Exact Controllability of the Time Discrete Wave Equation: a Multiplier Approach

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Abstract. In this paper we summarize recent results on the exact boundary controllability of a trapezoidal time discrete wave equation in a bounded domain. It is shown that the projection of the solution in an appropriate space in which the high frequencies have been filtered is exactly controllable with uniformly bounded controls (with respect to the time-step). By classical duality arguments, the problem is reduced to a boundary observability inequality for a time-discrete wave equation. Using multiplier techniques the uniform observability property is proved in a class of filtered initial data. The optimality of the order of filtering parameter is also established, although a careful analysis of the expected velocity of propagation of time discrete waves indicates that its actual value could be improved.

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1. Introduction

Let Ω be an open bounded domain in $\mathbb{R}^d$ ($d \in \mathbb{N}^*$) with C^2 boundary Γ. Let $T > 0$ be a given time duration. We consider the following wave equation with a controller acting on the nonempty subset Γ_0 of the boundary $\Gamma = \partial\Omega$:

$$\begin{cases} y'' - \Delta_x y = 0 & \text{in} \quad (0,T)\times\Omega \\ y = u1_{\Gamma_0} & \text{on} \quad (0,T)\times\Gamma \\ y(0) = y_0, \quad y'(0) = y_1 & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

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Here and henceforth, $'$ denotes the partial derivative with respect to time t, 1_{Γ_0} is the characteristic function of the set Γ_0, and Δ_x the Laplacian in the space variable $x \in \Omega$.

This paper is devoted to analyze whether the known controllability result for (1.1) can be recovered as a consequence of similar result for the time-discrete versions. This kind of problems has been the object of intensive research in the past few years but mainly in the context of space semi-discretizations. In the present paper we summarize the main results by the authors [15] in the time discrete case.

The exact controllability of (1.1) requires that the subset Γ_0 of the boundary fulfills some geometric conditions. It holds in particular for those subsets that are obtained through the multiplier method. More precisely, fix some $x_0 \in \mathbb{R}^d$, and put

$$\begin{cases} R \stackrel{\triangle}{=} \max_{x\in\Omega}|x - x_0|, \\ \\ \Gamma_0 \stackrel{\triangle}{=} \{x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0\}, \end{cases} \tag{1.2}$$

where $\nu(x)$ is the unit outward normal vector of Ω at $x \in \Gamma$. For these subsets Γ_0 the exact controllability property of (1.1) holds provided $T > 2R$.

To be more precise, the following exact controllability result for (1.1) is well known (see [8]): *For any* $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *there exists a control* $u \in L^2((0,T) \times \Gamma_0)$ *such that the solution* $y = y(t,x)$ *of* (1.1)*, defined by the classical transposition method* ([8])*, satisfies:*

$$y(T) = y'(T) = 0 \qquad \text{in } \Omega. \tag{1.3}$$

By classical duality arguments ([8]), the above controllability property is equivalent to a (boundary) observability of the following uncontrolled wave equation:

$$\begin{cases} \varphi'' - \Delta_x \varphi = 0, & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \Gamma \\ \varphi(T) = \varphi_0, \quad \varphi'(T) = \varphi_1, & \text{in } \Omega, \end{cases} \tag{1.4}$$

i.e., to the fact that *solutions of* (1.4) *satisfy*

$$E(0) \le C \int_0^T \int_{\Gamma_0} \Big|\frac{\partial \varphi}{\partial \nu}\Big|^2 d\Gamma_0 dt, \qquad \forall\ (\varphi_0, \varphi_1) \in H_0^1(\Omega) \times L^2(\Omega). \tag{1.5}$$

Here and thereafter, we will use C to denote a generic positive constant (depending only on T, Ω and Γ_0) which may vary from line to line. On the other hand, $E(0)$ stands for the energy $E(t)$ of (1.4) at $t = 0$, with

$$E(t) = \frac{1}{2} \int_\Omega \Big[|\varphi_t(t,x)|^2 + |\nabla \varphi(t,x)|^2\Big] dx, \tag{1.6}$$

which remains to be constant, i.e.

$$E(t) = E(0), \qquad \forall\, t \in [0,T].$$

Inequality (1.5) can be proved by several methods including multiplier techniques ([8]), microlocal analysis ([1]) and Carleman inequalities ([14]). In the particular case of subset Γ_0 as above and $T > 2R$, inequanlity (1.5) can be provided easily by the method of multipliers ([8]) that in the present paper we adapt to time-discrete equations.

Note however that the subsets Γ_0 of the boundary and the times of control obtained on this way are not optimal. The obtention of optimal control subsets and times requires the use of methods of geometric optics (*see* [1]).

In this paper, we analyze the time semi-discretization schemes for systems (1.1) and (1.4). We are thus replacing the continuous dynamics (1.1) and (1.4) by time-discrete ones and analyze their controllability/observability properties. Here we take the point of view of numerical analysis and, therefore, we analyze the limit behavior as the time-step tends to zero.

More precisely, we set the time step h by $h = T/K$, where $K > 1$ is a given integer. Denote by y^k and u^k respectively the approximations of the solution y and the control u of (1.1) at time $t_k = kh$ for any $k = 0, \cdots, K$. We then introduce the following trapezoidal time semi-discretization of (1.1):

$$\begin{cases} \dfrac{y^{k+1} + y^{k-1} - 2y^k}{h^2} - \Delta_x \Big(\dfrac{y^{k+1} + y^{k-1}}{2} \Big) = 0, \\ \qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \quad k = 1, \cdots, K-1 \\ y^k = u^k 1_{\Gamma_0}, \qquad\qquad\qquad \text{on } \Gamma, \ \ k = 0, \cdots, K \\ y^0 = y_0, \ \ y^1 = y_0 + hy_1, \qquad \text{in } \Omega. \end{cases} \tag{1.7}$$

Here $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ are the data given in system (1.1) that allow determining the initial data for the time-discrete system too.

We will establish the well-posedness of system (1.7) by means of the transposition method which is the first result of [15], as we shall see in Theorem 2.6 below.

