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How much information is needed in quantized nonlinear control?

ZHENG Chuang[1], LI Lin[1], WANG Leyi[2] & LI Chanying[3*]

[1]*School of Mathematical Sciences, Beijing Normal University, Beijing*100875*, China;*
[2] *Department of Electrical and Computer Engineering, Wayne State University, Detroit*48202*, USA;*
[3]*Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing*100190*, China*

Abstract Quantization rate is a crucial measure of complexity in determining stabilizability of control systems subject to quantized state measurements. This paper investigates quantization complexity for a class of nonlinear systems which are subjected to disturbances of unknown statistics and unknown bounds. This class of systems includes linear stablizable systems as special cases. Two lower bounds on the quantization rates are derived which guarantee input-to-state stabilizability for continuous-time and sampled-data feedback strategies, respectively. Simulation examples are provided to validate the results.

Keywords nonlinear systems, disturbances, input-to-state stabilizability, sampled systems, quantization rate

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1 Introduction

With the rapid development of networked control systems, control and communication systems become increasingly integrated. Due to communication resource constraints, such as bandwidth and power, packet delivery rates and data throughput are fundamentally limited. Consequently, feedback control under information complexity limitation becomes a critical issue. Achieving stabilization of a feedback system under quantized signals is mandatory in such frameworks.

A signal quantizer maps an infinite set of continuous values into a finite set of quantized values. Since information is inevitably lost during this process, taking full advantage of limited resources to achieve stability and performance in feedback control is of essential importance in control design for networked systems. This problem has drawn great interest and research effort in theoretical and methodology development in quantized control during the past several decades, resulting in significant advances in new methodologies and their applications [1, 3–8, 10, 15, 22, 23].

Theoretically, a fundamental difficulty rising in quantized feedback control is that an originally stabilizable system under full state feedback can become non-stabilizable under a given scheme of signal quantization. In principle, the less the information is available during quantization, the less the ability of a feedback system has in stabilizing an unstable plant. This intuition brings about a fundamental

* Corresponding author (email: cyli@amss.ac.cn)

question: to guarantee the stabilizability of a quantized feedback system, how much information must the quantizer provide? Much effort has been directed to understand this issue. For noise-free systems, [16], [17] and [20] derived certain critical data rates for stabilizability in auto-regressive moving average (ARMA) systems and linear time-invariant (LTI) state space systems. [12] revealed an interesting relationship between the quantization level and the growth rate of the underlying nonlinear systems. Plants with uncertain structures were further studied in [9, 11, 14, 24] for stability conditions under quantized feedback. Communication uncertainties are typically modeled in stochastic frameworks. Under random noises of known bounds on variance, a lower bound on quantization rates was found in [18] for the mean square stabilizability of a class of stochastic LTI systems. See also the related results on stochastic systems under quantization constraints [19, 21].

However, when systems are perturbed by external disturbances with unknown bounds, the situation becomes far more complicated. It implies that no prior knowledge about the noises is available to assist feedback design from the outset. In such circumstances, an adaptive quantization scheme with an adjustable parameter was introduced in [2] to overcome uncertainties caused by completely unknown disturbances. This elaborative scheme enables [13] to compute a quantization rate which is sufficient to achieve the stabilizability of a LTI system under quantized feedback.

This paper aims to study the quantization rates for the stabilizability of Lipschitz nonlinear systems. It is well understood that nearly all practical control systems are nonlinear. At present, to the best of our knowledge, quantization rates for nonlinear systems under noises of unknown bounds and unknown statistics are not available in the literature. Our results present two lower bounds on quantization rate for stabilizability of a class of nonlinear systems and corresponding control strategies for continuous-time and sampled-data quantized feedback systems, respectively. More precisely, our first result claims that the unknown disturbance does not affect the quantization rate in the continuous-time feedback case, provided it is bounded during the process. For the sampled-data feedback, the main result (Theorem 2) derives a lower bound on quantization rate, in terms of the sampling rate and the appropriate linear approximation of the nonlinear system. Especially, for sufficiently high-frequency sampling, the assumption of Theorem 2 is equivalent to a Schur matrix, which is the standard stabilizability condition for discrete-time linear systems. Based on a trajectory-based analysis, we give a control strategy similar in spirit to that of [13] but is more difficult, due to the sampling of control. Furthermore, compared with [10], our assumption for Theorem 2 exerts on the system function $f(x)$. The theoretical findings are evaluated and confirmed by simulation examples.

