Controllability of the time discrete heat equation

Chuang Zheng
Laboratory of Mathematics and Complex Systems, Ministry of Education, School of Mathematical Science, Beijing Normal University, 100875 Beijing, China
and
Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain
E-mail: chuang.zheng@uam.es

Abstract. In this paper we study the controllability of an Euler Implicit time discrete heat equation in a bounded domain with a local internal controller. We prove that, based on Lebeau-Robbiano's time iteration method, the projection in appropriate filtered space is null controllable with uniformly bounded control. In this way, the well-known null-controllability property of the heat equation can be proven as the limit, as $\triangle t \to 0$, of the controllability of projections of the time-discrete one. Consequently we prove the uniform approximate controllability after filtering with bounded control. A further study is made and analogous results are obtained for other discrete schemes, i.e. Euler Explicit schemes, θ-method schemes. We also discuss the null controllability of the Euler Implicit time discrete parabolic equation of fractional order.

1 Introduction and main results

Let $T > 0$ be given, Ω be a given open bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) with C^∞ boundary $\partial\Omega$, and ω a given non-empty open subset of Ω. Denote by 1_ω the characteristic function of ω. We consider the following heat equation with local internal controller:

$$\begin{cases} y_t - \Delta y = u 1_\omega, & (t,x) \in (0,T) \times \Omega \\ y = 0, & (t,x) \in (0,T) \times \partial\Omega \\ y(0,x) = y^0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

In (1.1), $y = y(t,x)$ is the state, $u = u(t,x)$ is the control function acting on the subset ω.

System (1.1) is said to be *approximately controllable* in time $T > 0$ if for any initial state $y^0 \in L^2(\Omega)$, any final state $y_1 \in L^2(\Omega)$ and any $\varepsilon > 0$, there exists a control $u \in L^2((0,T)\times\omega)$ such that the solution of (1.1) satisfies

$$\|y(T) - y_1\|_{L^2(\Omega)} \le \varepsilon.$$

System (1.1) is said to be *null controllable* in time $T > 0$ if for any $y^0 \in L^2(\Omega)$ there exists a control $u \in L^2((0,T) \times \omega)$, called henceforth a null control of (1.1), such that the corresponding solution satisfies

$$y(T,x) = 0, \qquad \forall \ x \in \Omega.$$

It is well known that system (1.1) is both approximately and null controllable for any $T > 0$ and for any non-empty open subset $\omega \subset \Omega$ (see, for instance, [5, 12]). Moreover, one has the following estimate for the minimal L^2-norm null control u of system (1.1):

$$\|u\|_{L^2((0,T)\times\omega)} \leq C \left\|y^0\right\|_{L^2(\Omega)},$$

where C is a positive constant depending only on T, Ω and ω.

In [12], a time iteration method is introduced by Lebeau and Robbiano to show the above mentioned controllability result. A simplified presentation of this method was given by Lebeau and Zuazua ([13]) where the linear system of thermoelasticity was addressed. More recently, Miller has applied it systematically to the analysis of other models: heat equations ([21]), Schrödinger equations ([20]) and linear system of thermoelasticity ([22]). In the sequel, we call it the L-R method.

In this paper we shall study first the uniform null controllability of the time discrete version of (1.1) by means of the L-R method. Then we analyze the uniform approximate controllability by means of the uniform null control. The classical known controllability results for (1.1) are recovered.

For this purpose, for any given $K \in \mathbb{N}^*$, we set $\triangle t = T/K$ and introduce the net

$$t_0 = 0 < t_1 < \cdots < t_K = T$$

with $t_k = k\triangle t$ and $k = 0, 1, \cdots, K$.

The time discrete counterpart of system (1.1) reads as follows:

$$\begin{cases} \dfrac{y^{k+1} - y^k}{\triangle t} - \Delta y^{k+1} = u^k 1_\omega, & x \in \Omega, \ \ k = 0, 1, \cdots, K-1 \\ y^k = 0, & x \in \partial\Omega, \ k = 1, \cdots, K \\ y^0 \in L^2(\Omega) \text{ given.} \end{cases} \tag{1.2}$$

System (1.2) is an implicit time discretization of the heat equation with control on the subset $\omega \subset \Omega$. Here $\{y^k\}_{0 \leq k \leq K}$ stands for the state and $\{u^k\}_{0 \leq k \leq K-1}$ the control. There are many methods to discretize system (1.1). We choose the Implicit Euler schemes (1.2) simply because it avoids the instability of the solutions in the whole space.

As the analogue of the continuous case, we introduce the following two definitions for the discrete schemes (1.2):

Definition 1.1 *System (1.2) is said to be approximately controllable at $k = K$ (for any given $\triangle t > 0$) if for any $y^0 \in L^2(\Omega)$, any final state $y^T \in L^2(\Omega)$ and $\varepsilon > 0$, there exists a control $\{u^k \in L^2(\omega)\}_{0 \leq k \leq K-1}$ such that the solution $\{y^k\}_{k=0,\cdots,K}$ of (1.2) satisfies*

$$\left\|y^K - y^T\right\|_{L^2(\Omega)} < \varepsilon.$$

Definition 1.2 *System (1.2) is said to be null controllable at $k = K$ (for any given $\triangle t > 0$) if for any $y^0 \in L^2(\Omega)$ there exists a control $\{u^k \in L^2(\omega)\}_{0\le k\le K-1}$, called henceforth a discrete null control, such that the solution $\{y^k\}_{k=0,\cdots,K}$ of (1.2) satisfies*

$$y^K(x) = 0, \qquad \forall\ x \in \Omega.$$

For any fixed $\triangle t > 0$, (1.2) is a system of controlled elliptic equations. The following result shows that, whatever $\Omega \subset \mathbb{R}^d$ is, controllability properties of system (1.2) fail unless ω covering the whole domain, i.e. $\omega = \Omega$:

Theorem 1.1 *Let $\Omega \setminus \overline{\omega} \neq \emptyset$. For any given $\triangle t > 0$, system (1.2) is neither null nor approximately controllable.*

Remark 1.1 *In fact, the Hilbert Uniqueness Method (HUM, see in [15]) provides that the null controllability of system (1.2) is equivalent to the observability of its adjoint system*

$$\begin{cases} -\dfrac{\varphi^{k+1}-\varphi^k}{\triangle t} - \Delta\varphi^k = 0, & x \in \Omega,\ \ k = 0,1,\cdots,K-1 \\ \varphi^k = 0, & x \in \partial\Omega,\ \ k = 1,\cdots,K \\ \varphi^K \ \ \text{given}. \end{cases} \tag{1.3}$$

The key point of the lack of null controllability is that the adjoint system (1.3) is not observable for $\varphi^K \in L^2(\Omega)$, i.e.

$$\sup_{\varphi^K \in L^2(\Omega)} \frac{\|\varphi^0\|^2_{L^2(\Omega)}}{\triangle t \displaystyle\sum_{k=0}^{K-1} \int_\omega |\varphi^k|^2 dx} = \infty \tag{1.4}$$

for any fixed $\triangle t > 0$, except for the trivial case $\omega = \Omega$.

In view of the lack of controllability of system (1.2) it is natural to relax the controllability requirement by considering the projections of solutions over a suitable class of low frequency Fourier components. In fact, it is by now well known that, often, numerical approximations schemes that are stable develop instabilities when applied to controllability problems. It is due to the presence of spurious high frequencies numerical solutions that the control mechanisms are not able to control uniformly as the mesh-size tends to zero. Hence, it is reasonable to cut off the high frequencies and only consider the lower part. This filtering method has been applied successfully in the context of controllability of numerical approximate schemes for wave equations (see [24] and the references therein).

Let $\Phi_j \in H_0^1(\Omega)$ be an orthonormal basis of $L^2(\Omega)$ consisting on the eigenvectors of the Dirichlet Laplacian:

$$\begin{cases} -\Delta\Phi_j = \mu_j\Phi_j, & x \in \Omega \\ \Phi_j = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

For any $s > 0$, we introduce the following subspace of $L^2(\Omega)$:

$$\mathcal{C}_s = \text{span}\,\{\Phi_j,\ \text{the corresponding eigenvalue } \mu_j \text{ satisfies } \mu_j \le s\}. \tag{1.6}$$

We denote by π_s the projection operator from $L^2(\Omega)$ to the filtered space $\mathcal{C}_s$. More precisely, let $f(x) \in L^2(\Omega)$, the projection $\pi_s f(x)$ is as the form of $\pi_s f(x) = \sum_{\mu_j \leq s} a_j \Phi_j$ where $f(x) = \sum_{j \geq 1} a_j \Phi_j$.

The following result shows that the projection of the solution of system (1.2) on $\mathcal{C}_s$ is uniformly null controllable, with appropriate choice of s:

Theorem 1.2 *For any fixed $T > 0$ and $r \in (0, 2)$, there exists a positive constant $\Lambda = \Lambda(r, T, \Omega, \omega)$ such that for all $y^0 \in L^2(\Omega)$, there exists a control $\{u^k \in L^2(\omega)\}_{k=0,\cdots,K-1}$, so that*

(1) *The solution of system (1.2) satisfies*

$$\pi_{\Lambda(\triangle t)^{-r}} y^K(x) = 0, \qquad \forall \; x \in \Omega; \tag{1.7}$$

(2) *There exists a constant $C = C(r, T, \Omega, \omega) > 0$, independent of $\triangle t$, such that*

$$\triangle t \sum_{k=0}^{K-1} \int_\omega |u^k|^2 dx \leq C \int_\Omega |y^0|^2 dx \tag{1.8}$$

holds for any $\triangle t > 0$ and $y^0 \in L^2(\Omega)$.

Some remarks are in order.

Remark 1.2

1. *Note that when $\triangle t$ is fixed, system (1.2) is also null controllable for any filtering parameter $s > 0$. This is due to the duality and the fact that its adjoint system is observable after filtering, no matter what "s" is. However, from the proof of Theorem 1.2, we shall see that in order to keep the controllability property uniformly as $\triangle t \to 0$, one needs to choose the filtering parameters to be of the order $\Lambda(\triangle t)^{-r}$ with*

$$\Lambda \sim \left(\frac{T}{8D}\right)^r. \tag{1.9}$$

In (1.9), T is the control-time and D is a constant depending on Ω and ω. Moreover, for any two controllers ω_1 and ω_2 in Ω, $D(\omega_1) > D(\omega_2)$ whenever $\omega_1 \subset \omega_2$, and, accordingly, $\Lambda(\omega_1) < \Lambda(\omega_2)$. Furthermore, Λ increases as the time T increases.

2. *Note that for any $r \in (0, 2)$ fixed, when $\triangle t$ tends to zero, the filtering parameter $s = \Lambda(\triangle t)^{-r}$ tends to infinity and the filtered space $\mathcal{C}_s$ tends to cover the whole space $L^2(\Omega)$. We imposed the restriction $r \in (0, 2)$ for technical reasons. However, it is likely that, when $r = 2$, the result fails because of the lack of sufficient dissipation, as it happens in the critical fractional order heat equation (see [19]). This is an interesting problem.*

3. *The results of this paper concern the interior control problem. The same issues make sense in the context of the boundary controllability. Recall that the time-continuous heat equation is controllable for all time* $T > 0$ *and an arbitrarily support* Γ_0*, open nonempty subset of* Γ *(see, for instance, [12]). However, even in the time continuous case, one can not derive the boundary controllability directly by means of the analogue introduced here since* (2.9) *is false when the observation set is a subset of the boundary. This is obvious, in particular, in the* $1-d$ *case.*

   A rather standard method to derive the boundary controllability from the interior one is based on extension-restriction argument and it is as follows. One first extends by zero the solution to an outer neighborhood of Γ_0*. The arguments for interior controls allow to control the system in the whole domain by means of a control supported in this small added domain. The restriction of the solution to the original domain satisfies all of the requirements and its restriction or trace to the subset of the boundary where the control had to be supported, provides the boundary control one is looking for. However, this argument does not work well for the present discrete problem. Indeed, by doing this, of course one can obtain a uniform (partial) null controllability of the system after filtering. However, the filtering space is spanned by the eigenvectors of the Dirichlet Laplacian in the extended domain rather than the original domain* Ω*.*

   Very likely, in this time-discrete setting, the most promising technique seems to be that based on the use of Carleman inequalities. But so far there have not been addressed in the time-discrete setting.

4. *Note that system (1.2) is a scheme discrete in time and continuous in space. Naturally, as a further study, one could consider the control problem for fully discrete schemes, both on time and space variables, for instance that one replaces* Δ *in (1.2) by a finite-difference space discretization operator. As far as we know, the controllability of such a fully discrete scheme is an open problem. The difficulty is that it is not clear whether the space discrete version of* (2.9) *holds or not, which seems to be a challenging problem. Indeed, the proof of* (2.9) *is based on doubling properties or Carleman inequalities for the space-continuous elliptic equations. However, none of these tools are developed well in the discrete settings. At this level it is very likely that one possible method to be explored could be the time-discrete biorthogonal sequences, as a discrete counterpart of the existing theorem for time-continuous* $1-d$ *parabolic problem ([2]).*

5. *Similar discrete controllability results can also be obtained in a more general setting. For instance, the operator* Δy *can be replaced by* $\sum_{i,j=1}^{d}(a_{ij}(x)y_{x_i})_{x_j}$*, where* $\big(a_{ij}\big)_{1\le i,j\le d} \in C^\infty(\overline{\Omega};\mathbb{R}^{d\times d})$ *is a symmetric and uniformly positive definite matrix. Indeed, in this case, as shown in [13], the counterpart of* (2.9) *holds true. Very likely, one may even adding a zero-order term ay (with* $a \in L^\infty(\Omega)$*) to* $\sum_{i,j=1}^{d}(a_{ij}(x)y_{x_i})_{x_j}$ *since the counterpart of* (2.9) *for the resulting elliptic operator should hold true (Recall that* (2.9) *can be proved by means of Carleman estimate, which is "robust" with*

respect to bounded perturbations). However, when adding any nonzero first-order term to $\sum_{i,j=1}^{d}(a_{ij}(x)y_{x_i})_{x_j}$, the resulting operator is not self-adjoint any more, and therefore the corresponding problem is beyond the setting of this paper. As for other boundary conditions, the problem is, again, whether or not, the counterpart of (2.9) *remains to be true, which, as far as we know, is an unsolved problem.*

Let us analyze further the convergence and error estimate for the discrete null controls. We have the following result:

Theorem 1.3 *For the discrete null control $\{u^k\}_{0\le k\le K-1}$ given in the proof of Theorem 1.2, it holds*

$$U^K(\cdot,x)\triangleq\sum_{k=0}^{K-1}u^k(x)1_{[t_k,t_{k+1})}(\cdot)\longrightarrow u(\cdot,x)\ \ \textit{strongly in}\ \ L^2((0,T)\times\omega)\ \ \textit{as}\ \triangle t\to 0,$$

where u is a null control of system (1.1). Moreover, there exists a constant $C>0$, independent of $\triangle t$ and y^0, such that U^K and u satisfy

$$\left\|U^K-u\right\|_{L^2((0,T)\times\omega)}\le C\sqrt{\triangle t}\left\|y^0\right\|_{L^2(\Omega)}. \tag{1.10}$$

Now we discuss the approximate controllability. Without loss of generality, we assume that $y^0=0$. We have the following approximate controllability with uniform bounded control:

Theorem 1.4 *Let T,r and Λ be given by Theorem 1.2. Then for any $y^0=0$, a final state $y_1\in L^2(\Omega)$ and $\varepsilon>0$, there exists a control $\{u^k\}_{k=0,\cdots,K-1}$ such that y^K satisfies:*

$$\left\|\pi_{\Lambda(\triangle t)^{-r}}(y^K-y_1)\right\|_{L^2(\Omega)}\le\varepsilon,\qquad\forall\ \triangle t>0. \tag{1.11}$$

Moreover, for

$$U^K=\sum_{k=0}^{K-1}u^k1_{[t_k,t_{k+1})}(t), \tag{1.12}$$

there exists a constant $C=C(r,T,\Omega,\omega)>0$, independent of $\triangle t$, such that

$$\left\|U^K\right\|_{L^2((0,T)\times\omega)}\le C\exp\left(\frac{\|\Delta y_1\|_{L^2(\Omega)}}{\varepsilon}\right)\|y_1\|_{L^2(\Omega)} \tag{1.13}$$

holds for any $\triangle t>0$ and any $y_1\in H^2(\Omega)\cap H_0^1(\Omega)$.

