psi_j + \frac{h}{2}b_1(d_h\psi_j)^2),
\end{aligned}
$$

with $b_1 \in (0, e^{\mu(\psi_{j+1}-\psi_j)})$. The Taylor expansion formula is used in the last step.

With the similar process we give the form of $\Delta_h\alpha_j$:

$$
\begin{aligned}
\Delta_h\alpha_j &= \frac{1}{h^2}\varphi_j\Big(e^{\mu(\psi_{j+1}-\psi_j)} + e^{\mu(\psi_{j-1}-\psi_j)} - 2\Big)\\
&= \frac{1}{h^2}\varphi_j\Big[\mu(\psi_{j+1}-\psi_j) + \frac{1}{2}\mu^2(\psi_{j+1}-\psi_j)^2 + \frac{1}{6}b_2\mu^3(\psi_{j+1}-\psi_j)^3\Big]\\
&\quad + \frac{1}{h^2}\varphi_j\Big[\mu(\psi_{j-1}-\psi_j) + \frac{1}{2}\mu^2(\psi_{j-1}-\psi_j)^2 + \frac{1}{6}b_3\mu^3(\psi_{j-1}-\psi_j)^3\Big],
\end{aligned} \tag{B.72}
$$

with $b_2 \in (0, e^{\mu(\psi_{j+1}-\psi_j)})$ and $b_3 \in (0, e^{\mu(\psi_{j-1}-\psi_j)})$. This is equivalent to the second identity in the lemma.

**Lemma B.8.** *We denote* $\max_j \theta_j^2(0)$ *by*

$$
\max_j \theta_j^2(0) = \exp 2\frac{e^{\mu\|\psi\|} - e^{2\mu\|\psi\|}}{s(h)(T+s(h))}. \tag{B.73}
$$

*Then we have*

$$
\begin{aligned}
&\Big|\int_0^T h\sum_{j=1}^N \Big((1+\lambda\frac{h^2}{2}\mathcal{A}_j)(d_hv_j)^2 - \lambda\mathcal{A}_jv_j^2\Big)' dt\Big| \le \\
&\frac{4}{h^2}\max_j\theta_j^2(0)h\sum_{j=1}^N u_j^2(0) + C\lambda\frac{\max_j\theta_j^2(0)}{s^2(h)}\int_0^T h\sum_{j=1}^N(\varphi_j^3\theta_j^2u_j^2 + g_j^2)dt.
\end{aligned} \tag{B.74}
$$

*Proof :* We divide the left side of (B.74) to two parts

$$
A = \int_0^T \Big(h\sum_{j=1}^N(1+\frac{\lambda h^2}{2}\mathcal{A}_j)(d_hv_j)^2\Big)' dt, \qquad B = -\int_0^T \Big(h\sum_{j=1}^N \lambda\mathcal{A}_jv_j^2\Big)' dt,
$$

respectively.

First we compute $A$. We have

$$\begin{aligned} A &= h\sum_{j=1}^{N}\Big(1+\frac{\lambda h^2}{2}\mathcal{A}_j(T)\Big)\Big(d_h v_j(T)\Big)^2 \\ &= -h\sum_{j=1}^{N}\Big(d_h v_j(0)\Big)^2 \\ &= -h\sum_{j=1}^{N}\Big(\frac{\theta_{j+1}u_{j+1}-\theta_j u_j}{h}(0)\Big)^2 \\ &\geq \sum_{j=1}^{N}\frac{2}{h^2}(\theta_{j+1}^2 u_{j+1}^2+\theta_j^2 u_j^2)(0) \\ &\geq \sum_{j=1}^{N}\frac{2}{h}\max_j \theta_j^2(u_j^2+u_{j+1}^2)(0) \\ &= \frac{2}{h}\max_j \theta_j^2(0)\sum_{j=1}^{N}u_j^2(0). \end{aligned} \tag{B.75}$$

It is obvious that $A$ tends to zero when the mesh size $h$ tends to zero.

We compute $B$ as

$$\begin{aligned} B &= -\int_0^T\Big(h\sum_{j=1}^{N}\lambda\mathcal{A}_j v_j^2\Big)' dt \\ &= h\sum_{j=1}^{N}\lambda\mathcal{A}_j(0)v_j^2(0)-h\sum_{j=1}^{N}\lambda\mathcal{A}_j(T)v_j^2(T) \\ &\geq -\lambda\max_j\mathcal{A}_j(T)h\sum_{j=1}^{N}v_j^2(T) \\ &\geq -\lambda\max_j\mathcal{A}_j(T)\max_j\theta_j^2(0)h\sum_{j=1}^{N}u_j^2(T). \end{aligned} \tag{B.76}$$

Now we deduce an estimate for $\sum_{j=1}^{N}u_j^2(T)$ by means of (B.57). Set $0<T/4<s<T$ and integrate (B.57) from $s$ to $T$ respect to $t$, then we have

$$e^{MT}h\sum_{j=1}^{N}u_j^2(T)-e^{Ms}h\sum_{j=1}^{N}u_j^2(s)\leq\int_s^T\frac{1}{\varepsilon}h\sum_{j=1}^{N}g_j^2 dt.$$

Integrate the above inequality from $T/4$ to $T/2$ respect to $t$, we obtain

$$\int_{T/4}^{T/2}h\sum_{j=1}^{N}u_j^2(T)dt\leq C\int_{T/4}^{T/2}h\sum_{j=1}^{N}\varphi_j^3\theta_j^2u_j^2dt+C\int_{T/4}^{T}h\sum_{j=1}^{N}g_j^2dt. \tag{B.77}$$

Combing (B.75),(B.76) and (B.77), noting tha the estimate of $\mathcal{A}_j$ in (B.40), we obtain the desired inequality (B.74).