The rest of the paper is organized as follows. Section 2 states the two main results with explicit lower bounds on quantization rate for stabilizability of a class of quantized nonlinear systems with noises of unknown bounds, under continuous-time and sampled-data feedback, respectively. The proofs are given separately in Section 3 and Section 4 along with the design of an appropriate signal quantizer under the derived quantization rate and a corresponding feedback strategy. Some simulation examples are given in Section 5 to validate the theoretical results. The paper concludes with some remarks in Section 6.

2 Main results

We first consider the following class of nonlinear systems under quantized feedback control in continuous-time:

$$\dot{x} = f(x) + Bu + Dd, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the control input, and $d \in \mathbb{R}^s$ a bounded disturbance whose bound is unknown *a priori*. u and d are Lebesgue measurable and locally bounded. $B \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{n\times s}$ are constant matrices, $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz and satisfies

$$||f(x)|| \leqslant L||x|| \quad \forall x \in \mathbb{R}^n. \tag{2}$$

In a quantized feedback loop, the state is measured by a quantizer $q : \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$. The quantization rate R of quantizer q is defined as the cardinal number of $\mathcal{Q}$. Acting on the quantized state $q(x)$, a quantized state feedback is designed in the form of $u = g(q(x))$, where $g : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous function. In this section, the state is quantized by q continuously in time. Therefore, the goal is to find an appropriate quantization rate R under which the system (1) can be stabilized by a quantized feedback controller of quantization rate R in the following sense.

Definition 1. The closed-loop system (1) is said to be input-to-state stable (ISS), if there exist some functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$ such that for every initial state x_0 and every bounded disturbance d,

$$||x(t)|| \leqslant \gamma_1(||x_0||) + \gamma_2(||d||_{[0,+\infty)}) \quad \forall t \geqslant 0 \tag{3}$$

with $||d||_{[0,+\infty)} = \sup_{t\in[0,+\infty)} ||d(t)||$ and

$$\lim_{t\to+\infty} \sup ||x(t)|| \leqslant \gamma_3 \left(\lim_{t\to+\infty} \sup ||d(t)|| \right). \tag{4}$$

The open-loop system

$$\dot{x} = f(x(t)) + Dd(t) \tag{5}$$

is assumed to be forward complete, namely for every initial state $x_0 = x(0)$, its solution exists and is unique for all $t \geqslant 0$.

Furthermore, the system is assumed to be stabilizable without quantization.

Assumption 1. There exist a matrix $K \in \mathbb{R}^{m\times n}$ and a positive constant N such that

$$x^T(f(x) + BKx) \leqslant -N||x||^2 \quad \forall x \in \mathbb{R}^n. \tag{6}$$

Remark 1. Note that for linear systems, $f(x) = Ax$ and Assumption 1 is reduced to the standard condition that the pair (A, B) is stabilizable and the inequality

$$x^T(A + BK)x \leqslant -N||x||^2 \quad \forall\ x \in \mathbb{R}^n,$$

implies that $A + BK$ is Hurwitz.

We first present a simple theorem for the continuous-time feedback.

Theorem 1. Under Assumption 1, system (1) is input-to-state stabilizable under quantized feedback whenever

$$R > n^{\frac{n}{2}}(5 + \frac{2||BK||}{N})^n. \tag{7}$$

Remark 2. Function γ_3 in (4) derived for Theorem 1, as shown in the proof below (see (18)), is in fact a constant gain.