The controllability problem for system (1.7) may be formulated as follows: *For any* $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$, *to find a control* $\{u^k \in L^2(\Gamma_0)\}_{k=1,\cdots,K-1}$ *such that the solution* $\{y^k\}_{k=0,\cdots,K}$ *of* (1.7) *satisfies:*

$$y^{K-1} = y^K = 0 \qquad \text{in } \Omega. \tag{1.8}$$

Note that (1.8) is equivalent to the condition $y^{K-1} = (y^K - y^{K-1})/h = 0$ that is a natural discrete version of (1.3).

As in the context of the above continuous wave equation, we also consider the uncontrolled system

$$\begin{cases} \dfrac{\varphi^{k+1} + \varphi^{k-1} - 2\varphi^k}{h^2} - \Delta_x \left(\dfrac{\varphi^{k+1} + \varphi^{k-1}}{2} \right) = 0, \\ \qquad\qquad\qquad\qquad\qquad \text{in } \Omega, \quad k = 1, \cdots, K-1 \\ \varphi^k = 0, \qquad\qquad\qquad\qquad \text{on } \Gamma, \quad k = 0, \cdots, K \\ \varphi^K = \varphi_0^h + h\varphi_1^h, \quad \varphi^{K-1} = \varphi_0^h, \qquad \text{in } \Omega, \end{cases} \tag{1.9}$$

where $(\varphi_0^h, \varphi_1^h) \in (H_0^1(\Omega))^2$. In particular, to guarantee the convergence of the solutions of (1.9) towards those of (1.4) one considers convergent data such that

$$\begin{cases} \varphi_0^h \to \varphi_0 \text{ strongly in } H_0^1(\Omega), \\ \varphi_1^h \to \varphi_1 \text{ strongly in } L^2(\Omega). \end{cases} \quad \text{as } K \to \infty \text{ (or } h \to 0), \tag{1.10}$$

with $h\varphi_1^h$ being bounded in $H_0^1(\Omega)$. Obviously because of the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ this choice is always possible.

Remark 1.1. Note that the choice of the values of φ^K and φ^{K-1} in (1.9) is motivated by the transposition arguments that are needed to define the solution of the time-discrete non-homogenous problem (1.7) (see in [15]).

The energy of system (1.9) is given by

$$E_h^k \triangleq \frac{1}{2}\int_\Omega \Big(\Big|\frac{\varphi^{k+1}-\varphi^k}{h}\Big|^2 + \frac{|\nabla\varphi^{k+1}|^2 + |\nabla\varphi^k|^2}{2}\Big) dx, \;\; k = 0, \cdots, K-1,$$

which is a discrete counterpart of the continuous energy E in (1.6). By multiplying the first equation of (1.9) by $(\varphi^{k+1} - \varphi^{k-1})/2h$ and integrating it in Ω, using integration by parts, it is easy to show that E_h^k is conserved in the discrete time variable $k = 0, \cdots, K-1$. Consequently the scheme under consideration is stable and its convergence (in the classical sense of numerical analysis) is guaranteed in an appropriate functional setting (in particular in the finite-energy space $H_0^1(\Omega) \times L^2(\Omega)$, under the condition (1.10)).

As we mentioned above, the controllability/observability of numerical approximation schemes for the wave equation has been the object of intensive studies. However most analytical results concern the case of space semi-discretizations (see [18] and the references cited therein). In practical applications, fully discrete schemes need to be used. The most typical example is the classical fully-discrete central scheme which converges under a suitable CFL condition ([3, 4, 12]). However, in the present setting in which the Laplacian Δ_x is kept continuous, without discretizing it, this scheme is unsuitable since it is unstable. Indeed, it is easy to see that the scheme

$$\frac{\varphi^{k+1} + \varphi^{k-1} - 2\varphi^k}{h^2} - \Delta_x \varphi^k = 0 \tag{1.11}$$

is unstable since $-\Delta_x$, with homogenous Dirichlet conditions, is a positive self-adjoint operator with an infinite sequence of eigenvalues $\{\mu_j^2\}_{j\geq 1}$ tending to infinity. The stability of (1.11) would be equivalent to the stability of the scheme

$$\frac{\varphi^{k+1} + \varphi^{k-1} - 2\varphi^k}{h^2} + \mu_j^2 \varphi^k = 0$$

for all values of μ_j^2, $j \geq 1$. The stability property fails clearly, regardless how small h is, when μ_j^2 is large enough. Hence, we choose the trapezoidal schemes (1.9) for the time-discrete problem, which is stable (due to the property of conservation of energy), as mentioned before.

Let us now return to the analysis of (1.7) and (1.9) and present the second results in [15]. It is of negative nature: *For any given $h > 0$ and any nonempty* *open subset* Γ_* *of* Γ*, system* (1.9) *is not observable and system* (1.7) *is not exactly controllable.* Indeed, similar to [16], for any fixed h, it is easy to find a series of initial data $\{(\varphi_n^0, \varphi_n^1)\}_{n=0,\cdots,\infty}$ of (1.9) with finite energies such that the corresponding observed quantities on Γ vanish as n tends to infinity. Thus, the observability fails. By duality, the exact controllability property of system (1.7) fails, too.

Obviously, these negative results of observability and controllability are related to the fact that the spaces in which the solutions evolve are infinite dimensional; while the number of time-steps is finite. Accordingly, to make the observability inequality possible one has to restrict the class of solutions of the adjoint system (1.9) under consideration by filtering the high frequency components. Similarly, since the property of exact controllability of system (1.7) fails, the final requirement (1.8) has to be relaxed by considering only low frequency projections of the solutions. Controlling such a projection can be viewed as a *partial* controllability problem. This filtering method has been applied successfully in the context of controllability of time discrete heat equations in [16] and space semi-discretization schemes for wave equations in [6, 17, 18].

Here we sketch the discrete version of the classical multiplier approach developed in [15] which allows to derive the uniform observability estimate (with respect to the time step h) for system (1.9) with initial data in a suitable filtered space, which, in turn, by duality, implies the partial controllability of (1.7), uniformly on h.