Remark 1.3 *Note that when $\triangle t$ is fixed, system (1.2) is approximate controllable for any filtering parameter $s>0$. This is due to the fact that it is a finite dimensional control problem no matter what "s" is. Also when $\triangle t\to 0$, the filtering parameter $s=\Lambda(\triangle t)^{-r}$ tends to infinity and a continuous approximate control is obtained as the limit of the discrete control. Moreover, the cost of the control is uniformly bounded by $\|\Delta y_1\|_{L^2(\Omega)}$ and ε as the form in (1.13), which is similar to the continuous one (see the continuous result in [3]).*

The rest of the paper is organized as follows. In Section 2 we list some preliminaries: the controllability properties of finite dimensional time-discrete systems, the description of the L-R time iteration method and some heuristics on the application of the L-R method in the time discrete case. In Section $3-6$ we give the proofs of Theorem 1.1−1.4, respectively. Some other discrete schemes are discussed in Section 7, i.e. Euler Explicit method, θ-method. We also analyze the time implicit semi-discrete fractional order parabolic system in the last Section and similar result is obtained.

2 Preliminaries

2.1 Controllability of finite dimensional systems

This subsection is devoted to recall some basic controllability results for time-discrete ordinary differential equations.

Let $n, m \in \mathbb{N}^*$. We consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0,T) \\ x(0) = x^0. \end{cases} \tag{2.1}$$

In (2.1), A is a real $n \times n$ matrix, B a real $n \times m$ matrix and x^0 a vector in $\mathbb{R}^n$. The function $x : [0,T] \to \mathbb{R}^n$ represents the state and $u : [0,T] \to \mathbb{R}^m$ the control.

System (2.1) is said to be controllable in time T if every initial datum $x^0 \in \mathbb{R}^n$ can be driven to any final datum $x_1 \in \mathbb{R}^n$ at time T. It is well known that the *Kalman condition* is a necessary and sufficient condition for controllability of finite dimension system: *System (2.1) is controllable in $T > 0$ if and only if*

$$rank[B, AB, \cdots, A^{n-1}B] = n. \tag{2.2}$$

Note that, according to this result, a system is controllable for some time $T > 0$ if and only if it is controllable for all $T > 0$.

Now we discretize system (2.1) with respect to time t as follows:

$$\begin{cases} \dfrac{x^{k+1} - x^k}{\triangle t} = Ax^{k+1} + Bu^{k+1}, & k = 0, \cdots, K-1 \\ x^0 \in \mathbb{R}^n \quad \text{given.} \end{cases} \tag{2.3}$$

The following result is also well known:

Theorem 2.1 *([9, 14]) Assume that A, B satisfy the Kalman condition (2.2) and $\lambda_1, \lambda_2, \cdots, \lambda_k$, $(k \leq n)$ to be the distinct eigenvalues of A. Then system (2.3) is controllable if*

$$\triangle t \neq \frac{2\nu\pi i}{\lambda_\mu - \lambda_l}, \quad \mu \neq l \tag{2.4}$$

holds for any $\nu \in \mathbb{N}^*$.

Remark 2.1 *Note that if A, B satisfy the Kalman condition, the discrete system (2.3) is controllable for $\triangle t$ sufficiently small. More precisely, (2.4) is fulfilled when $\triangle t$ satisfies*

$$\triangle t < \min_{\mu \neq l} \left| \frac{2\nu\pi}{\lambda_\mu - \lambda_l} \right|. \tag{2.5}$$

Remark 2.2 *Note that in Theorem 2.1, A is a finite dimensional operator. Assume that A_σ is an approximation of the unbounded operator Δ in 1-d, with the eigenvalues $\lambda_l = -l^2 \leq \sigma, l = 1, \cdots, \sqrt{\sigma}$. Thus, (2.5) can be rewritten as:*

$$\triangle t < \min_{\mu \neq l} \left| \frac{2\nu\pi}{\lambda_\mu - \lambda_l} \right| < \left| \frac{2\pi}{\sigma - 1} \right|. \tag{2.6}$$

When A_σ contains more and more eigenvalues of Δ the right side of (2.6) tends to zero. This is in agreement with the fact that system (2.3) is not null controllable for any fixed $\triangle t > 0$, if A is an unbounded operator. But it also shows that the controllability of the time-continuous system can be recovered as a consequence of the controllability of time-discrete one by letting $\triangle t \to 0$ and, simultaneously, $\sigma^2 \to \infty$.

The null controllability of system (1.1) is equivalent to an observability estimate for the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0, & (t, x) \in (0, T) \times \Omega \\ \varphi = 0, & (t, x) \in (0, T) \times \partial\Omega \\ \varphi(T, x) = \varphi^0(x), & x \in \Omega. \end{cases} \tag{2.7}$$

System (1.1) is null controllable if and only if there exists a positive constant $C > 0$ such that

$$\|\varphi(0)\|_{L^2(\Omega)} \leq C \|\varphi\|_{L^2((0,T)\times\omega)} \tag{2.8}$$

holds for all solutions of (2.7) with initial data $\varphi^0 \in L^2(\Omega)$. Inequality (2.8) is the observability inequality of adjoint system (2.7) and also can be proved directly via Carleman estimate (see, for instance, [4]).

The L-R method is based on an observability estimate for the eigenfunctions of the Dirichlet Laplacian, which is stated as follows:

Theorem 2.2 *([12, 13]) Let Ω be a bounded domain of class C^∞. Let $\{\mu_j\}_{j\geq 1}$ and $\{\Phi_j\}_{j\geq 1}$ be defined by system (1.5). Then for any open non-empty subset ω of Ω, there exist two positive constants $C_j(\Omega, \omega)$, $j = 1, 2$, such that*

$$\int_\omega \Big| \sum_{\mu_j \leq \sigma} a_j \Phi_j(x) \Big|^2 dx \geq C_1 e^{-C_2\sqrt{\sigma}} \sum_{\mu_j \leq \sigma} |a_j|^2 \tag{2.9}$$

holds for every $\sigma > 0$ and every sequence $\{a_j\}_{\mu_j \leq \sigma}$ of complex numbers.

Remark 2.3 *Note that Theorem 2.2 is related to an interior subset ω and similar result does not hold true for the boundary case. More precisely, let Γ_0 be a subset of the boundary $\partial\Omega$, then (2.9) is no longer true if one replaces $\int_\omega \left| \sum_{\mu_j \leq \sigma} a_j \Phi_j(x) \right|^2 dx$ by $\int_{\Gamma_0} \left| \sum_{\mu_j \leq \sigma} a_j \frac{\partial \Phi_j(x)}{\partial \nu} \right|^2 d\Gamma_0$. One can easily prove it by constructing an counterexample in $1-d$, for instance, $\Omega = (0,1)$ and $\Gamma_0 = \{1\}$.*

This result was implicitly used in [12] and proven in [13] by means of Carleman inequalities. Also, the assumption $\partial\Omega \in C^\infty$ can be relaxed to be $\partial\Omega \in C^2$, as remarked by L. Escauriaza (see [17, Remark 1.1]).

As a consequence of (2.9) one can prove that the observability inequality (2.8) holds for the solution of system (2.7) with initial data in $\mathcal{C}_\sigma = \text{span}\,\{\Phi_j\}_{\mu_j \leq \sigma}$, with the observability constant being of the order of $\exp(C\sqrt{\sigma})$. This allows us to show that the projection of solutions of (1.1) over $\mathcal{C}_\sigma$ can be controlled to zero with a control of size $\exp(C\sqrt{\sigma})$. Thus, when controlling the frequencies with $\mu_j \leq \sigma$ one increases the $L^2(\Omega)$-norm of the high frequencies with $\mu_j \geq \sigma$ by a multiplicative factor of the order of $\exp(C\sqrt{\sigma})$. However, solutions of system (1.1) without control ($f = 0$) but with a vanishing projection of the initial data over $\mathcal{C}_\sigma$, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\sigma t)$. This can be easily seen by means of the Fourier-type series expansion of the solution. Thus, if we divide the time interval $[0,T]$ into two parts $[0,T/2]$ and $[T/2,T]$, we control to zero the frequencies $\mu_j \leq \sigma$ in the first interval and then allow the equation to evolve without control in the second interval, it follows that, at time $t = T$, the projection of the solution u over $\mathcal{C}_\sigma$ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0.

This argument allows us to control to zero the projection of the solutions of (1.1) over $\mathcal{C}_\sigma$ for any $\sigma > 0$ but not the whole solution. For the later an iterative argument is needed in which the interval $[0,T]$ has to be decomposed in a suitable chosen sequences of subintervals $[T_l, T_{l+1}]$ and the argument above is applied in each subinterval to control an increasing range of frequencies with $\mu_j \leq \sigma_l$ and σ_l going to infinity at suitable rate. We refer to [12, 13] for more details in this respect.

It is important to underline that the strong dissipativity of the heat equation is essential to make this argument work. Indeed, one needs to make sure that the dissipation rate is stronger than the increase of the size of the controls as frequencies increase. This is so for the heat equation where the dissipation rate is $e^{-\mu_j^2 t}$ while the increase of the controls, in view of (2.9), is of the order of $e^{C_2 \mu_j}$.

Actually, the optimality of this kind of argument has been proven in [19] where it was shown that the fractional order parabolic equation $y_t + (-\Delta)^\alpha y = 0$ is null controllable for $\alpha > 1/2$ but that this fails to be true in the limiting case $a = 1/2$.

2.2 Heuristics on the time discrete system by L-R method

Let us consider now the time discrete heat equation (1.2). Recall that y^k is the solution and u^k is the corresponding control at time step k. When $u^k = 0$ for $0 \le k \le K-1$, i.e. no control acts on the domain ω, the solution of (1.2) at $t = T$, i.e. at $k = K$, can be written by means of the Fourier expansion as

$$y^K = \sum_{j \ge 1} a_j (1 + \mu_j \triangle t)^{-K} \Phi_j = \sum_{j \ge 1} a_j (1 + \mu_j \triangle t)^{-T/\triangle t} \Phi_j. \tag{2.10}$$

Assume σ is a positive constant, then y^K decays at the rate $(1 + \sigma \triangle t)^{-T/\triangle t}$ with initial data $y^0 = \sum_{\mu_j \ge \sigma} a_j \Phi_j$.

From the L-R time iteration method, we see that the key point to obtain a bounded control is that the norm of the solution of the continuous heat equation decays exponentially at the order of $\exp(-\sigma t)$. This phenomena compensates the exponential increase of the norm of control at the order of $\exp(C\sqrt{\sigma})$. However, the solution of the time discrete system (1.2) decays at the order of $(1 + \sigma \triangle t)^{-T/\triangle t}$. We need to know, under which condition this decay of solutions can compensate the exponential increase of the norm of control at the order of $\exp(C\sqrt{\sigma})$ in the time discrete case.

For this purpose, we show the following elementary result:

Lemma 2.1 *Let T and $C > 0$ be two positive constants and $0 < \triangle t < \frac{T^2}{4\sqrt{2}C^2}$. Then function*

$$f(\sigma) = e^{C\sqrt{\sigma}} (1 + \sigma \triangle t)^{-[\frac{T}{\triangle t}]} \tag{2.11}$$

has the following properties:

i). $f(\sigma)$ is decreasing in the interval $\Big((\frac{C}{T})^2, (\frac{T}{2C\triangle t})^2 \Big)$.

ii). $f(\sigma) < e^{-\delta\sqrt{\sigma}}$ in the interval $\Big((\frac{2C+\delta}{T-\triangle t})^2, (\frac{T}{2(C+\delta)\triangle t})^2 \Big)$ for any $0 \le \delta \le C$.

iii). It holds

$$\lim_{\triangle t \to 0^+} f(\sigma)\Big|_{\sigma = (\frac{T}{C\triangle t})^2} = 0. \tag{2.12}$$

Proof: Replacing $\sqrt{\sigma}$ by x and setting $f(x^2) = g(x)$, we have

$$g(x) = \exp\Big(Cx - [\frac{T}{\triangle t}] \ln(1 + \triangle t\, x^2) \Big).$$

The derivative of $f(x^2)$ with respect to x reads

$$g'(x) = g(x) \Big(C - [\frac{T}{\triangle t}] \frac{2\triangle t x}{1 + \triangle t x^2} \Big).$$

$g(x)$ is a decreasing function if and only if $g'(x)$ is negative. Since $[\frac{T}{\triangle t}] > \frac{T}{\triangle t} - 1$, by solving the inequality

$$C - \frac{2(T-\triangle t)x}{1+\triangle t\, x^2} < 0,$$

we conclude that $g(x)$ decreases in the interval

$$\Big(\frac{(T-\triangle t) - \sqrt{(T-\triangle t)^2 - C^2\triangle t}}{C\triangle t},\ \frac{(T-\triangle t) + \sqrt{(T-\triangle t)^2 - C^2\triangle t}}{C\triangle t}\Big).$$

Moreover, taking into account that $f(\sigma) = f(x^2) = g(x)$ and

$$\begin{aligned}
&\max_{0<\triangle t<(\frac{T}{2C})^2} \frac{(T-\triangle t) - \sqrt{(T-\triangle t)^2 - C^2\triangle t}}{C\triangle t} \\
=\ &\max_{0<\triangle t<(\frac{T}{2C})^2} \frac{C}{(T-\triangle t) + \sqrt{(T-\triangle t)^2 - C^2\triangle t}} \le \frac{C}{T}, \\
&\Big(\min_{0<\triangle t<(\frac{T}{2C})^2} \frac{(T-\triangle t) + \sqrt{(T-\triangle t)^2 - C^2\triangle t}}{C}\Big)\frac{1}{\triangle t} \ge \frac{T}{2C}\frac{1}{\triangle t},
\end{aligned}$$

we conclude that $f(\sigma)$ is decreasing in the interval $\Big((\frac{C}{T})^2, (\frac{T}{2C\triangle t})^2\Big)$ for any $\triangle t \in (0, (\frac{T}{C})^2)$.