For practical implementation, we now consider system (1) with the feedback control designed and implemented by using sampled and quantized data. Let $T > 0$ be a given sampling period. Denote $I_k := [kT, (k+1)T)$ for $k \in \mathbb{N}$. By the standard ZOH D/A (zero-order-hold digital-to-analog) conversion, u in (1) is a piecewise constant control, i.e., $u(t) = u_k := u(Tk)$ for $t \in [Tk, T(k+1))$. We use the notation $x_k := x(kT)$, and similarly for other variables. Let $d_k := ||d||_{I_k}$, $d_{[j_1,j_2]} := \sup\limits_{k\in[j_1,j_2]} d_k$ for $j_1, j_2 \in \mathbb{N}$, and $d_{[0,+\infty)} := \sup\limits_{k\in[0,+\infty)} d_k$.

First, we define ISS in sampled-data systems.

Definition 2. The closed-loop system (1) with a sampled-data feedback is said to be input-to-state stable (ISS) if there exist three functions $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}_\infty$, such that for every initial condition x_0 and every bounded disturbance d,

$$||x_k|| \leqslant \gamma_1(||x_0||) + \gamma_2(d_{[0,+\infty)}) \quad \forall k \in \mathbb{N}$$

and

$$\lim_{k\to+\infty} \sup ||x_k|| \leqslant \gamma_3 \left(\lim_{k\to+\infty} \sup d_k \right).$$

In the framework of sampled-data feedback control systems, we will focus on the following class of the nonlinear term $f(x)$.

Assumption 2. There exist two matrices $A \in \mathbb{R}^{n\times n}$ and $K \in \mathbb{R}^{n\times m}$ such that

$$c_2 := ||\Phi + \Gamma K|| + (L + ||A||)Te^{LT} \max_{t\in[0,T]} \left|\left| e^{At} + \int_0^t e^{As} BK ds \right|\right| e^{-Lt} < 1, \tag{8}$$

where $\Phi := e^{AT}$, $\Gamma := \int_0^T e^{As} Bds$ and L is the Lipschitz constant of $f(x)$ defined in (2).

Remark 3. The matrix A in Assumption 2 acts as a linear approximation of the nonlinear system function $f(x)$. With high-frequency sampling (T is sufficiently close to zero), Assumption 2 is equivalent to that $\Phi + \Gamma K$ is Schur. In other words, it becomes the standard stabilizability condition for discrete-time linear systems. Note that (8) may require the sampling interval T being relatively small to ensure the existence of A and K.

Our main result is stated as follows.

Theorem 2. Under Assumption 2, the system (1) is input-to-state stabilizable under quantized and sampled feedback whenever

$$R > n^{\frac{n}{2}} \left(3 + \frac{c_3}{1 - c_2}\right)^n, \tag{9}$$

where c_2 is defined in (8) and

$$c_3 = ||\Gamma K|| + (L + ||A||)Te^{LT} \max_{t\in[0,T]} \left|\left| \int_0^t e^{As} BK ds \right|\right| e^{-Lt}. \tag{10}$$

Remark 4. Formula (9) indicates that the lower bound of R is small when c_2 and c_3 are small. When T is fixed, according to (8) and (10), it is crucial to find appropriate A and K, which are the linearization and the control gain of the nonlinear system.

Remark 5. Theorems 1 and 2 could be extended to the local Lipschitz condition by a slight modification of Assumptions 1 and 2 accordingly. The ideas remain almost the same as those for the global Lipschitz condition in this paper (see, for instance, [12]).

3 Proof of Theorem 1

Given a quantization rate R satisfying (7), the corresponding quantizer and feedback scheme based on the ideas of [13] are present in Appendix. We will verify Theorem 1 by employing this feedback controller. We first prove some technique lemmas. The physical meanings of those parameters designed for the quantizer can be found in [13] and Appendix A.

Lemma 1 claims that *capture* = "*no*" can be triggered only a finite number of times and the state x and the zoom variable μ are bounded at the end of *capture* = "*no*" interval. Moreover, Lemma 1 shows that the state $||x||$ cannot exceed $M\mu$ after the value of *capture* is switched to "*yes*".

Lemma 1. There exist a time $t_1 \geqslant 0$ and a continuous function $\rho : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ such that

$$\max \left\{ ||x||_{[0,t_1]}, \mu(t_1) \right\} \leqslant \rho(||x_0|| + ||D|| ||d||_{[0,+\infty)}). \tag{11}$$

Moreover, for all $t \geqslant t_1$, *capture* = "*yes*" and $||x(t)|| \leqslant M\mu(t)$.