As in the continuous case, the multiplier technique applies mainly to the case when the controller/observer Γ_0 is given in (1.2) and some variants ([10]), but does not work when (T, Ω, Γ_0) is assumed to satisfy the sharp Geometric Control Condition (GCC) in [1]. As we shall see, the main advantage of our multiplier approach is that the filtering parameter we use has the optimal scaling in what concerns the frequency of observed/controlled solutions with respect to h.

It is important to note that this kind of results can not be obtained by standard perturbation arguments that rely simply on measuring the distance between solutions of the time-discrete and continuous wave equations. Indeed, when proceeding that way, one needs much stronger filtering requirements. In other words, the optimal filtering can only be obtained by a careful analysis of the time evolution of the system under consideration. This is already well-known in the context of space semi-discretizations (see [18]). Our discrete multiplier approach can also be extended to other PDEs of conservative nature, and in particular to the Schrödinger, plate, Maxwell's equations, among others.

The rest of the paper is organized as follows. In Section 2, we show the well-posedness of system (1.7). In Section 3 we state the main results, i.e., the uniform controllability and observability of systems (1.7) and (1.9) after filtering, respectively. The key ingredients in the proof of the uniform observability results will be sketched in Sections 4 and 5. Finally, in Section 6, we shall briefly discuss some open problems and closely related issues.

2. Hidden regularity and well-posedness

This section is devoted to establish the well-posedness of system (1.7).

First of all, for any given $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$ and $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$, suppose $\{\theta^k \in H_0^1(\Omega)\}_{k=0,\cdots,K}$ solves the system

$$\begin{cases} \dfrac{\theta^{k+1}+\theta^{k-1}-2\theta^k}{h^2} - \Delta_x\left(\dfrac{\theta^{k+1}+\theta^{k-1}}{2}\right) = f_k + \dfrac{g^{k+1}-g^k}{h}, \\ \qquad\qquad\qquad\qquad \text{in } \Omega, \quad k=1,\cdots,K-1 \\ \theta^k = 0, \qquad\qquad\qquad \text{on } \Gamma, \quad k=0,\cdots,K. \end{cases} \tag{2.1}$$

We define the energy of system (2.1) by

$$\mathcal{E}_h^k \triangleq \frac{1}{2}\int_\Omega \left(\left| \frac{\theta^{k+1}-\theta^k}{h}\right|^2 + \frac{|\nabla\theta^{k+1}|^2 + |\nabla\theta^k|^2}{2}\right) dx. \tag{2.2}$$

Following the multiplier techniques introduced in [8] in the continuous level, one can establish the following discrete version of the energy estimate:

Lemma 2.1. *For any $h>0$, it holds*

$$\max_{0\le k\le K-1}\mathcal{E}_h^k \le C\left\{ \min\left(\mathcal{E}_h^0, \mathcal{E}_h^{K-1}\right) + \left[h\sum_{k=1}^{K-1}\left(|f^k|_{L^2(\Omega)} + |g^k|_{H_0^1(\Omega)}\right)\right]^2 \right. \\ \left. + h\sum_{k=1}^{K-1}\int_\Omega \left| f^k \frac{g^{k+1}+g^k}{2}\right| dx \right\}. \tag{2.3}$$

Consequently, we have the following hidden regularity property of solutions of system (2.1):

Theorem 2.2. *For any $h>0$, any $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$ in Ω, any $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$, and any $\{\theta^k \in H_0^1(\Omega)\}_{k=0,\cdots,K}$ satisfying (2.1), it holds*

$$h\sum_{k=1}^{K-1}\int_\Gamma \left| \frac{\partial}{\partial\nu}\left(\frac{\theta^{k+1}+\theta^{k-1}}{2}\right)\right|^2 d\Gamma \le C\left\{ \min\left(\mathcal{E}_h^0, \mathcal{E}_h^{K-1}\right) \right. \\ \left. + \left[h\sum_{k=1}^{K-1}\left(|f^k|_{L^2(\Omega)} + |g^k|_{H_0^1(\Omega)}\right)\right]^2 + h\sum_{k=1}^{K-1}\int_\Omega \left| f^k \frac{g^{k+1}+g^k}{2}\right| dx \right\}. \tag{2.4}$$

Remark 2.3. When h tends to zero, the limit of the system (2.1) is

$$\begin{cases} \theta_{tt} - \Delta_x\theta = f + g_t, & \text{in } (0,T)\times\Omega \\ \theta = 0, & \text{in } (0,T)\times\Gamma. \end{cases} \tag{2.5}$$

Inequality (2.4) is a time discrete analogue of the following boundary estimate of (2.5):

$$\int_0^T\int_\Gamma\left|\frac{\partial\varphi}{\partial\nu}\right|^2 d\Gamma dt\le C\left\{\left[|f|_{L^1(0,T;L^2(\Omega))}+|g|_{L^1(0,T;H_0^1(\Omega))}\right]^2+\int_0^T\int_\Omega|fg|dxdt\right.$$
$$\left.+\min\left(\int_\Omega\left[|\nabla\theta(0)|^2+|\theta_t(0)|^2\right]dx,\int_\Omega\left[|\nabla\theta(T)|^2+|\theta_t(T)|^2\right]dx\right)\right\}.$$

We now establish the well-posedness of system (1.7) by means of a discrete version of the classical transposition approach ([8]). For this purpose, for any $\{f^k\in L^2(\Omega)\}_{k=1,\cdots,K-1}$, and any $\{g^k\in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1=g^K=0$, we consider the following adjoint problem of system (1.7):

$$\begin{cases}\dfrac{\zeta^{k+1}+\zeta^{k-1}-2\zeta^k}{h^2}-\Delta_x\left(\dfrac{\zeta^{k+1}+\zeta^{k-1}}{2}\right)=f_k+\dfrac{g^{k+1}-g^k}{h},\\ \qquad\qquad\qquad\qquad\qquad\text{in }\Omega,\quad k=1,\cdots,K-1\\ \zeta^k=0,\qquad\qquad\qquad\qquad\text{on }\Gamma,\quad k=0,\cdots,K\\ \zeta^K=\zeta^{K-1}=0,\qquad\qquad\quad\text{in }\Omega.\end{cases}\tag{2.6}$$

It is easy to see that (2.6) admits a unique solution $\{\zeta^k\in H_0^1(\Omega)\}_{k=0,\cdots,K}$. By Theorem 2.2, this solution has the regularity property $\frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right)\in L^2(\Gamma)$, for $k=1,\cdots,K-1$.