Now we consider the function $H(x) = f(x^2)e^{\delta\sqrt{\sigma}}$. Clearly, $H(x) < 1$ if and only if

$$\ln H(x) = C\,x - [\frac{T}{\triangle t}]\ln(1+\triangle t\, x^2) + \delta\, x < 0.$$

By the Taylor expansion, $\ln H(x)$ can be rewritten as:

$$\ln H(x) = (C+\delta)\,x - [\frac{T}{\triangle t}]\triangle t\, x^2 + [\frac{T}{\triangle t}]\frac{(\triangle t)^2}{2}x^4 - [\frac{T}{\triangle t}]\frac{(\triangle t)^3 x^6}{3(1+\xi\triangle t\, x^2)^3},$$

for some $\xi \in (0, \triangle t\, x^2)$. Since $x > 0$, we deduce that $\ln H(x) < 0$ if

$$C + \delta - (T-\triangle t)\,x + \frac{T\triangle t}{2}x^3 < 0. \tag{2.13}$$

We claim that (2.13) is satisfied when $x \in \Big(\frac{2C+\delta}{T-\triangle t}, (\frac{2C}{T\triangle t})^{1/3}\Big)$. This is due to the fact that

$$C + \delta - (T-\triangle t)\,x + \frac{T\triangle t}{2}x^3 < -C + \frac{T\triangle t}{2}x^3 < 0.$$

Similarly, it is easy to show that $H(x)$ is decreasing for $x \in (\frac{C+\delta}{T}, \frac{T}{2(C+\delta)\triangle t})$. Hence, $H(x) < 1$ when x satisfies

$$\frac{2C+\delta}{T-\triangle t} < x < \max\Big((\frac{2C}{T\triangle t})^{1/3}, \frac{T}{2(C+\delta)\triangle t}\Big). \tag{2.14}$$

Moreover, taking $\triangle t \in (0, \frac{T^2}{\sqrt{2}(C+\delta)^2})$, we get that the right side of (2.14) equals to $\frac{T}{2(C+\delta)\triangle t}$ and consequently $H(x) < 1$ in the interval

$$\frac{2C+\delta}{T-\triangle t} < x < \frac{T}{2(C+\delta)\triangle t}.$$

Hence, $ii)$ is satisfied when $\triangle t < \min_{0<\delta<C} \dfrac{T^2}{\sqrt{2}(C+\delta)^2}$, i.e. $\triangle t < \frac{T^2}{4\sqrt{2}C^2}$.

Next, by

$$\lim_{\triangle t\to 0} f(\sigma)\Big|_{\sigma=(\frac{T}{C\triangle t})^2} = \lim_{\triangle t\to 0} \exp\Big(\frac{T-[\frac{T}{\triangle t}]\triangle t\ln(1+\frac{T^2}{C^2\triangle t})}{\triangle t}\Big) = 0, \tag{2.15}$$

we get (2.12). $\square$

Remark 2.4 *The function $f(\sigma)$ comes from the L-R estimate and indicates the compensation between the increase of control and the decay of the solution. Lemma 2.1 tells that the increase of the norm of the control can be compensated uniformly with respect to $\triangle t$, provided that eigenvalues σ satisfy $\sigma \le (\frac{T}{2C\triangle t})^2$.*

Remark 2.5 *Now we denote the time parameter T by T_l which indicates the length of the time interval of the l-th step iteration in the L-R time iteration method in our proof of Theorem 1.2. Moreover, let $T>0$ be the control time in Theorem 1.2, we have the following identity between T and T_l:*

$$T_l = 2^{-l-1}T,$$

where l is a positive integer. Obviously $T_l < T$ and we will see, by careful analysis in Section 4, function (2.11) makes sense as σ increasing as the order of $(\triangle t)^{-r}$ with $r\in(0,2)$, instead of $(\triangle t)^{-2}$.

3 Lack of controllability

In this section we will prove Theorem 1.1, which shows the lack of controllability of system (1.2) and the necessity of introducing filtering.

Proof of Theorem 1.1: First, let us prove that system (1.2) is not null controllable.

We use contradiction argument. If (1.2) is null controllable, then for any $y^0 \in L^2(\Omega)$, we can find a control $\{u^k\}_{0\le k\le K-1}$ such that the solution $\{y^k\}_{0\le k\le K}$ of system (1.2) vanishes at $k=K$. Multiplying the first equation in (1.2) by the solution φ^k of (1.3) and summing up in k, using integration by parts, we get

$$0 = \int_\Omega y^K\varphi^K dx = \triangle t\sum_{k=0}^{K-1}\int_\omega u^k\varphi^k dx + \int_\omega y^0\varphi^0 dx.$$

Hence

$$\int_\Omega y^0\varphi^0 dx = -\triangle t\sum_{k=0}^{K-1}\int_\omega u^k\varphi^k dx, \qquad \forall\ y^0\in L^2(\Omega). \tag{3.1}$$

We now show that (3.1) is impossible.

Since $\Omega \setminus \overline{\omega} \neq \emptyset$, there exists a point x_0 in $\Omega \setminus \overline{\omega}$ and consequently one can find a ball

$$B(x_0, A) = \{x : |x - x_0| < A\} \subset \Omega \setminus \overline{\omega},$$

with some positive constant A. We choose a function $\psi \in C_0^\infty(B)$ with positive L^2-norm and let $\varphi^0 = \psi$. From system (1.3) we compute

$$\varphi^{k+1} = \varphi^k - \triangle t \Delta \varphi^k, \qquad k = 0, \cdots, K-1. \tag{3.2}$$

Multiplying (3.2) by φ^k and integrating in Ω, we obtain

$$\int_\Omega \varphi^{k+1}\varphi^k dx = \int_\Omega |\varphi^k|^2 dx + \triangle t \int_\Omega |\nabla \varphi^k|^2 dx. \tag{3.3}$$

By (3.3) we deduce that φ^{k+1} does not vanish in Ω when φ^k has positive L^2-norm. Consequently the initial data $\varphi^K \in L^2(\Omega)$ has positive L^2-norm.

Moreover, taking into account that $\varphi^k \in C_0^\infty(B)$ for any $k \geq 0$ and $B(x_0, A) \cap \omega = \emptyset$, we find that the right side of (3.1) vanishes, i.e.

$$-\triangle t \sum_{k=0}^{K-1} \int_\omega u^k \varphi^k dx = 0. \tag{3.4}$$

Hence, by taking $y^0 = \varphi^0$ in (3.1), we deduce that $\varphi^0 \equiv 0$, which is a contradiction.

Next, we will prove that system (1.2) is not approximately controllable.

It is easy to deduce that approximate controllability of (1.2) is equivalent to the unique continuation of (1.3).

However, it is obvious that the adjoint system (1.3) is not observable since as shown above, there exists a special initial data $\varphi^K \in C_0^\infty(\Omega \setminus \overline{\omega})$ such that the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K-1}$ of (1.3) satisfies:

- $\left\|\varphi^k\right\|_{L^2(\omega)} = 0$ for any $k = 0, \cdots, K-1$;
- $\|\varphi^0\|_{L^2(\Omega)} > 0$.

This fact shows that the unique continuation of (1.3) fails. Roughly speaking, the information of φ^K never appears in the subdomain ω. The observability of system (1.3) fails even with initial data containing in $C_0^\infty(\Omega \setminus \overline{\omega})$.

Consequently, the approximate controllability of (1.2) fails. □

Remark 3.1 *There is an equivalent assertion of the approximate controllability property of the control system, named as "Unique Continuation" for solutions of its adjoint system (see in [21]). It is easy to show the equivalence also holds in the time discrete level. More precisely, one can prove that, the approximate controllability of (1.2) is equivalent to the unique continuation for solutions of system (1.3), which is defined by:*

- *The solution of system (1.3) is said to be of unique continuation if and only if the solution $\{\varphi^k\}_{k=0,\cdots,K}$ has the following property:*

$$\varphi^k = 0 \ \ in \ \Omega, \forall\, k = 0, \cdots, K, \qquad \Leftrightarrow \qquad \varphi^k = 0 \ \ in \ \omega, \forall\, k = 0, \cdots, K.$$

Another way to prove Theorem 1.1 is to find a counterexample such that the above assertion fails. It is similar to the present proof and we omit the details.

However, the counterexample in the proof of Theorem 1.1 disappears if we consider φ^K has the form as a finite combination of the Fourier expansion, i.e.

$$\varphi^K = \sum_{\mu_j \leq \sigma} a_j \Phi_j(x), \tag{3.5}$$

with a positive constant $\sigma > 0$. It is well known that the function φ^K defined in (3.5) is analytic in Ω and only contains finite number of zero points (see [11]). Hence φ^K no longer belongs to $C_0^\infty(\Omega \setminus \bar{\omega})$.

We will see in the next section, if φ^K has the form as in (3.5), the solutions of (1.3) are observable. Consequently, it is possible to discuss the controllability of system (1.2) by the duality argument. This technique is called "filtering method", and has been successfully applied in the context of controllability of numerical approximate schemes for wave equations (see [24] and the references therein).

4 Proof of the Theorem 1.2

This section is devoted to prove Theorem 1.2. The main technique used in the proof is the L-R method, which also plays a key role in the proof of the continuous case.

Proof of Theorem 1.2: We use the L-R time iteration method. The proof is split into several steps.

Step *I*. Let us show the partial controllability property for system (1.2).

We consider first a partial observability of (1.3), i.e. the adjoint system of (1.2). Denote by $\{a_j\}_{j\geq 1}$ the Fourier coefficients of φ^K, i.e.

$$\varphi^K = \sum_{j\geq 1} a_j \Phi_j.$$

The solution of system (1.3) is given by:

$$\varphi^k = \sum_{j\geq 1} a_j (1 + \mu_j \triangle t)^{k-K} \Phi_j.$$

It is easy to see that, for any $\varphi^K \in \mathcal{C}_\sigma$, its Fourier coefficients $a_j = 0$ whenever $\mu_j \geq \sigma$. For any $\sigma > 0$, we claim that there exist two positive constants $C_j = C_j(\Omega, \omega) > 0, j = 1, 2$, independent of σ, such that

$$\triangle t \sum_{k=0}^{K-1} \int_\omega |\varphi^k|^2 dx \geq C_1 T e^{-C_2\sqrt{\sigma}} \int_\Omega |\varphi^0|^2 dx, \qquad \forall\ \varphi^K \in \mathcal{C}_\sigma. \tag{4.1}$$

In fact, using inequality (2.9) in Theorem 2.2 and recalling $K\triangle t = T$, we have

$$\begin{aligned}
\triangle t \sum_{k=0}^{K-1} \int_\omega |\varphi^k|^2 dx \ &= \triangle t \sum_{k=0}^{K-1} \int_\omega \Big| \sum_{\mu_j \leq \sigma} a_j (1 + \mu_j \triangle t)^{k-K} \Phi_j \Big|^2 dx \\
&\geq \triangle t \sum_{k=0}^{K-1} C_1 e^{-C_2\sqrt{\sigma}} \sum_{\mu_j \leq \sigma} |a_j (1 + \mu_j \triangle t)^{k-K}|^2 \\
&= C_1 e^{-C_2\sqrt{\sigma}} \sum_{\mu_j \leq \sigma} a_j^2 \triangle t \sum_{k=1}^{K} (1 + \mu_j \triangle t)^{-2k} \\
&\geq C_1 T e^{-C_2\sqrt{\sigma}} \sum_{\mu_j \leq \sigma} a_j^2 (1 + \mu_j \triangle t)^{-2K} \\
&= C_1 T e^{-C_2\sqrt{\sigma}} \int_\Omega |\varphi^0|^2 dx.
\end{aligned}$$

Formula (4.1) is a partial observability of system (1.3). By means of the classical duality argument we conclude that there exist $\{u^k\}_{0\leq k\leq K}$ and two positive constants $C_1, C_2 > 0$, such that the corresponding solution $\{y^k\}_{0\leq k\leq K}$ of (1.2) satisfies

$$\pi_\sigma y^K = 0, \tag{4.2}$$

and

$$\triangle t \sum_{k=0}^{K} \int_\omega |u^k|^2 dx \leq \frac{1}{C_1 T} e^{C_2\sqrt{\sigma}} \int_\Omega |y^0|^2 dx. \tag{4.3}$$

Step II**.** We now construct the desired control by means of the L-R method.

For $l = 1, 2, ...,$ we set $\sigma_l = A\sigma_{l-1}$ with two parameters $\sigma_0 > 0$ and $A > 1$ to be determined later. Set $T_0 = 0,\ T_{2l} - T_{2l-1} = T_{2l-1} - T_{2l-2} = 2^{-l-1}T$. More precisely, we choose

$$T_{2l} = (1 - 2^{-l})T, \qquad T_{2l-1} = (1 - 3 \cdot 2^{-l-1})T, \qquad l = 1, 2, \cdots.$$

The time interval $(0, T)$ is divided to a series of subintervals

$$I_1 = (T_0, T_2),\ I_2 = (T_2, T_4), \cdots,\ I_l = (T_{2l-2}, T_{2l}), \cdots. \tag{4.4}$$

We choose the control for system (1.2) as follows:

• **In the time interval** (T_0, T_1)**.** This is the first half part of I_1. Set $K_0 = [\frac{T_0}{\triangle t}]$ and $K_1 = [\frac{T_1}{\triangle t}]$. From **Step** I, especially recalling (4.2) and (4.3), we can find a control

$$\{u^k\}_{K_0 \le k \le K_1} \tag{4.5}$$

such that the corresponding solution $\{y^k\}_{K_0 \le k \le K_1}$ of (1.2) satisfies

$$\pi_{\sigma_1} y^{K_1} = 0, \tag{4.6}$$

and

$$\triangle t \sum_{k=K_0}^{K_1} \int_\omega |u^k|^2 dx \le \frac{1}{C_1(T_1 - T_0)} e^{C_2\sqrt{\sigma_1}} \int_\Omega |y^0|^2 dx. \tag{4.7}$$

By means of the usual energy method, noting (4.7) and $T_1 - T_0 = 2^{-2}T$, it is clear that

$$\int_\Omega |y^{K_1}|^2 dx \le e^{D\sqrt{\sigma_1}} \int_\Omega |y^0|^2 dx, \tag{4.8}$$

where the constant $D > 0$ is independent of $\triangle t$.