Proof. The process starts at *capture* = "*no*". Thus, the continuous dynamic is given by $\dot{x}(t) = f(x(t)) + Dd(t)$. Consequently

$$x(t) = \int_0^t f(x(s)) + Dd(s)ds + x_0.$$

Under the assumption that $f(x)$ is a Lipschitz function with the constant L defined in (2), it is easy to show that

$$||x(t)|| \leqslant e^{Lt} \left(||x_0|| + \frac{1}{L} ||D|| ||d||_{[0,t]} \right). \tag{12}$$

During this initial time interval, the zoom variable μ is updated every T_{out} units of time, i.e.,

$$\mu(kT_{out}) = \Omega_{out}^k \mu_0 \quad k = 0, 1, 2, \cdots .$$

By choosing $T_{out} < \log \Omega_{out}/L$, the inequality (12) implies that $\mu(kT_{out})$ grows faster than $||x||$ due to (12) and boundedness of $d(t)$. Hence, there exists a t_1 at which the value of *capture* is switched to "yes".

Moreover, since we use the same algorithm of updating μ as [13] for linear systems, the rest of the proof is nearly verbatim to that in [13], and will not be repeated. We conclude that $||x(t)|| \leqslant M\mu(t)$ for all $t \geqslant t_1$. □

Now, we define

$$R_1(\mu(t)) := \{x : x^T x < (M - 2\Delta)^2 \mu(t)^2\}.$$

Lemma 2 shows that $R_1(\mu(t))$ is an invariant region for the dynamic system when $\mu(t)$ satisfies (13). As a result, a zoom-out cannot occur. Moreover, Lemma 2 claims that a zoom-in will be triggered with certainty unless μ is already small enough relative to the disturbance.

Lemma 2. Suppose that $\mu(t)$ satisfies

$$(M - 2\Delta)\mu(t) > \frac{2||BK||}{N}\Delta\mu(t) + \frac{2||D||}{N}||d||_{[t,+\infty)}. \tag{13}$$

For some $t \geqslant t_1$ such that $x(t) \in R_1(\mu(t))$, the next event can only be a zoom-in. Furthermore, if $\mu(t)$ satisfies

$$(\ell_{in} - \Delta)\mu(t) > \frac{2||BK||}{N}\Delta\mu(t) + \frac{2||D||}{N}||d||_{[t,+\infty)}, \tag{14}$$

the zoom-in will happen in finite time.

Proof. For $t \geqslant t_1$, we have *capture* = "*yes*" and $||x(t)|| \leqslant M\mu(t)$ by Lemma 1. Hence, the feedback control law (A5) leads to the closed-loop system

$$\dot{x} = f(x) + BKq_\mu(x) + Dd. \tag{15}$$

The measurement error $e := q_\mu(x) - x$, (15) can be rewritten as

$$\dot{x} = f(x) + BKx + BKe + Dd.$$

Consequently, taking into account Assumption 1 and (A1), we obtain

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}x^T x\right) &= x^T(f(x) + BKx) + x^T BKe + x^T Dd \\ &\leqslant -N||x||^2 + ||x||||BK|||e| + ||x||||D|||d| \\ &\leqslant -N||x||^2 + ||x||||BK||\Delta\mu + ||x||||D||||d|| \\ &\leqslant -\frac{N}{2}||x||^2 \quad \forall\ ||x|| > \frac{2||BK||}{N}\Delta\mu + \frac{2||D||}{N}||d||. \end{aligned} \tag{16}$$

Set

$$R_2(\mu(t)) := \{x : x^T x < (\ell_{in} - \Delta)^2 \mu(t)^2\},$$
$$B(\mu(t)) := \left\{x : ||x|| \leqslant \tfrac{2||BK||}{N}\Delta\mu(t) + \tfrac{2||D||}{N}||d||\right\}.$$

If (13) holds, we have $B(\mu(t)) \subseteq R_1(\mu(t))$. Therefore, $R_1(\mu(t))$ is an invariant region for the dynamic system as long as $\mu(t)$ remains constant by using (16). Moreover, $R_1(\mu(t)) = \{x : ||x|| < (\ell_{out} - \Delta)\mu(t)\}$. This, combined with (A1), implies that a zoom-out cannot be triggered.