In order to give a reasonable definition for the solution of the non-homogenous boundary problem (1.7) in terms of the transposition method, we consider first the case when the control $\{u^k\}_{k=0,\cdots,K}$ and the initial data (y^0,y^1) are sufficiently smooth. The following result holds:

Lemma 2.4. *Assume that* $\{y^k\in H^2(\Omega)\}_{k=0,\cdots,K}$ *satisfies* (1.7). *Then*

$$h\sum_{k=1}^{K-1}\int_\Omega f^k\frac{y^{k+1}+y^{k-1}}{2}dx-\frac{h}{2}\sum_{k=2}^{K-1}\int_\Omega g^k\left(\frac{y^{k+1}-y^k}{h}+\frac{y^{k-1}-y^{k-2}}{h}\right)dx$$
$$=\int_\Omega\zeta^0\frac{y^1-y^0}{h}dx-\int_\Omega\frac{\zeta^1-\zeta^0}{h}y^0dx-h\sum_{k=1}^{K-1}\int_{\Gamma_0}\frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right)u^kd\Gamma_0.\tag{2.7}$$

Multiplying both sides of (2.1) by $(y^{k+1}+y^{k-1})/2$, integrating the resulting identity in Ω, summing it for $k=1,\cdots,K-1$, one can easily obtain the desired identity (2.7).

Note that (2.7) still makes sense even if the regularity of $\{y^k\}_{k=0,\cdots,K}$ is relaxed as follows

$$\begin{cases}y^{k+1}+y^{k-1}\in L^2(\Omega),\qquad k=1,\cdots,K-1,\\ \dfrac{y^{k+1}-y^k}{h}+\dfrac{y^{k-1}-y^{k-2}}{h}\in H^{-1}(\Omega),\qquad k=2,\cdots,K-1.\end{cases}\tag{2.8}$$

This is consistent with the existence result for (1.1) (in terms of the transposition method). Indeed, under the condition $u \in L^2(\Gamma \times (0,T))$ it is well-known that the solution of (1.1) satisfies $y \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$. Note that formally, letting $h \to 0$, (2.8) leads to $y(t,\cdot) \in L^2(\Omega)$ and $y_t(t,\cdot) \in H^{-1}(\Omega)$. This observation motivates the definition of solution for system (1.7).

More precisely, set

$$\mathcal{H} = \left\{ \{y^k\}_{k=0,\cdots,K} \;\middle|\; y^0, \cdots, y^K \text{ satisfy (2.8)} \right\}. \tag{2.9}$$

We introduce the following

Definition 2.5. We say $\{y^k\}_{k=0,\cdots,K} \in \mathcal{H}$ to be a solution of (1.7), in the sense of transposition, if $y^0 = y_0$, $y^1 = y_0 + hy_1$, and for any $\{f^k \in L^2(\Omega)\}_{k=1,\cdots,K-1}$, and $\{g^k \in H_0^1(\Omega)\}_{k=1,\cdots,K}$ with $g^1 = g^K = 0$, it holds

$$h\sum_{k=1}^{K-1}\int_\Omega f^k \frac{y^{k+1}+y^{k-1}}{2}dx - \frac{h}{2}\sum_{k=2}^{K-1}\left\langle g^k, \frac{y^{k+1}-y^k}{h} + \frac{y^{k-1}-y^{k-2}}{h}\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)}$$
$$= \left\langle \zeta^0, y_1\right\rangle_{H_0^1(\Omega),H^{-1}(\Omega)} - \int_\Omega \frac{\zeta^1-\zeta^0}{h} y_0 dx - h\sum_{k=1}^{K-1}\int_{\Gamma_0}\frac{\partial}{\partial\nu}\left(\frac{\zeta^{k+1}+\zeta^{k-1}}{2}\right)u^k d\Gamma_0,$$

where $\{\zeta^k \in H_0^1(\Omega)\}_{k=0,\cdots,K}$ is the unique solution of (2.6).

We now show the following well-posedness result for system(1.7):

Theorem 2.6. *Assume* $(y_0, y_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *and* $\{u^k \in L^2(\Gamma_0)\}_{k=1,\cdots,K-1}$. *Then system* (1.7) *admits one and only one solution* $\{y^k\}_{k=0,\cdots,K} \in \mathcal{H}$ *in the sense of Definition* 2.5. *Moreover,*

i) *When K is odd,* $\left(y^{2\ell}, \frac{y^{2\ell+1}-y^{2\ell}}{h}\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ *for* $\ell = 0, 1, \cdots, [\frac{K}{2}]$, *and*

$$\begin{aligned}&\max_{\ell=0,1,\cdots,[\frac{K}{2}]}\left\|\left(y^{2\ell}, \frac{y^{2\ell+1}-y^{2\ell}}{h}\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\\ &\leq C\left(\|(y_0,y_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} + h\sum_{k=1}^{K-1}\left\|u^k\right\|^2_{L^2(\Gamma_0)}\right).\end{aligned} \tag{2.10}$$

ii) *When K is even,* $\left(y^{2\ell}, \frac{y^{2\ell-1}-y^{2\ell-2}}{h}\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ *for* $\ell = 1, \cdots, \frac{K}{2}$, *and*

$$\begin{aligned}&\max_{\ell=1,\cdots,\frac{K}{2}}\left\|\left(y^{2\ell}, \frac{y^{2\ell-1}-y^{2\ell-2}}{h}\right)\right\|^2_{L^2(\Omega)\times H^{-1}(\Omega)}\\ &\leq C\left(\|(y_0,y_1)\|^2_{L^2(\Omega)\times H^{-1}(\Omega)} + h\sum_{k=1}^{K-1}\left\|u^k\right\|^2_{L^2(\Gamma_0)}\right).\end{aligned} \tag{2.11}$$

Furthermore, the constant $C > 0$ *in the estimates* (2.10) *and* (2.11) *is independent of the time-step* h.