• **In the time interval** (T_1, T_2)**.** This is the last half part of I_1. Set $K_2 = [\frac{T_2}{\triangle t}]$. We choose the control as

$$u^k = 0, \qquad k = K_1 + 1, \cdots, K_2. \tag{4.9}$$

Taking (4.6) into account and $T_2 - T_1 = 2^{-2}T$, it is easy to show that

$$\int_\Omega |y^{K_2}|^2 dx \le (1 + \sigma_1 \triangle t)^{-[\frac{2^{-2}T}{\triangle t}]} \int_\Omega |y^{K_1}|^2 dx. \tag{4.10}$$

• **In the time interval** (T_{2l-2}, T_{2l-1}), **with** $l = 2, 3, \cdots$**.** This is the first half part of I_l. Set $K_{2l-2} = [\frac{T_{2l-2}}{\triangle t}]$ and $K_{2l-1} = [\frac{T_{2l-1}}{\triangle t}]$. Similarly, one can find a control

$$\{u^k\}_{K_{2(l-1)} < k \le K_{2l-1}} \tag{4.11}$$

such that

$$\pi_{\sigma_l} y^{K_{2l-1}} = 0, \tag{4.12}$$

and

$$\triangle t \sum_{k=K_{2(l-1)}}^{K_{2l-1}} \int_\omega |u^k|^2 dx \le \frac{1}{C_1(T_{2l-1} - T_{2(l-1)})} e^{C_2\sqrt{\sigma_l}} \int_\Omega |y^{K_{2(l-1)}}|^2 dx. \tag{4.13}$$

By means of usual energy method, noting (4.13) and $T_{2l-1} - T_{2(l-1)} = 2^{-l-1}T$, it is clear that

$$\int_\Omega |y^{K_{2l-1}}|^2 dx \le e^{D\sqrt{\sigma_l}} \int_\Omega |y^{K_{2(l-1)}}|^2 dx, \tag{4.14}$$

with the same constant D in (4.8).

• **In the time interval** (T_{2l-1}, T_{2l})**.** This is the last half part of I_l. Set $K_{2l} = [\frac{T_{2l}}{\triangle t}]$. We choose the control as

$$u^k = 0, \qquad k = K_{2l-1}+1, \cdots, K_{2l}. \tag{4.15}$$

Taking (4.12) and (4.14) into account and recalling $T_{2l} - T_{2l-1} = 2^{-l-1}T$, it is easy to show that

$$\begin{aligned} \int_\Omega |y^{K_{2l}}|^2 dx &\le (1+\sigma_l \triangle t)^{-[\frac{2^{-l-1}T}{\triangle t}]} \int_\Omega |y^{K_{2l-1}}|^2 dx \\ &\le (1+\sigma_l \triangle t)^{-[\frac{2^{-l-1}T}{\triangle t}]} e^{D\sqrt{\sigma_l}} \int_\Omega |y^{K_{2(l-1)}}|^2 dx. \end{aligned} \tag{4.16}$$

By induction, we have

$$\int_\Omega |y^{K_{2l}}|^2 dx \le \prod_{s=1}^{l} (1+\sigma_s \triangle t)^{-[\frac{2^{-s-1}T}{\triangle t}]} e^{D\sqrt{\sigma_s}} \int_\Omega |y^0|^2 dx. \tag{4.17}$$

Replacing l by $l-1$ in (4.17) and recalling (4.13) we deduce that there exists a constant $C > 0$, depending only on C_1, such that

$$\begin{aligned} \triangle t \sum_{k=K_{2(l-1)}}^{K_{2l-1}} \int_\omega |u^k|^2 dx &\le C e^{C_2\sqrt{\sigma_l}} \prod_{s=1}^{l-1} (1+\sigma_s \triangle t)^{-[\frac{2^{-s-1}T}{\triangle t}]} e^{D\sqrt{\sigma_s}} \int_\Omega |y^0|^2 dx \\ &\le C e^{D\sqrt{\sigma_1}} \prod_{s=1}^{l-1} (1+\sigma_s \triangle t)^{-[\frac{2^{-s-1}T}{\triangle t}]} e^{D\sqrt{\sigma_{s+1}}} \int_\Omega |y^0|^2 dx. \end{aligned} \tag{4.18}$$

Step III**.** Recalling that $\sigma_l = A\sigma_{l-1}$ with $A > 1$, we may rewrite the product $\prod_{s=1}^{l-1}(1+\sigma_s \triangle t)^{-[\frac{2^{-s-1}T}{\triangle t}]} e^{D\sqrt{\sigma_{s+1}}}$ in (4.18) as a function

$$R(\sigma_l) = \prod_{s=1}^{l-1} \exp\Big(D\sqrt{A\sigma_s} - \Big[\frac{2^{-s-1}T}{\triangle t}\Big] \ln(1+\sigma_s \triangle t)\Big). \tag{4.19}$$

By Lemma 2.1 we claim that for $0 \le \delta \le D\sqrt{A}$, there exists a positive constant C_3, depending on σ_0 and A, such that

$$R(\sigma_l) \le \prod_{s=1}^{l-1} e^{-\delta\sigma_s} \le C_3 e^{-\frac{\delta}{A-1}\sigma_l} \tag{4.20}$$

holds under the restrictions

$$\Big(\frac{2D\sqrt{A}+\delta}{2^{-s-1}T - \triangle t}\Big)^2 \le \sigma_s \le \Big(\frac{2^{-s-1}T}{2(D\sqrt{A}+\delta)\triangle t}\Big)^2. \tag{4.21}$$

We now choose σ_0 and A such that (4.21) holds. Letting $\delta = D\sqrt{A}$. First, it is not difficult to see that, the left inequality of (4.21) is satisfied if

$$\triangle t \le 2^{-s-2}T, \qquad A \ge 4 \qquad \text{and} \qquad \sigma_0 \ge (\frac{4}{A})^{l-1}\frac{256D^2}{T^2}. \tag{4.22}$$

Secondly, by the second inequality in (4.21), we need

$$\sigma_l = A^l \sigma_0 \leq \frac{4^{-l} T^2}{64 D^2 A (\triangle t)^2}, \tag{4.23}$$

or, equivalently,

$$l \leq \log_{4A} \Big(\frac{T^2}{64 \sigma_0 D^2 (\triangle t)^2} \Big).$$

Denoting by L the maximum of l, i.e.

$$L = \Big[\log_{4A} \Big(\frac{T^2}{64 \sigma_0 D^2 (\triangle t)^2} \Big) \Big], \tag{4.24}$$

one has the estimate

$$\sigma_L = A^L \sigma_0 \geq A^{\log_{4A} \Big(\frac{T^2}{64\sigma_0 D^2 (\triangle t)^2} \Big) - 1} \sigma_0 = \sigma_0^{\log_{4A} 4} \Big(\frac{T}{8D} \Big)^{2(1-\log_{4A} 4)} (\triangle t)^{-2(1-\log_{4A} 4)}.$$

Hence, for any $r \in (0,2)$, we choose

$$A = \max(4, \frac{1}{4} e^{\frac{2\ln 2}{1-r/2}}) \qquad \text{and} \qquad \sigma_0 \geq \frac{256 D^2}{T^2}. \tag{4.25}$$

Consequently we have $\sigma_L \geq \Lambda (\triangle t)^{-r}$, with a constant $\Lambda = \Lambda(r)$ independent of $\triangle t$. More precisely,

- For $r \in [1,2)$, we have $\sigma_L \geq \Lambda(\triangle t)^{-r}$, with $\Lambda = \Lambda(r) = \sigma_0^{1-\frac{r}{2}} \Big(\frac{T}{8D} \Big)^r$.
- For $r \in (0,1)$, we have $\sigma_L \geq \Lambda(\triangle t)^{-1} \geq \Lambda(\triangle t)^{-r}$, with $\Lambda = \Lambda(1) = \sigma_0^{1/2} \Big(\frac{T}{8D} \Big)$.

Step *IV***.** From the analysis above, we know that each $\triangle t$ corresponds to a filtering parameter σ_L. Combining all of the control (4.5), (4.9), (4.11), (4.15) and setting

$$u^k = 0, \qquad K_{2L} < k \leq T/\triangle t,$$

we conclude that for any $r \in (0,2)$ there exist a series of $\{u^k\}_{k=0,\cdots,K-1}$ and a positive constant $\Lambda = \Lambda(r, T, \Omega, \omega)$ such that

$$\pi_{\Lambda(\triangle t)^{-r}} y^K(x) = 0, \qquad \forall\, x \in \Omega. \tag{4.26}$$

Moreover, combining (4.7),(4.18) and taking (4.19) into account, we conclude that

$$\triangle t \sum_{k=0}^{K-1} \int_\omega |u^k|^2 dx \leq C_1 e^{D\sqrt{\sigma_1}} \Big(1 + \sum_{l=1}^{L-1} R(\sigma_l) \Big) \int_\Omega |y^0|^2 dx. \tag{4.27}$$

The analysis in the third step yields a constant C, independent of $\triangle t$, such that

$$\triangle t \sum_{k=0}^{K-1} \int_\omega |u^k|^2 dx \leq C \int_\Omega |y^0|^2 dx \tag{4.28}$$

for any $\triangle t > 0$. More precisely, the constant is given by

$$\begin{aligned} C &= \sup_{\triangle t>0} C_1 e^{D\sqrt{\sigma_1}}\Big(1 + C_3\sum_{l=1}^{L-1} e^{-\frac{D\sqrt{A}}{A-1}\sigma_l}\Big) \\ &\leq C_1 e^{D\sqrt{\sigma_1}}\Big(1 + C_3\sum_{l=1}^{\infty} e^{-\frac{D\sqrt{A}}{A-1}\sigma_l}\Big) \\ &= C(A, \sigma_0, T, C_1, C_2), \end{aligned}$$

which dependents only on r, T, Ω and ω. □

5 Error estimate of the control

In this section we discuss the error estimate for the discrete control $\{u^k\}_{k=0,\cdots,K-1}$ and prove Theorem 1.3.

Proof of Theorem 1.3: Let T_l, I_l, $\{u^k\}_{k=0,\cdots,K}$ be the same as in (4.4). From the proof of Theorem 1.2 we know that the following properties of the discrete control $\{u^k\}_{k=0,\cdots,K-1}$ hold:

- For any fixed $\triangle t > 0$, only L steps of time iterations are given by L-R method, where L can be deduced by (4.24);
- L tends to infinity as $\triangle t$ tends to zero;
- The control is separately located in the subintervals $(T_{2(l-1)}, T_{2l-1})$, with $l = 1, \cdots, L$.

Hence, we consider the convergence of control in each subinterval $(T_{2(l-1)}, T_{2l-1})$, separately.

Step 1: First we consider the discrete control $\{u^k\}_{k=0,\cdots,K_1}$ with $K_1 = [\frac{T_1}{\triangle t}]$, i.e. the control in the time subinterval (T_0, T_1).

• **Discrete control.** Let $\psi^{K_1} \in \mathcal{C}_{\sigma_1}$ be the initial data of (1.3), i.e. the adjoint system of (1.2). We denote the corresponding solution by $\{\psi^k\}_{k=0,\cdots,K_1}$.

Given $y^0 \in L^2(\Omega)$, the control $\{u^k \in L^2(\omega)\}_{k=0,\cdots,K_1}$ is given by

$$u^k = \hat{\psi}^k, \qquad \forall\ k = 0, \cdots, K_1. \tag{5.1}$$

Here $\{\hat{\psi}^k\}_{k=0,\cdots,K_1}$ is the solution of system (1.3) corresponding to the initial data $\hat{\psi}^{K_1}$ which minimizes the functional

$$J_{\triangle t}(\psi^{K_1}) = \frac{1}{2}\triangle t\sum_{k=0}^{K_1-1}\int_\omega |\psi^k|^2 dx + \int_\Omega y^0\psi^0 dx \tag{5.2}$$

in the space $\mathcal{C}_{\sigma_1}$. The minimizer of $J_{\triangle t}$ is well defined since observability of the system (1.3) is provided if its initial data is taken in $\mathcal{C}_{\sigma_1}$.

Taking into account that $\hat{\psi}^{K_1}$ is the minimizer, we deduce that $\{\hat{\psi}^k\}_{k=0,\cdots,K_1}$ satisfies

$$\triangle t\sum_{k=0}^{K_1-1}\int_\omega \hat{\psi}^k\psi^k dx+\int_\Omega y^0\psi^0 dx=0,\qquad \forall\ \psi^{K_1}\in\mathcal{C}_{\sigma_1}. \tag{5.3}$$

We define two functions $\hat{\psi}^{\triangle t}$ and $\psi^{\triangle t}$ by

$$\hat{\psi}^{\triangle t}(t,\cdot)=\sum_{k=0}^{K_1-1}\hat{\psi}^k(\cdot)1_{[k\triangle t,(k+1)\triangle t)}(t),\qquad \psi^{\triangle t}(t,\cdot)=\sum_{k=0}^{K_1-1}\psi^k(\cdot)1_{[k\triangle t,(k+1)\triangle t)}(t),$$

respectively. Consequently, formula (5.3) can be rewritten as a time-continuous form

$$\int_0^{T_1}\int_\omega \hat{\psi}^{\triangle t}\psi^{\triangle t}dxdt+\int_\Omega y^0\psi^0 dx=0,\qquad \forall\ \psi^{K_1}\in\mathcal{C}_{\sigma_1}. \tag{5.4}$$

• **Continuous control.** Let $\Psi^{T_1}\in\mathcal{C}_{\sigma_1}$ be the initial data of (2.7), i.e. the adjoint system of (1.1), with T replaced by T_1. We denote the corresponding solution by $\Psi(t,x)$ with $t\in(0,T_1)$.

Similarly, given the same initial data $y(0,x)=y^0$ for the continuous system (1.1), the control $u(t,x)\in L^2((0,T_1)\times\omega)$ is given by

$$u(t,x)=\hat{\Psi}(t,x),\qquad (t,x)\in(0,T_1)\times\omega, \tag{5.5}$$

where $\hat{\Psi}$ is the solution of (2.7) corresponding to the initial data $\hat{\Psi}^{T_1}$ which minimizes the functional

$$J(\Psi^{T_1})=\frac{1}{2}\int_0^{T_1}\int_\omega \Psi^2 dxdt+\int_\Omega y^0\Psi(0)dx \tag{5.6}$$

in the space $\mathcal{C}_{\sigma_1}$.

Taking into account that $\hat{\Psi}^{T_1}$ is the minimizer of $J(\Psi^{T_1})$, we conclude that $\hat{\Psi}$ satisfies

$$\int_0^{T_1}\int_\omega \hat{\Psi}\Psi dxdt+\int_\Omega y^0\Psi(0)dx=0,\qquad \forall\ \Psi^{T_1}\in\mathcal{C}_{\sigma_1}. \tag{5.7}$$

Now we compute the difference between $\hat{\psi}^{\triangle t}$ and $\hat{\Psi}$.