**Lemma B.9.** *We have*

(B.78)

$$\begin{array}{lcll}
\dfrac{\theta_{j+1}-\theta_j}{h} & = & \theta_j[\lambda d_h\alpha_j + \dfrac{\lambda^2}{2}he^{\lambda c_1}(d_h\alpha_j)^2], & c_1\in(0,\alpha_{j+1}-\alpha_j),\\
 & = & \theta_j[\lambda d_h\alpha_j + \dfrac{\lambda^2}{2}h(d_h\alpha_j)^2 + \dfrac{\lambda^3}{6}h^2e^{\lambda c_1'}(d_h\alpha_j)^3], & c_1'\in(0,\alpha_{j+1}-\alpha_j),\\
\dfrac{\theta_j-\theta_{j-1}}{h} & = & \theta_j[\lambda d_h\alpha_{j-1} - \dfrac{\lambda^2}{2}he^{\lambda c_2}(d_h\alpha_{j-1})^2], & c_2\in(0,\alpha_j-\alpha_{j-1}),\\
 & = & \theta_j[\lambda d_h\alpha_{j-1} - \dfrac{\lambda^2}{2}h(d_h\alpha_{j-1})^2 + \dfrac{\lambda^3}{6}h^2e^{\lambda c_2'}(d_h\alpha_{j-1})^3], & c_2'\in(0,\alpha_j-\alpha_{j-1}),\\
\Delta_h\theta_j & = & \theta_j\Big(\lambda\Delta_h\alpha_j + \dfrac{\lambda^2}{2}[(d_h\alpha_j)^2+(d_h\alpha_{j-1})^2]\Big) & \\
 & & +\theta_j\dfrac{\lambda^3}{6}h\Big[e^{\lambda c_1'}(d_h\alpha_j)^3 - e^{\lambda c_2'}(d_h\alpha_{j-1})^3\Big]. &
\end{array}$$

*Proof:* First we give out the Taylor series of $e^{\lambda(\alpha_{j+1}-\alpha_j)}$ :

$$\begin{aligned}
e^{\lambda(\alpha_{j+1}-\alpha_j)} &= 1+\lambda(\alpha_{j+1}-\alpha_j) + \frac{\lambda^2}{2}e^{\lambda c_1}(\alpha_{j+1}-\alpha_j)^2\\
&= 1+\lambda(\alpha_{j+1}-\alpha_j) + \frac{\lambda^2}{2}(\alpha_{j+1}-\alpha_j)^2 + \frac{\lambda^3}{6}e^{\lambda c_1'}(\alpha_{j+1}-\alpha_j)^3
\end{aligned} \tag{B.79}$$

with

$$c_1, c_1' \in (0,\alpha_{j+1}-\alpha_j).$$

Moreover, for $e^{\lambda(\alpha_j-\alpha_{j-1})}$ we have

$$\begin{aligned}
e^{\lambda(\alpha_j-\alpha_{j-1})} &= 1+\lambda(\alpha_j-\alpha_{j-1}) + \frac{\lambda^2}{2}e^{\lambda c_2}(\alpha_j-\alpha_{j-1})^2\\
&= 1+\lambda(\alpha_j-\alpha_{j-1}) + \frac{\lambda^2}{2}(\alpha_j-\alpha_{j-1})^2 + \frac{\lambda^3}{6}e^{\lambda c_2'}(\alpha_j-\alpha_{j-1})^3,
\end{aligned} \tag{B.80}$$

with

$$c_2, c_2' \in (0,\alpha_j-\alpha_{j-1}).$$

Hence,

$$\begin{aligned}\frac{\theta_{j+1}-\theta_j}{h} &= \frac{e^{\lambda\alpha_{j+1}}-e^{\lambda\alpha_j}}{h} = e^{\lambda\alpha_j}\frac{e^{\lambda(\alpha_{j+1}-\alpha_j)}-1}{h}\\ &= \theta_j[\lambda d_h\alpha_j + \frac{\lambda^2}{2}he^{\lambda c_1}(d_h\alpha_j)^2]\\ &= \theta_j[\lambda d_h\alpha_j + \frac{\lambda^2}{2}h(d_h\alpha_j)^2 + \frac{\lambda^3}{6}e^{\lambda c_1'}(d_h\alpha_j)^3],\end{aligned}$$

and

$$\begin{aligned}\frac{\theta_j-\theta_{j-1}}{h} &= \frac{e^{\lambda\alpha_j}-e^{\lambda\alpha_{j-1}}}{h} = e^{\lambda\alpha_j}\frac{1-e^{-\lambda(\alpha_j-\alpha_{j-1})}}{h}\\ &= \theta_j[\lambda d_h\alpha_{j-1} + \frac{\lambda^2}{2}e^{\lambda c_2}(d_h\alpha_{j-1})^2]\\ &= \theta_j[\lambda d_h\alpha_{j-1} - \frac{\lambda^2}{2}h(d_h\alpha_{j-1})^2 + \frac{\lambda^3}{6}e^{\lambda c_2'}(d_h\alpha_{j-1})^3].\end{aligned}$$

Consequently, we obtain $\Delta_h\theta_j$ directly.

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