3. Main Results

Let $\{\Phi_j\}_{j\geq 1} \subset H_0^1(\Omega)$ be an orthonormal basis of $L^2(\Omega)$ consisting of the eigenvectors (with eigenvalues $\{\mu_j^2\}_{j\geq 1}$) of the Dirichlet Laplacian:

$$\begin{cases} -\Delta_x \Phi_j = \mu_j^2 \Phi_j, & \text{in } \Omega \\ \Phi_j = 0, & \text{on } \Gamma. \end{cases}$$

For any $s > 0$, we set

$$\mathcal{C}_{1,s} = \{f(x) \mid f(x) = \sum_{\mu_j^2 < s} a_j \Phi_j(x), \ \ a_j \in \mathbb{C}\} \subset H_0^1(\Omega), \tag{3.1}$$

$$\mathcal{C}_{0,s} = \{g(x) \mid g(x) = \sum_{\mu_j^2 < s} b_j \Phi_j(x), \ \ b_j \in \mathbb{C}\} \subset L^2(\Omega), \tag{3.2}$$

and

$$\mathcal{C}_{-1,s} = \{z(x) \mid z(x) = \sum_{\mu_j^2 < s} c_j \Phi_j(x), \ \ c_j \in \mathbb{C}\} \subset H^{-1}(\Omega), \tag{3.3}$$

subspaces of $H_0^1(\Omega)$, $L^2(\Omega)$ and $H^{-1}(\Omega)$, respectively, with the induced topologies. It is clear that $\bigcup_{k=1}^{\infty} \mathcal{C}_{1,k}$ is dense in $H_0^1(\Omega)$, and the same can be said for $\bigcup_{k=1}^{\infty} \mathcal{C}_{0,k}$ in $L^2(\Omega)$ and $\bigcup_{k=1}^{\infty} \mathcal{C}_{-1,k}$ in $H^{-1}(\Omega)$. Denote by $\pi_{1,s}$, $\pi_{0,s}$ and $\pi_{-1,s}$ the projection operators from $H_0^1(\Omega)$, $L^2(\Omega)$ and $H^{-1}(\Omega)$ to $\mathcal{C}_{1,s}$, $\mathcal{C}_{0,s}$ and $\mathcal{C}_{-1,s}$, respectively.

Our main results are stated as follows:

Theorem 3.1. *Let $T > 2R$. Then there exist three constants $h_0 > 0$, $\delta > 0$ and $C > 0$, depending only on T, R and the dimension d, such that for all $(\varphi_0, \varphi_1) \in \mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$, the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.9) satisfies*

$$E_h^0 \leq Ch \sum_{k=1}^{K-1} \int_{\Gamma_0} \Big| \frac{\partial}{\partial \nu} \Big(\frac{\varphi^{k+1} + \varphi^{k-1}}{2} \Big) \Big|^2 d\Gamma_0, \qquad \forall\, h \in (0, h_0]. \tag{3.4}$$

Remark 3.2. We refer to (5.4) for the exact form of δ, which depends only on d, T and R. In particular it indicates that δ decreases as T decreases. This is natural since, as T decreases, less and less time-step iterations are involved in system (1.9) and, consequently, less Fourier components of the solutions may be observed. Further, δ tends to zero as T tends to $2R$. This is natural too since our proof of (3.4) is based on the method of multipliers which works at the continuous level for all $T > 2R$ but that, at the time-discrete level, due to the added dispersive effects, may hardly work when T is very close to $2R$, except if the filtering is strong enough.

Remark 3.3. Note that taking a filtering parameter of the order of h^{-2} is optimal. This corresponds to filtering precisely numerical solutions whose wave length is of the order of the mesh-size h, for which resonance phenomena may arise. However, an analysis of the dispersion diagram for (1.9) does not exclude that, whatever $\delta > 0$ is, inequality (3.4) holds within the class $\mathcal{C}_{1,\delta h^{-2}} \times \mathcal{C}_{0,\delta h^{-2}}$ for $T > 0$ large enough (see [15]). The multiplier method we develop here does not give such result and we need to impose a smallness condition on δ. But it is well known, even at the continuous level, that the method of multipliers is often unable to yield observability results that can be obtained by other ways. We refer to [5] where sharp observability estimates (that can not be derived by means of the method of multipliers) are obtained by means of Carleman inequalities.

As a consequence of the partial observability result in Theorem 3.1, by duality, we can derive the following uniform partial controllability result:

Theorem 3.4. *Let T, h_0 and δ be given as in Theorem 3.1, and $K > 1$ be an odd integer. Then for any $h \in (0, h_0]$ and any $(y^0, \frac{y^1-y^0}{h}) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists a control $\{u^k \in L^2(\Gamma_0)\}_{k=0,\cdots,K}$ such that the solution of (1.7) satisfies*

i)

$$\pi_{0,\delta h^{-2}} y^{K-1} = \pi_{-1,\delta h^{-2}} \Big(\frac{y^K - y^{K-1}}{h} \Big) = 0 \quad \text{in } \Omega; \tag{3.5}$$

ii) There exists a constant $C > 0$, independent of h, y^0 and y^1, such that

$$h \sum_{k=1}^{K-1} \int_{\Gamma_0} \Big| \frac{u^{k+1} + u^{k-1}}{2} \Big|^2 d\Gamma_0 \le C \Big\| \Big(y^0, \frac{y^1 - y^0}{h} \Big) \Big\|^2_{L^2(\Omega) \times H^{-1}(\Omega)};$$

iii) When $h \to 0$,

$$U_h \triangleq \sum_{k=1}^{K-1} u^k(x) 1_{[kh,(k+1)h)}(t) \longrightarrow u \text{ strongly in } \ L^2((0,T) \times \Gamma_0), \tag{3.6}$$

where u is a control of system (1.1), fulfilling (1.3);

iv) When $h \to 0$,

$$\begin{aligned} y_h \triangleq y^0 1_{\{0\}}(t) + \frac{1}{h} \sum_{k=0}^{K-1} \Big[&(t - kh) y^{k+1} \\ &- \Big(t - (k+1)h \Big) y^k \Big] 1_{(kh,(k+1)h]}(t) \\ \longrightarrow y \text{ strongly in } \ & C([0,T]; L^2(\Omega)) \cap H^1([0,T]; H^{-1}(\Omega)), \end{aligned} \tag{3.7}$$

where y is the solution of system (1.1) with the limit control u as above.