Taking $\psi^{K_1}=\Psi^{T_1}\in\mathcal{C}_{\sigma_1}$, combining (5.4) and (5.7) we get

$$\int_0^{T_1}\int_\omega(\hat{\psi}^{\triangle t}-\hat{\Psi})\Psi dxdt=\int_\Omega y^0(\Psi(0)-\psi^0)dx+\int_0^{T_1}\int_\omega \hat{\psi}^{\triangle t}(\Psi-\psi^{\triangle t})dxdt. \tag{5.8}$$

Hence,

$$\begin{aligned}&\Big|\int_0^{T_1}\int_\omega(\hat{\Psi}-\hat{\psi}^{\triangle t})\Psi dxdt\Big|\\ &\le \|y^0\|_{L^2(\Omega)}\|\Psi(0)-\psi^0\|_{L^2(\Omega)}+\left\|\hat{\psi}^{\triangle t}\right\|_{L^2((0,T_1)\times\omega)}\left\|\Psi-\psi^{\triangle t}\right\|_{L^2((0,T_1)\times\omega)}.\end{aligned} \tag{5.9}$$

Assuming $\psi^{K_1} = \Psi^{T_1} = \sum_{\mu_j \le \sigma_1} a_j \Phi_j$, we get

$$\Psi = \sum_{\mu_j \le \sigma_1} a_j e^{-\mu_j (T_1 - t)} \Phi_j, \qquad \psi^k = \sum_{\mu_j \le \sigma_1} a_j (1 + \mu_j \triangle t)^{-(K_1 - k)} \Phi_j. \tag{5.10}$$

The square of $\|\Psi(0) - \psi^0\|_{L^2(\Omega)}$ reads

$$\begin{aligned}
\left\|\Psi(0) - \psi^0\right\|^2_{L^2(\Omega)} &= \int_\Omega \Big(\sum_{\mu_j \le \sigma_1} a_j \Big[e^{-\mu_j T_1} - (1 + \mu_j \triangle t)^{-K_1} \Big] \Phi_j \Big)^2 dx \\
&= \sum_{\mu_j \le \sigma_1} a_j^2 \Big[e^{-\mu_j T_1} - (1 + \mu_j \triangle t)^{-K_1} \Big]^2 \\
&= \sum_{\mu_j \le \sigma_1} a_j^2 e^{-2\mu_j T_1} \Big[1 - \exp(\mu_j T_1 - K_1 \ln(1 + \mu_j \triangle t)) \Big]^2 \\
&\le \Big(\frac{T_1}{2} \sigma_1^2 \triangle t + O((\triangle t)^2) \Big)^2 \sum_{\mu_j \le \sigma_1} a_j^2 e^{-2\mu_j T_1} \\
&\le C \Big(\sigma_1^2 \triangle t \Big)^2 \sum_{\mu_j \le \sigma_1} a_j^2 e^{-2\mu_j T_1} = C \Big(\sigma_1^2 \triangle t \Big)^2 \|\Psi(0)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.11}$$

Similarly, we estimate

$$\begin{aligned}
\left\|\Psi - \psi^{\triangle t}\right\|^2_{L^2((0,T_1)\times\omega)} &= \sum_{k=0}^{K_1 - 1} \int_{k\triangle t}^{(k+1)\triangle t} \int_\omega \Big(\sum_{\mu_j \le \sigma_1} a_j (e^{\mu_j (t - T_1)} - (1 + \mu_j \triangle t)^{k - K_1}) \Phi_j \Big)^2 dx \\
&\le C \Big(\sigma_1^2 \triangle t \Big)^2 \int_0^{T_1} \int_\Omega \Big(\sum_{\mu_j \le \sigma_1} a_j^2 e^{-2\mu_j (T_1 - t)} \Big) dx dt \\
&= C \Big(\sigma_1^2 \triangle t \Big)^2 \|\Psi\|^2_{L^2((0,T_1)\times\Omega)}.
\end{aligned} \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.9) and using the known observability estimate for Ψ, we conclude that for any $\psi^{K_1} = \Psi^{T_1} \in \mathcal{C}_{\sigma_1}$ it holds

$$\begin{aligned}
&\Big| \int_0^{T_1} \int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi}) \Psi dx dt \Big| \\
&\le C \sigma_1^2 \triangle t \Big(\|\Psi(0)\|_{L^2(\Omega)} \|y_0\|_{L^2(\Omega)} + \left\| \hat{\psi}^{\triangle t} \right\|_{L^2((0,T_1)\times\omega)} \|\Psi\|_{L^2((0,T_1)\times\Omega)} \Big) \\
&\le C \sigma_1^2 \triangle t \Big(\|\Psi\|_{L^2((0,T_1)\times\omega)} \|y_0\|_{L^2(\Omega)} + \left\| \hat{\psi}^{\triangle t} \right\|_{L^2((0,T_1)\times\omega)} \|\Psi\|_{L^2((0,T_1)\times\Omega)} \Big).
\end{aligned} \tag{5.13}$$

On the other hand, (5.8) can be also written as

$$\int_0^{T_1} \int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi}) \psi^{\triangle t} dx dt = \int_\Omega y^0 (\Psi(0) - \psi^0) dx + \int_0^{T_1} \int_\omega \hat{\Psi} (\Psi - \psi^{\triangle t}) dx dt. \tag{5.14}$$

With the same procedure as before we have

$$\begin{aligned}
&\forall\, \psi^{K_1} = \Psi^{T_1} \in \mathcal{C}_{\sigma_1}, \qquad \Big| \int_0^{T_1} \int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi}) \psi^{\triangle t} dx dt \Big| \\
&\le C \sigma_1^2 \triangle t \Big(\|\Psi\|_{L^2((0,T_1)\times\omega)} \|y_0\|_{L^2(\Omega)} + \left\| \hat{\Psi} \right\|_{L^2((0,T_1)\times\omega)} \|\Psi\|_{L^2((0,T_1)\times\Omega)} \Big).
\end{aligned} \tag{5.15}$$

Now we choose $\Psi^{T_1} = \hat{\Psi}^{T_1}$ in (5.13) and $\psi^{K_1} = \hat{\psi}^{K_1}$ in (5.15), then adding (5.13) and (5.15), we get

$$\begin{aligned}
&\Big|\int_0^{T_1}\int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi})\hat{\Psi} dxdt\Big| + \Big|\int_0^{T_1}\int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi})\hat{\psi}^{\triangle t} dxdt\Big| \\
&\le C\sigma_1^2 \triangle t\Big(\left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\omega)} \|y_0\|_{L^2(\Omega)} + \left\|\hat{\psi}^{\triangle t}\right\|_{L^2((0,T_1)\times\omega)} \left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\Omega)} + \\
&\qquad + \left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\omega)} \left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\Omega)}\Big).
\end{aligned} \tag{5.16}$$

In view of the Theorem 2.2 and using the observability (2.9), we directly get that

$$\left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\Omega)} \le C_1 e^{C_2\sqrt{\sigma_1}} \left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\omega)}, \tag{5.17}$$

where the two positive constants are independent of $\triangle t$ and T_1.

Since $\hat{\Psi}$ and $\hat{\psi}^k$ are controls, we deduce from Theorem 1.2 that there exists a constant C, independent of $\triangle t$, such that

$$\left\|\hat{\Psi}\right\|_{L^2((0,T_1)\times\omega)} \le C \|y_0\|_{L^2(\Omega)}, \qquad \left\|\hat{\psi}^{\triangle t}\right\|_{L^2((0,T_1)\times\omega)} \le C \|y_0\|_{L^2(\Omega)}. \tag{5.18}$$

Combining (5.16), (5.17) and (5.18) we conclude that

$$\begin{aligned}
&\Big|\int_0^{T_1}\int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi})\hat{\Psi} dxdt\Big| + \Big|\int_0^{T_1}\int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi})\hat{\psi}^{\triangle t} dxdt\Big| \\
&\le C \triangle t\ \sigma_1^2 e^{C_2\sqrt{\sigma_1}} \|y_0\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.19}$$

Consequently,

$$\int_0^{T_1}\int_\omega (\hat{\psi}^{\triangle t} - \hat{\Psi})^2 dxdt \le C \triangle t\ \sigma_1^2 e^{C_2\sqrt{\sigma_1}} \|y_0\|^2_{L^2(\Omega)}. \tag{5.20}$$

Step l, with $l = 2, \cdots,$: We redo the same proceeding as in **Step** 1. Recalling inequality (4.17) we get the following estimate for the initial data of the l step:

$$\left\|y^{K_{2(l-1)}}\right\|^2_{L^2(\Omega)} \le R(\sigma_{l+1}) \left\|y^0\right\|^2_{L^2(\Omega)}.$$

Hence, by carefully computation we obtain that

$$\begin{aligned}
&\Big|\int_{T_{2(l-1)}}^{T_{2l-1}}\int_\omega \Big(\sum_{k=K_{2(l-1)}}^{K_{2l-1}} \hat{\psi}^k 1_{[t_k,t_{k+1})}(t) - \hat{\Psi}\Big)^2 dxdt\Big| \\
&\le C \triangle t\ \sigma_l^2 e^{C_2\sqrt{\sigma_l}} \left\|y^{K_{2(l-1)}}\right\|^2_{L^2(\Omega)} \\
&\le C \triangle t\ \sigma_l^2 e^{C_2\sqrt{\sigma_l}} R(\sigma_{l+1}) \left\|y^0\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{5.21}$$

Conclusion: Taking into account that $u^k = \hat{\psi}^k$ for all $k = 0, \cdots, K-1$ and $u = \hat{\Psi}$, combining inequalities (5.20) and (5.21) (applying for all $2 \le l \le L$), recalling the estimation of $R(\sigma_l)$ in (4.20) we attain

$$\begin{aligned} \int_0^{T_{2l-1}} \int_\omega (U^K - u)^2 dxdt &= \sum_{l=1}^{L} \Big| \int_{T_{2(l-1)}}^{T_{2l-1}} \int_\omega \Big(\sum_{k=K_{2(l-1)}}^{K_{2l-1}} \hat{\psi}^k 1_{[t_k, t_{k+1})}(t) - \hat{\Psi} \Big)^2 dxdt \Big| \\ &\le \triangle t \sum_{l=1}^{L} C\, \sigma_l^2 e^{C_2\sqrt{\sigma_l}} R(\sigma_l) \left\| y^0 \right\|_{L^2(\Omega)}^2 \le C \triangle t \left\| y^0 \right\|_{L^2(\Omega)}^2 . \end{aligned} \tag{5.22}$$

Furthermore, since $U^K = 0$ for $t \in (T_{2L-1}, T)$ and $u(t) = 0$ for $t \in (T_{2L-1}, T_{2L})$, we estimate

$$\int_{T_{2L-1}}^{T} \int_\omega (U^K - u)^2 dxdt = \int_{T_{2L}}^{T} \int_\omega u^2 dxdt \le Ce^{-C\sigma_L} \left\| y^0 \right\|_{L^2}^2 \le Ce^{-\Lambda(\triangle t)^{-r}} \left\| y^0 \right\|_{L^2(\Omega)}^2 . \tag{5.23}$$

Combining (5.22) and (5.23), we arrive at

$$\int_0^T \int_\omega (U^K - u)^2 dxdt \le C \triangle t \left\| y^0 \right\|_{L^2(\Omega)}^2 , \tag{5.24}$$

which equals to (1.10).

On the other hand, from (4.28) we have

$$\lim_{\triangle t \to 0} \left\| U^K \right\|_{L^2((0,T)\times\omega)}^2 = \lim_{\triangle t \to 0} \triangle t \sum_{k=0}^{K-1} \int_\omega |u^k|^2 dx < C \int_\Omega |y^0|^2 dx < \infty. \tag{5.25}$$

Taking (5.24) and (5.25) into account, we arrive at

$$\lim_{\triangle t \to 0} U^K = \lim_{\triangle t \to 0} \sum_{k=0}^{K-1} u^k(x) 1_{[t_k, t_{k+1})}(t) \longrightarrow u \text{ strongly in } L^2((0,T)\times\omega),$$

where u is a control for the continuous heat equation (1.1) with initial data $y^0 \in L^2(\Omega)$.

In fact, from the construction of the control we have indicated that, the control u satisfies

$$u = \begin{cases} u_l, & t \in \text{the first half part of } I_l, \qquad l = 1, 2, \cdots, \\ 0, & t \in \text{the last half part of } I_l, \qquad l = 1, 2, \cdots, \end{cases} \tag{5.26}$$

with

$$u_l(t,x) = \lim_{\triangle t \to 0} \sum_{k=T_{2l-2}/\triangle t}^{T_{2l-1}/\triangle t} u^k(x) 1_{[t_k, t_{k+1})}(t).$$

□

6 Approximate controllability

In previous sections, we have proved that system (1.2) is uniformly null-controllable with appropriate filtering parameter s. More precisely, after filtering the final target y^K, i.e. if we only consider the projection of y^K on the filtered space $\mathcal{C}_{\Lambda(\triangle t)^{-r}}$, the null controllability holds uniformly with respect to $\triangle t$. As a direct consequence of Theorem 1.2, we have the following uniform observability of system (1.3) with $\varphi^K \in \mathcal{C}_{\Lambda(\triangle t)^{-r}}$:

Corollary 6.1 *Let $\{\varphi^k\}_{k=0,\cdots,K-1}$ be the solution of (1.3) with initial data φ^K. Then for any fixed $T > 0$ and $r \in (0,2)$, there exists two positive constants $\Lambda = \Lambda(r,T,\Omega,\omega)$ and $C = C(r,T,\Omega,\omega)$ such that*

$$\left\|\varphi^0\right\|^2_{L^2(\Omega)} \le C\triangle t \sum_{k=0}^{K-1} \int_\omega |\varphi^k|^2 dx \tag{6.1}$$

holds for any $\triangle t > 0$ and $\varphi^K \in \mathcal{C}_{\Lambda(\triangle t)^{-r}}$.

Proof: It is a direct consequence of Theorem 1.2 by means of the HUM method. □

Now we give a proof of Theorem 1.4:

Proof of Theorem 1.4: In the sequel we denote $\Lambda(\triangle t)^{-r}$ by s.

For given $y_1 \in L^2(\Omega)$ and $\varepsilon > 0$, we define a functional $J_\varepsilon : \mathcal{C}_s \longrightarrow \mathbb{R}$ by

$$J_\varepsilon(\varphi^K) = \frac{1}{2}\triangle t \sum_{k=0}^{K-1} \int_\omega |\varphi^k|^2 dx + \varepsilon \left\|\varphi^K\right\|_{L^2(\Omega)} - \int_\Omega y_1 \varphi^K dx, \tag{6.2}$$

where $\{\varphi^k\}_{k=0,\cdots,K-1}$ is the solution of the adjoint system (1.3) with initial data φ^K.

First, we prove the existence of the control:

• **Existence of the control.** The following Lemmas 6.1−6.2 ensure that the minimum of J_ε gives a control for our approximate controllability problem:

Lemma 6.1 *If $\hat\varphi^K$ is a minimizer of J_ε in $\mathcal{C}_s$ and $\{\hat\varphi^k\}_{k=0,\cdots,K-1}$ is the solution of the adjoint system (1.3) with initial data $\hat\varphi^K$, then $\{u^k = \hat\varphi^k|_\omega\}_{k=0,\cdots,K-1}$ is a control such that (1.11) holds.*

Proof: Suppose that J_ε attains its minimum value at $\hat\varphi^K \in \mathcal{C}_s$. Then for any $\psi^K \in \mathcal{C}_s$ and $h \in \mathbb{R}$ we have $J_\varepsilon(\hat\varphi^K) \le J_\varepsilon(\hat\varphi^K + h\psi^K)$. On the other hand,

$$\begin{aligned} &J_\varepsilon(\hat\varphi^K + h\psi^K) \\ &= \frac{1}{2}\triangle t \sum_{k=0}^{K-1} \int_\omega |\hat\varphi^k + h\psi^k|^2 dx + \varepsilon \left\|\hat\varphi^K + h\psi^K\right\|_{L^2(\Omega)} - \int_\Omega y_1(\hat\varphi^K + h\psi^K) dx \\ &= \frac{1}{2}\triangle t \sum_{k=0}^{K-1} \int_\omega |\hat\varphi^k|^2 dx + \frac{h^2}{2}\triangle t \sum_{k=0}^{K-1} \int_\omega |\psi^k|^2 dx + h\triangle t \sum_{k=0}^{K-1} \int_\omega \hat\varphi^k \psi^k dx + \\ &\qquad + \varepsilon \left\|\hat\varphi^K + h\psi^K\right\|_{L^2(\Omega)} - \int_\Omega y_1(\hat\varphi^K + h\psi^K) dx. \end{aligned} \tag{6.3}$$

Thus,

$$
\begin{aligned}
0 \leq \quad & \varepsilon\Big[\left\|\hat{\varphi}^K + h\psi^K\right\|_{L^2(\Omega)} - \left\|\hat{\varphi}^K\right\|_{L^2(\Omega)}\Big] + \frac{h^2}{2}\triangle t\sum_{k=0}^{K-1}\int_\omega |\psi^k|^2 dx \\
& + h\Big[\triangle t\sum_{k=0}^{K-1}\int_\omega \hat{\varphi}^k\psi^k dx - \int_\Omega y_1\psi^K dx\Big].
\end{aligned} \tag{6.4}
$$

Since

$$
\left\|\hat{\varphi}^K + h\psi^K\right\|_{L^2(\Omega)} - \left\|\hat{\varphi}^K\right\|_{L^2(\Omega)} \leq h\left\|\psi^K\right\|_{L^2(\Omega)},
$$

we obtain that

$$
0 \leq \varepsilon|h|\left\|\psi^K\right\|_{L^2(\Omega)} + \frac{h^2}{2}\triangle t\sum_{k=0}^{K-1}\int_\omega |\psi^k|^2 dx + h\triangle t\sum_{k=0}^{K-1}\int_\omega \hat{\varphi}^k\psi^k dx - h\int_\Omega \psi^K y_1 dx
$$

holds for all $h \in \mathbb{R}$ and $\psi^K \in \mathcal{C}_s$.

Dividing by $h > 0$ and by passing to the limit $h \to 0$ we get

$$
0 \leq \varepsilon\left\|\psi^K\right\|_{L^2(\Omega)} + \triangle t\sum_{k=0}^{K-1}\int_\omega \hat{\varphi}^k\psi^k dx - \int_\Omega y_1\psi^K dx. \tag{6.5}
$$

The same calculations with $h < 0$ gives that

$$
\Big|\triangle t\sum_{k=0}^{K-1}\int_\omega \hat{\varphi}^k\psi^k dx - \int_\Omega y_1\psi^{K_1} dx\Big| \leq \varepsilon\left\|\psi^K\right\|_{L^2(\Omega)}, \qquad \forall\ \psi^K \in \mathcal{C}_s. \tag{6.6}
$$

On the other hand, if we take the control $u^k = \hat{\varphi}^k$ in (1.2), by multiplying in (1.2) by ψ^k and adding from $k = 0$ to $K - 1$ we get that

$$
\triangle t\sum_{k=0}^{K-1}\int_\omega \hat{\varphi}^k\psi^k dx = \int_\Omega y^K\psi^K dx. \tag{6.7}
$$

Combining (6.6) and (6.7), it follows that

$$
\Big|\int_\Omega (y^K - y_1)\psi^K dx\Big| \leq \varepsilon\left\|\psi^K\right\|_{L^2(\Omega)}, \qquad \forall\ \psi^K \in \mathcal{C}_s,
$$

which is equivalent to

$$
\left\|\pi_s(y^K - y_1)\right\|_{L^2(\Omega)} \leq \varepsilon.
$$

The proof of the Lemma is now complete. □

Now we show that J attains its minimum in $\mathcal{C}_s$.

Lemma 6.2 *There exists a* $\hat{\varphi}^K \in \mathcal{C}_s$ *such that*

$$
J_\varepsilon(\hat{\varphi}^K) = \min_{\varphi^K \in \mathcal{C}_s} J_\varepsilon(\varphi^K). \tag{6.8}
$$

Proof: It is easy to see that J_ε is convex and continuous in $\mathcal{C}_s$. The existence of a minimum of J_ε is ensured if J_ε is coercive, i.e.

$$J_\varepsilon(\varphi^K) \to \infty \text{ when } \left\|\varphi^K\right\|_{\mathcal{C}_s} \to \infty. \tag{6.9}$$

In fact we shall prove that

$$\varliminf_{\|\varphi^K\|_{\mathcal{C}_s}\to\infty} \frac{J_\varepsilon(\varphi^K)}{\|\varphi^K\|_{\mathcal{C}_s}} \geq \varepsilon. \tag{6.10}$$

Obviously, (6.10) implies (6.9) and the proof of the Lemma is complete.

In order to prove (6.10) let $(\varphi_j^K) \in \mathcal{C}_s$ be a sequence of initial data for the adjoint system with $\left\|\varphi_j^K\right\|_{\mathcal{C}_s} \to \infty$ as $j \to \infty$. We normalize them

$$\tilde{\varphi}_j^K = \varphi_j^K / \left\|\varphi_j^K\right\|_{\mathcal{C}_s},$$

so that $\left\|\varphi_j^K\right\|_{\mathcal{C}_s} = 1$.

On the other hand, let $\tilde{\varphi}_j^k$ be the solutions of the (1.3) with initial data $\tilde{\varphi}_j^K$. Then

$$\frac{J_\varepsilon(\varphi_j^K)}{\left\|\varphi_j^K\right\|_{\mathcal{C}_s}} = \frac{1}{2}\left\|\varphi_j^K\right\|_{\mathcal{C}_s} \triangle t \sum_{k=0}^{K-1} \int_\omega |\tilde{\varphi}_j^k|^2 dx + \varepsilon - \int_\Omega y_1 \tilde{\varphi}_j^K dx.$$

The following two cases may occur:

1) $\displaystyle\varliminf_{j\to\infty} \triangle t \sum_{k=0}^{K-1} \int_\omega |\tilde{\varphi}_j^k|^2 dx > 0$. In this case we obtain immediately that

$$\frac{J_\varepsilon(\varphi_j^K)}{\left\|\varphi_j^K\right\|_{\mathcal{C}_s}} \to \infty, \qquad \text{as } j \to \infty.$$

2) $\displaystyle\varliminf_{j\to\infty} \triangle t \sum_{k=0}^{K-1} \int_\omega |\tilde{\varphi}_j^k|^2 dx = 0$. In this case since $\tilde{\varphi}_j^K$ is bounded in $\mathcal{C}_s$, by extracting a subsequence we can guarantee that $\tilde{\varphi}_j^K \rightharpoonup \psi_0^K$ weakly in $\mathcal{C}_s$ and $\tilde{\varphi}_j^k \rightharpoonup \psi_0^k$ weakly in $\mathcal{C}_s$, where $\{\psi_0^k\}_{k=0}^{K-1}$ is the solution of (1.3) with initial data ψ_0^K. Moreover, by lower semi-continuity,

$$\triangle t \sum_{k=0}^{K-1} \int_\omega |\psi_0^k|^2 dx \leq \varliminf_{j\to\infty} \triangle t \sum_{k=0}^{K-1} \int_\omega |\tilde{\varphi}_j^k|^2 dx = 0$$

and therefore $\psi_0^k = 0$ in ω for any $k = 0, \cdots, K-1$.

On the other hand, by (6.1) it is obvious that the unique continuation of (1.3) holds for any $\psi^K \in \mathcal{C}_s$. Hence $\psi^k \equiv 0$ in Ω for any $k = 0, \cdots, K$ and consequently $\psi_0^K = 0$.

Therefore, $\tilde{\varphi}_j^K \rightharpoonup 0$ weakly in $\mathcal{C}_s$ and consequently $\int_\Omega y_1 \tilde{\varphi}_j^K dx$ vanishes as well.

Hence

$$\frac{J_\varepsilon(\varphi_j^K)}{\left\|\varphi_j^K\right\|_{\mathcal{C}_s}} \geq \varliminf_{j\to\infty} (\varepsilon - \int_\Omega y_1 \tilde{\varphi}_j^K dx) = \varepsilon,$$

and (6.10) follows.

Now we estimate the bound of the control $\{u^k\}_{k=0,\cdots,K-1}$:

• **Uniformly bounded control**. It is an analogue of the continuous case (see, Section 6 of [3]). We assume that

$$z_1 = \sum_{j=1}^{m} b_j \Phi_j. \tag{6.11}$$

For each $\alpha > 0$, let us consider the functional $J^{z_1}_{\triangle t,\alpha}$, given by

$$J^{z_1}_{\triangle t,\alpha}(\varphi^K) = \frac{1}{2}\sum_{k=0}^{K-1}\int_\omega |\varphi^k|^2 dx + \alpha \left\|\varphi^K\right\|_{L^2(\Omega)} - \int_\Omega \varphi^K z_1 dx \tag{6.12}$$

for all $\varphi^K \in \mathcal{C}_s$. Let $\hat{\varphi}^K_\alpha$ be the unique minimizer of $J^{z_1}_{\triangle t,\alpha}$ in $\mathcal{C}_s$. Then $\{u^k_\alpha = \hat{\varphi}^k_\alpha\}_{k=0,\cdots,K-1}$, where $\{\hat{\varphi}^k_\alpha\}_{k=0,\cdots,K-1}$ is the solution of (1.3) with $\varphi^K = \hat{\varphi}^K_\alpha$, is such that the associate solution y^K_α of (1.2) satisfies

$$\left\|\pi_s(y^K_\alpha - z_1)\right\|_{L^2(\Omega)} \le \alpha, \qquad \forall\ \triangle t > 0 \tag{6.13}$$

Since $J^{z_1}_{\triangle t,\alpha}$ attains its minimum at $\hat{\varphi}^K_\alpha$, we have

$$\frac{1}{2}\triangle t\sum_{k=0}^{K-1}\int_\omega |\hat{\varphi}^k_\alpha|^2 dx + \alpha\left\|\hat{\varphi}^K_\alpha\right\|_{L^2(\Omega)} \le \int_\Omega \hat{\varphi}^K_\alpha z_1 dx.$$

Assuming $\hat{\varphi}^K_\alpha = \sum_{\mu_j\le s} \hat{a}_j \Phi_j$ and using (6.11) and (6.1), we find

$$\begin{aligned}
\triangle t\sum_{k=0}^{K-1}\int_\omega |\hat{\varphi}^k_\alpha|^2 dx &\le 2\sum_{j=1}^{\min(s,m)} \hat{a}_j b_j \\
&\le 2\Big(\sum_{\mu_j\le s}(1+\mu_j\triangle t)^{-2K}\hat{a}_j^2\Big)^{1/2}\Big(\sum_{j=1}^{m}(1+\mu_j\triangle t)^{2K} b_j^2\Big)^{1/2} \\
&\le C\Big(\triangle t\sum_{k=0}^{K-1}\int_\omega |\hat{\varphi}^k_\alpha|^2 dx\Big)^{1/2}(1+\mu_m\triangle t)^K \left\|z_1\right\|_{L^2(\Omega)}
\end{aligned}$$

holds for any $\triangle t > 0$.

Hence, recalling that $T = K\triangle t$, we have

$$\begin{aligned}
\Big(\triangle t\sum_{k=0}^{K-1}\int_\omega |u^k_\alpha|^2 dx\Big)^{1/2} &= \Big(\triangle t\sum_{k=0}^{K-1}\int_\omega |\hat{\varphi}^k_\alpha|^2 dx\Big)^{1/2} \\
&\le C(1+\mu_m\triangle t)^K \left\|z_1\right\|_{L^2(\Omega)} \le Ce^{\mu_m T}\left\|z_1\right\|_{L^2(\Omega)}.
\end{aligned} \tag{6.14}$$

For fixed $\triangle t$, taking (for instance) $\alpha = 1/n$ for each $n \ge 1$ and letting $n \to \infty$, we can obtain a bounded sequence of controls $\{u^k_n = \hat{\varphi}^k 1_\omega\}_{k=0,\cdots,K-1}$ such that the corresponding states y^K_n satisfy

$$\left\|\pi_s(y^K_n - z_1)\right\|_{L^2(\Omega)} \le 1/n. \tag{6.15}$$

Let $\{u^k\}_{k=0,\cdots,K-1}$ be the weak limit in $\mathcal{C}_s$ of a subsequence of $\{u_n^k\}_{k=0,\cdots,K-1}$. The corresponding solution of (1.2) is such that $\pi_s y^K = \pi_s z_1$. Furthermore the functionl U^K, defined in (1.12) is bounded in $L^2((0,T)\times\omega)$ as in (6.14). Thus we have proved that for any $\triangle t$, there exists a control U^K, such that the projection of the solution of (1.2) can be controlled exactly with cost

$$\left\|U^K\right\|_{L^2((0,T)\times\omega)} \leq Ce^{\mu_m T}\left\|z_1\right\|_{L^2(\Omega)}. \tag{6.16}$$

Now assume that $y_1 \in H^2(\Omega)\cap H_0^1(\Omega)$ and $\varepsilon > 0$ are given. Let us put

$$y_1 = \sum_{j\geq 1} b_j \Phi_j, \qquad \text{with} \quad \sum_{j\geq 1} \mu_j^2 b_j^2 < \infty.$$

Let us introduce

$$y_{1,\varepsilon} = \sum_{j=1}^{m(\varepsilon)} b_j \Phi_j, \tag{6.17}$$

where $m(\varepsilon)$ is such that $\sum_{j\geq m(\varepsilon)+1} b_j^2 \leq \varepsilon^2$. It is clear that

$$\left\|y_1 - y_{1,\varepsilon}\right\|_{L^2(\Omega)} \leq \varepsilon.$$

From (6.16), written for $z_1 = y_{1,\varepsilon}$ as the form of (6.17), we obtain that there exists a control U^K such that

$$\left\|\pi_s(y^K - y_1)\right\|_{L^2(\Omega)} \leq \left\|\pi_s(y^K - y_{1,\varepsilon})\right\|_{L^2(\Omega)} + \left\|\pi_s(y_{1,\varepsilon} - y_1)\right\|_{L^2(\Omega)} \leq \varepsilon$$

and, moreover, we have the following estimate

$$\left\|U^K\right\|_{L^2((0,T)\times\omega)} \leq Ce^{\mu_{m(\varepsilon)}T}\left\|y_{1,\varepsilon}\right\|_{L^2(\Omega)} \leq Ce^{\mu_{m(\varepsilon)}T}\left\|y_1\right\|_{L^2(\Omega)} \tag{6.18}$$

for any $\triangle t > 0$.