Remark 3.5. The above theorem contains two results: the uniform partial controllability and the convergence of the controls and states as $h \to 0$. The proof is standard. Indeed, the partial controllability statement follows from Theorem 3.1 and classical duality arguments ([8]); while for the convergence result, one may use the approach developed in [18].

It is important to note that, in the limit, one can recover the controllability of (1.1) for all $T > 2R$, i.e. the same results as the multiplier method applied directly to the time-continuous wave equation does, as we have shown in the last two properties of Theorem 3.4. Indeed, given any $T > 2R$, one can choose a sufficiently small δ such that Theorem 3.4 guarantees the controllability of the projections $\pi_{0,\delta h^{-2}}$ in time T. Since these projections involve the frequencies μ_j^2 such that $\mu_j^2 < \delta h^{-2}$, it is clear that, as $h \to 0$, this range of frequencies eventually covers the whole spectrum of the time-continuous wave equation. It is however important to underline that the filtering parameter δ has to be chosen depending on the value of T and that $\delta \to 0$ as T approaches $2R$, as indicated in Remark 3.2.

By duality, Theorem 3.4 is a consequence of Theorem 3.1. Hence, in the sequel we shall concentrate mainly on the proof of Theorem 3.1. To show Theorem 3.1, we shall develop a multiplier approach, which is a discrete analogue of the classical one for the time-continuous case ([7, 8]). There are two key ingredients when doing this. One is a basic identity for the solutions of (1.9) obtained by means of multipliers, which is a discrete version of the classical one on the time-continuous wave equation ([8]). The other one is the construction of the filtering operator to guarantee the uniform observability of (1.9) after filtering. We shall explain them in more detail later in this paper.

4. A key identity via multipliers

In this section we present the first key point of the proof of Theorem 3.1, i.e., an identity for the solutions of (1.9).

The desired identity is as follows:

Lemma 4.1. *For any $h > 0$ and any solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.9), it holds*

$$\begin{aligned} &\frac{h}{2}\sum_{k=0}^{K-1}\int_\Omega\Big(\Big|\frac{\varphi^{k+1}-\varphi^k}{h}\Big|^2+\frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}\Big)dx+X+Y+Z\\ &=\frac{h}{2}\sum_{k=1}^{K-1}\int_\Gamma(x-x_0)\cdot\nu\Big|\frac{\partial}{\partial\nu}\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\Big|^2d\Gamma, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} X=&\int_\Omega\Big[(x-x_0)\cdot\nabla\Big(\frac{\varphi^K+\varphi^{K-2}}{2}\Big)+\frac{d-1}{2}\varphi^K\Big]\,\frac{\varphi^K-\varphi^{K-1}}{h}dx\\ &-\int_\Omega\Big[(x-x_0)\cdot\nabla\Big(\frac{\varphi^2+\varphi^0}{2}\Big)+\frac{d-1}{2}\varphi^0\Big]\,\frac{\varphi^1-\varphi^0}{h}dx, \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
Y = \ & \frac{d}{2}\Big[h^2 \sum_{k=1}^{K-1} \int_\Omega \Delta_x \Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\frac{\varphi^k-\varphi^{k-1}}{h}\Big)dx \\
& \qquad -h\int_\Omega \Big|\frac{\varphi^K-\varphi^{K-1}}{h}\Big|^2 dx\Big] \\
& + \int_\Omega (x-x_0)\cdot\Big[\nabla\Big(\frac{\varphi^{K-1}-\varphi^{K-2}}{2}\Big)\frac{\varphi^K-\varphi^{K-1}}{h} \\
& \qquad\qquad +\nabla\Big(\frac{\varphi^2-\varphi^1}{2}\Big)\frac{\varphi^1-\varphi^0}{h}\Big]dx,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
Z = \ & \frac{(d-2)h}{8}\sum_{k=1}^{K-1}\int_\Omega \Big|\nabla(\varphi^{k+1}-\varphi^{k-1})\Big|^2 dx \\
& -\frac{(d-1)h}{4}\sum_{k=0}^{K-1}\int_\Omega \Big|\nabla\Big(\varphi^{k+1}-\varphi^k\Big)\Big|^2 dx \\
& -\frac{(d-1)h}{4}\int_\Omega \Big(\nabla\varphi^K\cdot\nabla\varphi^{K-1}+\nabla\varphi^1\cdot\nabla\varphi^0\Big)dx \\
& +\frac{(d-2)h}{4}\int_\Omega \Big(|\nabla\varphi^{K-1}|^2+|\nabla\varphi^1|^2\Big)dx.
\end{aligned}
\tag{4.4}
$$

Proof. Multiplying the first equation of (1.9) by $(x-x_0)\cdot\nabla(\varphi^{k+1}+\varphi^{k-1})/2$ (which is a discrete version of the classical multiplier $(x-x_0)\cdot\nabla\varphi$ for the wave equation), integrating it in Ω, summing it up from 1 to $K-1$ and using integration by parts, we obtain

$$
\begin{aligned}
& h\sum_{k=1}^{K-1}\int_\Omega (x-x_0)\cdot\nabla\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\frac{\varphi^{k+1}+\varphi^{k-1}-2\varphi^k}{h^2}dx \\
& = h\sum_{k=1}^{K-1}\int_\Omega (x-x_0)\cdot\nabla\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\Delta_x\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)dx.
\end{aligned}
\tag{4.5}
$$