Notice that (6.18) must hold whenever $m(\varepsilon)$ is such that $\sum_{j\geq m(\varepsilon)+1} b_j^2 \leq \varepsilon^2$. We are now going to make a particular choice of $m(\varepsilon)$ which leads to (1.13).

First, we claim that the unique case of interest is when

$$\frac{\left\|\Delta y_1\right\|_{L^2(\Omega)}}{\mu_1} > \varepsilon \tag{6.19}$$

(recall that μ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$). Otherwise $\left\|y_1\right\|_{L^2(\Omega)} \leq \varepsilon$ and the control $\{u^k = 0\}_{k=0,\cdots,K-1}$ is such that the solution of (1.2) with $y^0 = 0$ satisfies

$$\left\|\pi_s(y^K - y_1)\right\|_{L^2(\Omega)} = \left\|\pi_s y_1\right\|_{L^2(\Omega)} \leq \varepsilon.$$

Thus, the control can be zero when (6.19) is violated.

Let $m(\varepsilon)$ be the first integer m satisfying

$$\frac{\left\|\Delta y_1\right\|_{L^2(\Omega)}}{\mu_{m+1}} \leq \varepsilon$$

Because of (6.19), this is well defined. For this choice of $m(\varepsilon)$, we have

$$\sum_{j\geq m(\varepsilon)+1} b_j^2 \leq \frac{1}{\mu_{m(\varepsilon)+1}^2} \sum_{j\geq 1} \mu_j^2 b_j^2 \leq \frac{\|\Delta y_1\|_{L^2(\Omega)}^2}{\mu_{m(\varepsilon)+1}^2}$$

and, consequently, (6.18) has to be satisfied. We also have

$$\mu_{m(\varepsilon)} \leq \frac{\|\Delta y_1\|_{L^2(\Omega)}}{\mu_{m(\varepsilon)+1}}. \tag{6.20}$$

From (6.18) and (6.20), we obtain (1.13) for any $\triangle t > 0$. This finishe the proof of Theorem 1.4. $\square$

Remark 6.1 *Note that the filtering parameter $s = \Lambda(\triangle t)^{-r}$ tends to infinity as $\triangle t$ tends to zero. Consequently $\mathcal{C}_s$ tends to cover the whole space $L^2(\Omega)$. Hence, it is easy to derive the approximate controllability of the time continuous system (1.1), as a limit of the discrete system.*

7 Other discrete schemes

In this section, we consider the null controllability of another two time discrete schemes. First we address the null controllability of the Euler Explicit schemes, then we discuss the null controllability of the θ-method schemes.

7.1 Euler Explicit schemes

We state the Euler explicit time discretization of the system (1.1) as follows:

$$\begin{cases} \dfrac{y^{k+1}-y^k}{\triangle t} - \Delta y^k = u^k 1_\omega, & x\in\Omega,\ \ k=0,1,\cdots,K-1 \\ y^k = 0, & x\in\partial\Omega,\ k=1,\cdots,K \\ y^0 \in L^2(\Omega) \text{ given.} \end{cases} \tag{7.1}$$

Using the similar method as in the proof of Theorem 1.1, we claim that system (7.1) is not null controllable, even not approximate controllable, except for the trivial case $\Omega = \omega$:

Theorem 7.1 *Let $\Omega\backslash\overline{\omega} \neq \emptyset$. Then system (7.1) is neither null controllable nor approximate controllable for any given $\triangle t > 0$.*

Proof: The adjoint system of (7.1) is:

$$\begin{cases} -\dfrac{\varphi^{k+1}-\varphi^k}{\triangle t} - \Delta\varphi^{k+1} = 0, & x\in\Omega,\ \ k=0,1,\cdots,K-1 \\ \varphi^k = 0, & x\in\partial\Omega,\ \ k=0,1,\cdots,K \\ \varphi^K(x)\in L^2(\Omega)\ \text{ given,} & x\in\Omega. \end{cases} \tag{7.2}$$

We use the contradiction argument. If (7.1) is null controllable, then for any $y^0 \in L^2(\Omega)$ we can find a control $\{u^k\}_{0\le k\le K-1}$ such that the solution $\{y^k\}_{0\le k\le K}$ of system (7.1) vanishes at $k=K$. Multiplying the first equation in (7.1) by the solution φ^k of (7.2) and summing up in k, using integration by parts, we get

$$0=\int_\Omega y^K\varphi^K dx=\triangle t\sum_{k=0}^{K-1}\int_\omega u^k\varphi^k dx+\int_\omega y^0\varphi^0 dx.$$

Hence

$$\int_\Omega y^0\varphi^0 dx=-\triangle t\sum_{k=0}^{K-1}\int_\omega u^k\varphi^k dx,\qquad \forall\ y^0\in L^2(\Omega). \tag{7.3}$$

Since $\Omega\setminus\overline{\omega}\neq\emptyset$, there exists a point x_0 in $\Omega\setminus\overline{\omega}$ and consequently one can find a ball

$$B(x_0,A)=\{x:|x-x_0|<A\}\subset\Omega\setminus\overline{\omega},$$

with some positive constant A. We choose a non-trivial function $\psi\in C_0^\infty(B)$ and let $\varphi^K=\psi$. From system (7.2) we compute

$$\varphi^k=\varphi^{k+1}+\triangle t\Delta\varphi^{k+1},\qquad k=0,\cdots,K-1. \tag{7.4}$$

By induction we have

$$\varphi^0=(I+\triangle t\Delta)^K\varphi^K.$$

Moreover, taking into account that $\varphi^k\in C_0^\infty(B)$ for any $k\ge 0$ and $\bar{B}\cap\omega=\emptyset$, we find that the right side of (7.3) vanishes, i.e.

$$-\triangle t\sum_{k=0}^{K-1}\int_\omega u^k\varphi^k dx=0. \tag{7.5}$$

Hence, by taking $y^0=\varphi^0$ in (7.3), we conclude that $\varphi^0\equiv 0$. Consequently, we have $\varphi^K\equiv 0$ if $\varphi^0\equiv 0$ in Ω, which is a contradiction.

On the other hand, it is easy to prove that the unique continuation of the system (7.2) fails. Due to the fact of the equivalence between approximate controllability of control system and unique continuation of its adjoint system, system (7.1) is not approximately controllable. □

Let $y^0=\sum_{j\ge1}a_j^0\Phi_j$, $u^k=\sum_{j\ge1}b_j^k\Phi_j$. The solution of the system (7.1) is given by:

$$y^{k+1}(x)=\sum_{j\ge1}a_j^{k+1}\Phi_j(x);\qquad a_j^{k+1}=a_j^0(1-\mu_j\triangle t)^{k+1}+1_\omega\triangle t\sum_{s=0}^{k}(1-\mu_j\triangle t)^s b_j^{k-s},$$

for any $k=0,1,\cdots,K-1$.

To guarantee the stability of the scheme, we need the restriction of the eigenvalues as $\mu_j\le(\triangle t)^{-1}$. Under this new restriction, we redo the L-R time iteration as the same process as in the proof of Theorem 1.2, and attain a similar result. The only difference is that the range of parameter r is replaced by $(0,1)$, instead of the range $(0,2)$ in the Euler Implicit case.

Proposition 7.1 *Let $\{y^k\}_{k=0,\cdots,K}$ be the solution of system (7.1). Then Theorem 1.2 is true, by replacing $s=\Lambda(\triangle t)^{-r}$ with $r\in(0,1)$.*

Proof: The proof is an analogue of Section 5, under an extra restriction $\mu_j \le (\triangle t)^{-1}$. $\square$

7.2 The θ-method

Given $\theta\in(0,1)$, we discretize system (1.1) with the θ-method as follows:

$$\begin{cases} \dfrac{y^{k+1}-y^k}{\triangle t} - \Delta\Big(\theta y^k + (1-\theta)y^{k+1}\Big) = u^k 1_\omega, & x\in\Omega,\ \ k=0,1,\cdots,K-1 \\ y^k = 0, & x\in\partial\Omega,\ k=1,\cdots,K \\ y^0\in L^2(\Omega) \text{ given.} & \end{cases} \tag{7.6}$$

The corresponding adjoint system reads:

$$\begin{cases} -\dfrac{\varphi^{k+1}-\varphi^k}{\triangle t} - \Delta\Big(\theta\varphi^{k+1} + (1-\theta)\varphi^{k}\Big) = 0, & x\in\Omega,\ \ k=0,1,\cdots,K-1 \\ \varphi^k = 0, & x\in\partial\Omega,\ \ k=0,1,\cdots,K \\ \varphi^K(x)\in L^2(\Omega)\ \text{ given,} & x\in\Omega. \end{cases} \tag{7.7}$$

The solution of the system (7.7) is given by:

$$\varphi^k(x) = \sum_{j\ge1} a_j\Big(\frac{1-\theta\mu_j\triangle t}{1+(1-\theta)\mu_j\triangle t}\Big)^{K-k}\Phi_j(x), \qquad \forall x\in\Omega. \tag{7.8}$$

We have the following lemma:

Lemma 7.1 *Let T and $C>0$ be two positive constants and $0<\triangle t<\min(\frac{\theta T^2}{4C^2},\frac{2T}{\theta^2 C})$. Then function*

$$f(\sigma) = e^{C\sqrt{\sigma}}\Big(\frac{1-\theta\sigma\triangle t}{1+(1-\theta)\sigma\triangle t}\Big)^{-[\frac{T}{\triangle t}]} \tag{7.9}$$

has the following properties:

i). $f(\sigma)$ is decreasing in the interval $\Big((\frac{C}{T})^2, \frac{1}{\theta\triangle t}\Big)$.

ii). $f(\sigma) < e^{-\delta\sqrt{\sigma}}$ in the interval $\Big((\frac{2C+\delta}{T-\triangle t})^2, \frac{1}{\theta\triangle t}\Big)$ for any $0\le\delta\le C$.

iii). It holds

$$\lim_{\triangle t\to 0} f(\sigma)\Big|_{\sigma=\frac{1}{\theta\triangle t}} = 0. \tag{7.10}$$

Proof: Replacing $\sqrt{\sigma}$ by x and setting $f(x^2) = g(x)$, we have

$$g(x) = \exp\Big(Cx - [\frac{T}{\triangle t}]\ln\Big(\frac{1+(1-\theta)\triangle tx^2}{1-\theta\triangle tx^2}\Big)\Big).$$

The derivative of $g(x)$ with respect to x reads

$$g'(x) = g(x)\Big(C - [\frac{T}{\triangle t}]\frac{2\triangle t\, x}{(1+(1-\theta)\triangle tx^2)(1-\theta\triangle tx^2)}\Big) \triangleq g(x)\rho(x).$$

To prove the result we need to justify that $\rho(x)$ is negative in the interval I. It is necessary that $x^2 \leq \frac{1}{\theta\triangle t}$ if $\rho(x)$ is negative. Furthermore, we compute

$$\rho(x) = C - [\frac{T}{\triangle t}]\frac{2\triangle t\, x}{(1+(1-\theta)\triangle tx^2)(1-\theta\triangle tx^2)} < C - \frac{2Tx}{1+\triangle tx^2}.$$

Obviously, by Lemma 2.1, $\rho(x)$ is negative in the interval $x \in (\frac{C}{T}, \frac{T}{2C\triangle t})$. Hence $\rho(x)$ is negative in the interval $I = (\frac{C}{T}, \zeta)$ with $\zeta = \min(\frac{1}{\sqrt{\theta\triangle t}}, \frac{T}{2C\triangle t})$. Taking into account that $\triangle t < \frac{\theta T^2}{4C^2}$, we know that $\zeta = \frac{1}{\sqrt{\theta\triangle t}}$ and consequently, $f(\sigma)$ is decreasing in the interval $\Big((\frac{C}{T})^2, \frac{1}{\theta\triangle t}\Big)$.

Now we consider the function $H(x) = f(x^2)e^{\delta\, x}$. $H(x) < 1$ if and only if

$$\ln H(x) = C\, x - [\frac{T}{\triangle t}]\ln\Big(\frac{1+(1-\theta)\triangle tx^2}{1-\theta\triangle tx^2}\Big) + \delta\, x < 0.$$

By the Taylor expansion, $\ln H(x)$ can be rewritten as:

$$\ln H(x) = (C+\varepsilon)x - \frac{[\frac{T}{\triangle t}]\triangle t\, x^2}{1-\theta\triangle tx^2} + \frac{[\frac{T}{\triangle t}](\triangle t)^2x^4}{2(1-\theta\triangle tx^2)^2} - \frac{[\frac{T}{\triangle t}](\triangle t)^3\, x^6}{3(1-\theta\triangle tx^2)^3\Big(1+\xi\frac{\triangle t\, x^2}{1-\theta\triangle tx^2}\Big)^3}$$

for some $\xi \in (0, \frac{\triangle t\, x^2}{1-\theta\triangle tx^2})$. Since $x > 0$, we have $\ln H(x) < 0$ if

$$C + \varepsilon - \frac{(T-\triangle t)\, x}{1-\theta\triangle tx^2} + \frac{T\triangle tx^3}{2(1-\theta\triangle tx^2)^2} < 0. \tag{7.11}$$

We claim that (7.11) is satisfied when

$$x \in \Big(\frac{2C+\varepsilon}{T-\triangle t}, \Big(\frac{2C}{4C\theta\triangle t + T\triangle t}\Big)^{1/3}\Big).$$

This due to the fact that

$$\begin{aligned} C + \varepsilon - \frac{(T-\triangle t)\, x}{1-\theta\triangle tx^2} + \frac{T\triangle tx^3}{2(1-\theta\triangle tx^2)^2} &< -C + \frac{T\triangle tx^3}{2(1-\theta\triangle tx^2)^2} \\ < \frac{4C\theta\triangle t\, x^2 - 2C + T\triangle t\, x^3}{2(1-\theta\triangle tx^2)^2} &< \frac{4C\theta\triangle t\, x^3 - 2C + T\triangle t\, x^3}{2(1-\theta\triangle tx^2)^2} < 0. \end{aligned} \tag{7.12}$$

In (7.12) we use $x > \frac{2C+\varepsilon}{T-\triangle t}$ in the first inequality and $x^3 < \frac{2C}{4C\theta\triangle t + T\triangle t}$ in the last one. Similarly, it is easy to show that $H(x)$ is decreasing for $x \in (\frac{C+\delta}{T}, \frac{1}{\sqrt{\theta\triangle t}})$. Hence $H(x) < 1$ when x satisfies

$$\frac{2C+\varepsilon}{T-\triangle t} < x < \max\Big(\frac{1}{\sqrt{\theta\triangle t}}, \Big(\frac{2C}{4C\theta\triangle t + T\triangle t}\Big)^{1/3}\Big) \tag{7.13}$$

Moreover, taking into account that

$$\frac{1}{\sqrt{\theta\triangle t}} > \Big(\frac{2C}{4C\theta\triangle t + T\triangle t}\Big)^{1/3}$$

for $\triangle t < \frac{2T}{\theta^2 C}$ we finish the proof of ii).

Next, by

$$\lim_{\triangle t\to 0} f(\sigma)\Big|_{\sigma=\zeta} = \lim_{\triangle t\to 0} f(\zeta) = f(\frac{1}{\theta\triangle t}) = 0$$

we get (7.10). □

With Lemma 7.1, we obtain the similar conclusion as in Theorem 1.2:

Proposition 7.2 *Let $\{y^k\}_{k=0,\cdots,K}$ be the solution of system (7.6). Then Theorem 1.2 is true, by replacing $s = \Lambda(\triangle t)^{-r}$ with $r \in (0,1)$ and $\Lambda = \Lambda(r,\theta,T,\Omega,\omega) > 0$.*

Proof: The proof is an analogue of Section 5. The only difference is that $(\triangle t)^2$ is replaced by $\triangle t$ in the inequality (4.23). Consequently, we have $r \in (0,1)$. □

8 Time discrete fractional order parabolic equation

In this section, we discuss the controllability of the time semi-discrete fractional order parabolic equation with $\alpha > 1/2$:

$$\begin{cases} \dfrac{y^{k+1}-y^k}{\triangle t} + (-\Delta)^\alpha y^{k+1} = 1_\omega u^k, & k = 0,\cdots,K-1, \quad x \in \Omega \\ y^k = 0, & k = 0,\cdots,K-1, \quad x \in \partial\Omega \\ y^0 \in L^2(\Omega). & \end{cases} \tag{8.1}$$

Equation (8.1) is the time Euler Implicit semi-discretization of the continuous controlled fractional order parabolic equation

$$\begin{cases} y_t + (-\Delta)^\alpha y = 1_\omega u, & t \in (0,T), \quad x \in \Omega \\ y^k = 0, & t \in (0,T), \quad x \in \partial\Omega \\ y(0,x) = y^0 \in L^2(\Omega), & x \in \Omega. \end{cases} \tag{8.2}$$

The controllability of equation (8.2) has been solved in [1] for the case $\alpha > 1/2$ in one space-dimension. Moreover, Micu and Zuazua (see in [19]) proved that $\alpha = 1/2$ is sharp, i.e. equation (8.2) is not controllable for $\alpha \leq 1/2$.

Therefore, in this section, we are interested in the case $\alpha > 1/2$. We will prove that, the projection of the solution of equation (8.1) is null controllable (uniformly) in any given $\mathcal{C}_s$(recall the definition of $\mathcal{C}_s$ in (1.6)):

Theorem 8.1 *Let $\alpha > 1/2$. For any fixed $T > 0$ and $r \in (0,2)$, there exists a positive constant $\Lambda = \Lambda(r,\alpha,T,\Omega,\omega)$ such that for all $y^0 \in L^2(\Omega)$, there exists a control $\{u^k \in L^2(\omega)\}_{k=0,\cdots,K-1}$, so that*

(1) *The solution of system (8.1) satisfies*

$$\pi_{\Lambda(\triangle t)^{-r}} y^K(x) = 0, \qquad \forall\ x \in \Omega; \tag{8.3}$$

(2) *There exists a constant $C = C(r,\alpha,T,\Omega,\omega) > 0$, independent of $\triangle t$, such that*

$$\triangle t \sum_{k=0}^{K-1} \int_\omega |u^k|^2 dx \le C \int_\Omega |y^0|^2 dx \tag{8.4}$$

holds for any $\triangle t > 0$ and $y^0 \in L^2(\Omega)$;

(3) *The control $\{u^k\}_{k=0\le k\le K-1}$ of system (8.1) may be built such that*

$$U^K(\cdot,x) \triangleq \sum_{k=0}^{K-1} u^k(x) 1_{[t_k,t_{k+1})}(\cdot) \longrightarrow u(\cdot,x) \ \ \text{strongly in} \ \ L^2((0,T)\times\omega) \ \ \text{as} \ \triangle t \to 0,$$

where u is a null control of the corresponding continuous-time heat equation (8.2).

Remark 8.1 *Theorem 8.1 is the same as the Theorem 1.2, except for that the filtering constant Λ is replaced by*

$$\Lambda = \Big(\frac{\alpha T}{8D}\Big)^r.$$

Hence, we can control more frequencies when α increases.

Remark 8.2 *Note that when $\alpha = 1/2$, the function (8.5) is increasing for all $\sigma > 0$. It means that the increase of the control never could be compensated by the decay of the solution, the control of L-R method no longer converges. In fact, with the same method in Theorem 1.1, it is easy to show that, system (8.1) is not null controllable when $\alpha = 1/2$.*

To prove Theorem 8.1, we need the following Lemma, which is the analogue of Lemma 2.1:

Lemma 8.1 *Let $\alpha > 1/2$. Let T and $C > 0$ be two positive constants and $\triangle t$ be sufficiently small. Then function*

$$f(\sigma) = e^{C\sqrt{\sigma}}(1+\sigma^\alpha \triangle t)^{-[\frac{T}{\triangle t}]} \tag{8.5}$$

has the following properties:

i). $f(\sigma)$ is decreasing in the interval $\Big((\frac{C}{\alpha T})^{2/(2\alpha-1)}, (\frac{\alpha T}{C\triangle t})^2\Big)$.

ii). $f(\sigma) < e^{-\delta\sqrt{\sigma}}$ in the interval $\Big((\frac{2C+\delta}{T})^{2/(2\alpha-1)}, (\frac{\alpha T}{(C+\delta)\triangle t})^2\Big)$ for any $\delta > 0$.

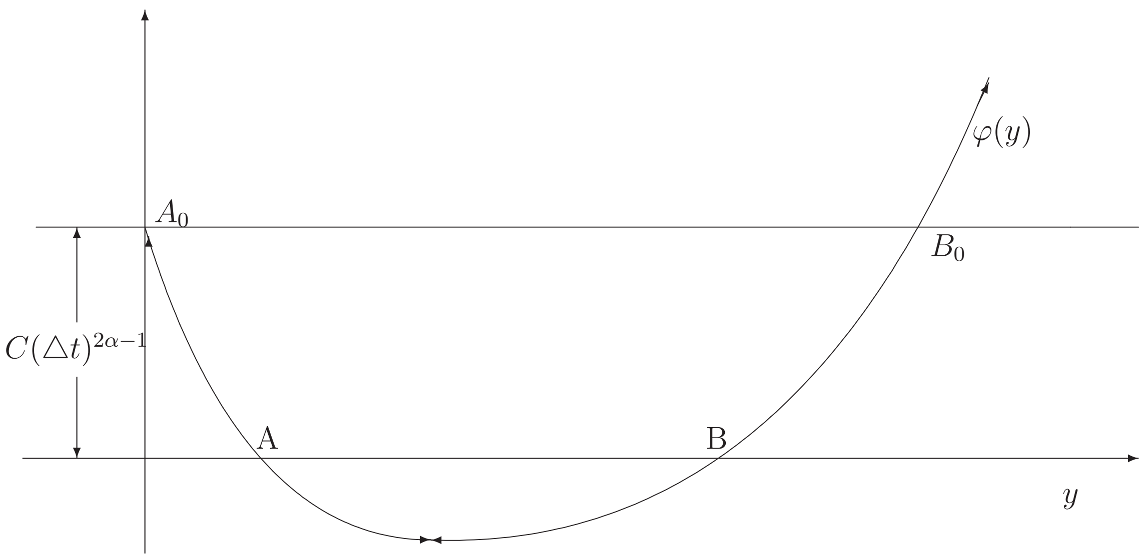

Figure 1: Figure of $\varphi(y)$

iii). It holds

$$\lim_{\triangle t\to 0} f(\sigma)\Big|_{\sigma=(\frac{\alpha T}{C\triangle t})^2} = 0. \tag{8.6}$$

Proof: Replacing $\sqrt{\sigma}$ by x and setting $f(x^2) = g(x)$, we have

$$g(x) = \exp\Big(Cx - [\frac{T}{\triangle t}]\ln(1 + \triangle t\, x^{2\alpha})\Big).$$

The derivative of $g(x)$ with respect to x reads

$$g'(x) = g(x)\Big(C - [\frac{T}{\triangle t}]\frac{2\alpha\triangle t x^{2\alpha-1}}{1+\triangle t x^{2\alpha}}\Big).$$

Here $g(x)$ is a decreasing function if and only if $g'(x)$ is negative. To find the decreasing part of $f(\sigma)$ we need to compute the interval in which $g'(x)$ has negative value. Since $g(x) > 0$, it is sufficient that

$$C\triangle t x^{2\alpha} - 2\alpha T x^{2\alpha-1} + C < 0. \tag{8.7}$$

Denote $\triangle tx$ by y. It is obvious from (8.7) that $y = \triangle tx$ should satisfy

$$\varphi(y) \stackrel{\triangle}{=} y^{2\alpha} - 2\alpha T y^{2\alpha-1} + C(\triangle t)^{2\alpha-1} < 0.$$

Figure 1 is the graph of $\varphi(y)$, with $A_0 = (0, C(\triangle t)^{2\alpha-1})$, $B_0 = (\frac{2\alpha T}{C}, C(\triangle t)^{2\alpha-1})$. We assume that $A = (\delta_0, 0)$ and $B = (\delta_1, 0)$ be the two points of intersection of $\varphi(y)$ and the y-axis.

For $\triangle t$ sufficiently small, we compute A and B, respectively.

- 1). Point A. Obviously, δ_0 tends to 0 as $\triangle t$ tends to 0. Hence, by

$$\varphi(\delta_0) = C\delta_0^{2\alpha} - 2\alpha T\delta_0^{2\alpha-1} + C(\triangle t)^{2\alpha-1} = 0,$$

we have

$$\delta_0 = \Big(\frac{C}{2\alpha T}\Big)^{1/(2\alpha-1)}\triangle t + o(\triangle t). \tag{8.8}$$

- 2). Point B. Obviously δ_1 tends to $\frac{2\alpha T}{C}$ as $\triangle t$ tends to 0. Let $\delta_1 = \frac{2\alpha T}{C} + \delta$ and we compute

$$\begin{aligned}
\varphi(\delta_1) = C(\frac{2\alpha T}{C} + \delta)^{2\alpha} - 2\alpha T(\frac{2\alpha T}{C} + \delta)^{2\alpha-1} + C(\triangle t)^{2\alpha-1} = 0,\\
\Leftrightarrow \qquad 0 = \Big(\frac{2\alpha T}{C}\Big)^{2\alpha-1}\delta + o(\delta) + (\triangle t)^{2\alpha-1},\\
\Leftrightarrow \qquad \delta = -\Big(\frac{C\triangle t}{2\alpha T}\Big)^{2\alpha-1} + o\Big((\triangle t)^{2\alpha-1}\Big).
\end{aligned}$$

Hence, we arrive at

$$\delta_1 = \frac{2\alpha T}{C} - \Big(\frac{C\triangle t}{2\alpha T}\Big)^{2\alpha-1} + o\Big((\triangle t)^{2\alpha-1}\Big). \tag{8.9}$$

Let (x_0, x_1) be the interval in which $g'(x)$ has negative value. Taking into account that $y = \triangle t x$, we have

$$\begin{aligned}
x_0 &= \Big(\frac{C}{2\alpha T}\Big)^{1/(2\alpha-1)} + O(\triangle t) < (\frac{C}{\alpha T})^{1/(2\alpha-1)},\\
x_1 &= \frac{2\alpha T}{C\triangle t} - \Big(\frac{C\triangle t}{2\alpha T}\Big)^{2\alpha-2} + o\Big((\triangle t)^{2\alpha-2}\Big) > \frac{\alpha T}{C\triangle t}.
\end{aligned}$$

Hence we conclude that This finishes the proof of i) by replacing x as $\sqrt{\sigma}$.

Now we consider the function $H(x) = f(x^2)e^{\delta\sqrt{\sigma}}$. $H(x) < 1$ if and only if

$$\ln H(x) = C\,x - [\frac{T}{\triangle t}]\ln(1 + \triangle t\,x^{2\alpha}) + \delta\,x < 0.$$

By the Taylor expansion, $\ln H(x)$ can be rewritten as:

$$\ln H(x) = (C+\delta)\,x - [\frac{T}{\triangle t}]\triangle t\,x^{2\alpha} + \frac{[\frac{T}{\triangle t}](\triangle t)^2}{2}x^{4\alpha} - \frac{[\frac{T}{\triangle t}](\triangle t)^3 x^{6\alpha}}{3(1+\xi\triangle t\,x^{2\alpha})^3},$$

for some$\xi \in (0, \triangle t\,x^{2\alpha})$. Since $x > 0$, we deduce that $\ln H(x) < 0$ if

$$C + \delta - (T - \triangle t)\,x^{2\alpha-1} + \frac{T\triangle t}{2}x^{4\alpha-1} < 0. \tag{8.10}$$

We claim that (8.10) is satisfied when $x \in \Big((\frac{2C+\delta}{T-\triangle t})^{1/(2\alpha-1)}, (\frac{2C}{T\triangle t})^{1/(4\alpha-1)} \Big)$. This is due to the fact that

$$C + \varepsilon - (T - \triangle t)\, x^{2\alpha-1} + \frac{T\triangle t}{2} x^{4\alpha-1} < -C + \frac{T\triangle t}{2} x^{4\alpha-1} < 0.$$

Similarly, it is easy to show that $H(x)$ is decreasing for $x \in \Big((\frac{C+\delta}{T})^{1/(2\alpha-1)}, \frac{\alpha T}{(C+\delta)\triangle t} \Big)$. Hence, $H(x) < 1$ when x satisfies

$$(\frac{2C+\delta}{T})^{1/(2\alpha-1)} < x < \max \Big((\frac{2C}{T\triangle t})^{1/(4\alpha-1)}, \frac{\alpha T}{(C+\delta)\triangle t} \Big). \tag{8.11}$$

Moreover, since $\triangle t$ is sufficiently small, we get that the right side of (8.11) equals to $\frac{\alpha T}{(C+\delta)\triangle t}$ and consequently $H(x) < 1$ in the interval

$$(\frac{2C+\delta}{T})^{1/(2\alpha-1)} < x < \frac{\alpha T}{(C+\delta)\triangle t}.$$

Hence, we finish to prove *ii*) by replacing x as $\sqrt{\sigma}$.

Next, by

$$\lim_{\triangle t \to 0} f(\sigma)\Big|_{\sigma = (\frac{\alpha T}{C\triangle t})^2} = \lim_{\triangle t \to 0} \exp \Big(\frac{\alpha T - [\frac{T}{\triangle t}]\triangle t \ln(1 + \frac{\alpha^2 T^2}{C^2 \triangle t})}{\triangle t} \Big) = 0, \tag{8.12}$$

we get (8.6). $\square$

Now we prove Theorem 8.1.

Sketch of the Proof of Theorem 8.1: The proof is similar to that in section 5. The only difference is that the function (4.19) is replaced by (8.5) and the corresponding Lemma is replaced by Lemma 8.1. $\square$

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