One can check that the left hand side term of (4.5) coincides with

$$
\begin{aligned}
& \frac{d}{2}h\sum_{k=0}^{K-1}\int_\Omega \Big|\frac{\varphi^{k+1}-\varphi^k}{h}\Big|^2 dx + Y \\
& + \int_\Omega (x-x_0)\cdot\nabla\Big[\Big(\frac{\varphi^K+\varphi^{K-2}}{2}\Big)\frac{\varphi^K-\varphi^{K-1}}{h}-\Big(\frac{\varphi^2+\varphi^0}{2}\Big)\frac{\varphi^1-\varphi^0}{h}\Big]dx,
\end{aligned}
\tag{4.6}
$$

where Y is defined as in (4.3). Further, applying the classical multiplier identity for the Laplacian

$$
\int_\Omega (x-x_0)\cdot\nabla\psi\Delta_x\psi dx = \frac{1}{2}\int_\Gamma (x-x_0)\cdot\nu\Big|\frac{\partial\psi}{\partial\nu}\Big|^2 d\Gamma - \frac{2-d}{2}\int_\Omega |\nabla\psi|^2 dx \tag{4.7}
$$

which holds for all $\psi\in H^2\cap H_0^1(\Omega)$, (see [8]). Then using the identity $(a+b)^2 = 2(a^2+b^2)-(a-b)^2$ for any $a,b\in\mathbb{R}$, the right hand side term of (4.5) may be written as

$$
\begin{aligned}
&\frac{h}{2}\sum_{k=1}^{K-1}\int_\Gamma (x-x_0)\cdot\nu\Big|\frac{\partial}{\partial\nu}\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\Big|^2 d\Gamma\\
&\quad+\frac{(d-2)h}{2}\Bigg\{\sum_{k=0}^{K-1}\int_\Omega\frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}dx\\
&\quad-\sum_{k=1}^{K-1}\int_\Omega\Big|\nabla\Big(\frac{\varphi^{k+1}-\varphi^{k-1}}{2}\Big)\Big|^2dx-\frac{1}{2}\int_\Omega\Big(|\nabla\varphi^{K-1}|^2+|\nabla\varphi^1|^2\Big)dx\Bigg\}.
\end{aligned}
\tag{4.8}
$$

On the other hand, multiplying the first equation of (1.9) by φ^k (which is a discrete version of the multiplier φ in the time-continuous setting which allows establishing the identity of equipartition of energy), integrating it in Ω, summing it up for $k=1,\cdots,K-1$ and using integration by parts, similar to the above, we obtain the following equipartition of energy identity:

$$
\begin{aligned}
&h\sum_{k=0}^{K-1}\int_\Omega\Big(\Big|\frac{\varphi^{k+1}-\varphi^k}{h}\Big|^2-\frac{|\nabla\varphi^{k+1}|^2+|\nabla\varphi^k|^2}{2}\Big)dx\\
&=-\frac{h}{2}\sum_{k=0}^{K-1}\int_\Omega\Big|\nabla(\varphi^{k+1}-\varphi^k)\Big|^2dx-\frac{h}{2}\int_\Omega\Big(\nabla\varphi^K\cdot\nabla\varphi^{K-1}+\nabla\varphi^1\cdot\nabla\varphi^0\Big)dx\\
&\qquad+\int_\Omega\Big(\frac{\varphi^K-\varphi^{K-1}}{h}\varphi^K-\frac{\varphi^1-\varphi^0}{h}\varphi^0\Big)dx.
\end{aligned}
\tag{4.9}
$$

By (4.5)–(4.9), recalling (4.2) and (4.4) respectively for X and Z, we arrive at the desired identity (4.1). □

Remark 4.2. Identity (4.1) is a time-discrete analogue of the following well-known identity for the wave equation (1.9) obtained by multipliers (see [8]):

$$
\frac{1}{2}\int_0^T\int_\Omega\Big[|\varphi_t|^2+|\nabla\varphi|^2\Big]dxdt+\mathcal{X}=\frac{1}{2}\int_0^T\int_\Gamma(x-x_0)\cdot\nu\Big|\frac{\partial\varphi}{\partial\nu}\Big|^2d\Gamma dt,
\tag{4.10}
$$

where

$$
\mathcal{X}=\int_\Omega\Big[(x-x_0)\cdot\nabla\varphi+\frac{d-1}{2}\varphi\Big]\varphi_t dx\Big|_{t=0}^T.
$$

There are clear analogies between (4.1) and (4.10). In fact the only major differences are that, in the discrete version (4.1), two extra reminder terms (Y and Z) appear, which are due to the time discretization. It is easy to see, formally, that Y and Z tend to zero as $h\to 0$. But this convergence does not hold uniformly for all solutions. Consequently, these added terms impose the need of using filtering of the high frequencies to obtain observability inequalities out of (4.1) and modify the observability time, as we shall see.

5. Filtering and uniform observability

In this section, we present the second key ingredient of the proof of Theorem 3.1, i.e., the choice of the filtering parameter which, combined with the identity in Lemma 4.1, leads to the desired uniform observability inequality in Theorem 3.1.

For this, we first derive the following result, which provides an estimate on the reminder term $X+Y+Z$ in Lemma 4.1 in terms of the energy:

Lemma 5.1. *Let K be an even integer, $s>0$ and $T>0$. Then, for any $(\varphi^0, \frac{\varphi^1-\varphi^0}{h}) \in \mathcal{C}_{1,s}\times\mathcal{C}_{0,s}$, for the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.9), it holds*

$$X+Y+Z \geq -\Big[2R+a_1h+3R\sqrt{s}h+T\Big(\frac{d}{2}\sqrt{s}h+a_2sh^2\Big)\Big]E_h^0, \tag{5.1}$$

where

$$a_1 = 3d-2+\max\left(\frac{d-1}{2},2\right), \qquad a_2=\min\big(1,(2-d)^+\big)+\frac{d-1}{2}. \tag{5.2}$$

Proof. For any $(\varphi^0, \frac{\varphi^1-\varphi^0}{h}) \in \mathcal{C}_{1,s}\times\mathcal{C}_{0,s}$, in view of the Fourier series decomposition of the corresponding solution $\{\varphi^k\}_{k=0,\cdots,K}$ of (1.9), one sees that, for any k, we have

$$\begin{aligned}&\int_\Omega |\nabla(\varphi^k-\varphi^{k-1})|^2dx \leq s\int_\Omega |\varphi^k-\varphi^{k-1}|^2dx,\\ &\int_\Omega \Big|\Delta_x\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\Big|^2 dx \leq s\int_\Omega \Big|\nabla\Big(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\Big)\Big|^2 dx.\end{aligned} \tag{5.3}$$

Recalling (4.2)–(4.4) and using (5.3), and noting $T=Kh$ and that the energy of system (1.9) is conservative, we can show that

$$\begin{aligned}&|X| \leq \Big[2R+2(d-1)h+Rh\sqrt{s}\Big]E_h^0,\ |Y|\leq h\left[d\Big(\frac{\sqrt{s}T}{2}+1\Big)+2R\sqrt{s}\right]E_h^0,\\ &Z\geq -h\left\{\left[\min(1,(2-d)^+)+\frac{d-1}{2}\right]shT+\max\left(\frac{d-1}{2},2\right)\right\}E_h^0,\end{aligned}$$

which gives (5.1). □

Finally, Theorem 3.1 follows from Lemmas 4.1 and 5.1 immediately. Indeed, combining (4.1) and (5.1) and recalling the definition of Γ_0 in (1.2) we deduce that

$$\begin{aligned}&\left\{T\Big(1-\frac{d}{2}\sqrt{s}h-a_2sh^2\Big)-\Big[2R+a_1h+3R\sqrt{s}h\Big]\right\}E_h^0\\ &\leq \frac{R}{2}h\sum_{k=1}^{K-1}\int_{\Gamma_0}\Big|\frac{\partial}{\partial\nu}\left(\frac{\varphi^{k+1}+\varphi^{k-1}}{2}\right)\Big|^2 d\Gamma_0.\end{aligned}$$

For this inequality to yield an estimate on E_h^0 we need to choose $s=\delta h^{-2}$ with h small enough such that

$$a_2\delta+\frac{d}{2}\sqrt{\delta}<1,$$

or, more precisely,

$$0 < \sqrt{\delta} < \frac{4}{\sqrt{d^2 + 16a_2} + d}. \tag{5.4}$$

Once this is done, for $h \in (0, h_0)$, T has to be chosen such that

$$T > \frac{2R + a_1 h_0 + 3R\sqrt{\delta}}{1 - \frac{d}{2}\sqrt{\delta} - a_2\delta} \geq 2R. \tag{5.5}$$

Hence, (3.4) holds for $h \in (0, h_0]$.

Conversely, for any $T > 2R$ one can always choose h_0 and δ small enough so that (5.4) and (5.5) hold and guaranteeing the uniform observability inequality.

6. Further comments and open problems

1. *Full discretization.* The analysis in this paper can be combined with previous works (*see, for instance,* [18]) concerning space semi-discretizations to deal with full discretization schemes. But a complete analysis of this issue is still to be done.

2. *Other equations.* The approach and results in this paper can be extended to other PDEs of conservative nature, as the Schrödinger, plate, Maxwell's equations, and so on. There is a fruitful literature on the use of multiplier techniques for these models in the continuous setting (*see, for instance,* [7]). But, the analysis of the corresponding time-discrete systems, adapting the techniques developed in this paper, remains to be done.

3. *Other techniques.* The problem addressed in this paper could have been addressed, in $1-d$, using discrete Ingham inequalities as those in [9]. When doing that, one gets the same results, i.e. when applying the results in [9], one needs to filter the high frequencies by keeping the eigenvalues such that $\mu_j^2 \leq Ch^{-2}$. In [2] the problem of observability of time-discrete linear conservative systems is addressed in an abstract context. the techniques employed in [2] are inspired in those in [11] based on resolvent estimates, which allow to derive, in a systematic way, observability results for time-discrete systems as consequences of those that are by now well-known for time-continuous ones. The results in [2] can be applied to the time-discrete wave equation considered in this article. The main drawback of the results in [2] is that the observability time one gets seems to be far from the expected optimal values.

A different approach, which gives weaker results, is viewing (by extension to continuous time) the solutions of (1.9) as perturbed solutions of the continuous conservative wave equation (1.4). Absorbing the remainder terms then requires stronger filtering than the multipliers method we have employed (more precisely one needs to take $\sigma = 1$). Therefore this approach is not satisfactory and, consequentially, one needs to treat the time-discrete wave equation as such.

4. *Optimality.* The optimality of the results in this paper is also worth studying. We have chosen the filtering parameter of the order of h^{-2}. As we indicated in Remark 3.3, which is further explained in [15] by means of the analysis of the dispersion relation, it can be proved that it is optimal, in the sense that uniform observability fails when we deal with frequencies of the order of $\mu_j^2 > O(h^{-\sigma})$ with any $\sigma > 2$. However, in Theorems 3.1 and 3.4 we need to assume δ to be small enough (by (5.4)) which does not seem to be needed. In fact the abstract results in [2] when applied in this setting are valid for all $\delta > 0$ without any size restriction.

5. *Variable coefficients and nonlinear problems.* It is well-known that, in the continuous case, the multiplier approach can be applied to obtain the controllability/observability of the conservative PDEs with constant coefficients. As for the problems with variable coefficients and/or the nonlinear ones, one has to use microlocal analysis ([1]) and/or Carleman estimates ([14]) to get sharp results. In this time-discrete setting, it would be interesting to develop these other approaches to cover the same class of models as in the PDE setting. This is still to be done.

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