Next, it is easy to see that $B(\mu(t)) \subseteq R_2(\mu(t)) \subseteq R_1(\mu(t))$ if (14) holds. In view of (16), there exists a time $t' \geqslant t$ when $x(t') \in R_2(\mu(t))$, unless a zoom-in occurs earlier. Meanwhile, $R_2(\mu(t))$ is also invariant and $R_2(\mu(t)) \subseteq \{x : ||x|| \leqslant (\ell_{in} - \Delta)\mu(t)\}$. Together with (A1) and the algorithm of updating μ, this implies that a zoom-in must occur prior to time $t' + T_{in}$. □

We now prove Theorem 1.

Proof of Theorem 1: The proof is similar to the case of linear systems in [13]. However, due to nonlinearity and modified assumptions, the upper bound on $\mu(t)$ is different, thank to Assumption 1. For completeness, we briefly state the proof as follows.

First, by defining

$$\hat{\mu} := \frac{2||D||}{N(M-2\Delta)-2||BK||\Delta}||d||_{[0,+\infty)},$$

we obtain $\mu(t) \leqslant \Omega_{out} \max\{\hat{\mu}, \mu(t_1)\}$ for all $t \geqslant t_1$. This implies that

$$||x(t)|| \leqslant M\Omega_{out} \max\{\hat{\mu}, \mu(t_1)\} \quad \forall t \geqslant t_1 \tag{17}$$

by Lemma 1. In view of (11) and (17), the first ISS estimate (3) holds with some continous functions γ_1 and γ_2.

Next, for any $\varepsilon > 0$, there exists a time $t_\varepsilon \geqslant t_1$ such that $||d(t)|| \leqslant \lim_{t\to\infty} \sup ||d(t)|| + \varepsilon$ for all $t \geqslant t_\varepsilon$. Define

$$\widetilde{\mu} := \frac{2||D||}{N(\ell_{in}-\Delta)-2||BK||\Delta}\left(\lim_{t\to\infty} \sup ||d(t)|| + \varepsilon\right).$$

Recall the fact that zoom-in will be trigged in finite time by Lemma 2, we claim that there exists a time $\widetilde{t}_\varepsilon \geqslant t_\varepsilon$ such that $\mu(t) \leqslant \Omega_{out}\widetilde{\mu}$ for all $t \geqslant \widetilde{t}_\varepsilon$. This implies that

$$||x(t)|| \leqslant M\Omega_{out}\widetilde{\mu} \quad \forall\, t \geqslant \widetilde{t}_\varepsilon$$

by Lemma 1. Hence, the second ISS estimate (4) is valid with a constant gain function

$$\gamma_3(r) := \frac{2M\Omega_{out}||D||}{N(\ell_{in}-\Delta)-2||BK||\Delta}r. \tag{18}$$

This completes the ISS of the system (1). Furthermore, by Lemma 5 in Appendix, the relationship between the quantization rate R and M/Δ is given by

$$\frac{M}{\Delta} \leqslant \frac{\sqrt[n]{R}}{\sqrt{n}}. \tag{19}$$

Combining (A4) and (19), we arrive at (7), which completes the proof. □

4 Proof of Theorem 2

We now design an appropriate quantizer and a corresponding linear feedback control law for the nonlinear system (1). Let $\ell_{out} > \ell_{in} > 0, \Omega_{out} > 1 > \Omega_{in} > 0$ be some positive constants that will be determined later. We define the following control law and the scheme of updating μ :

$$u_k := \begin{cases} 0, & \Omega_k = \Omega_{out}, \\ Kq_k, & \Omega_k = \Omega_{in}. \end{cases} \tag{20}$$

$$\mu_{k+1} := \begin{cases} \Omega_{out}(\mu_k + 1), & \Omega_k = \Omega_{out}, \\ \Omega_{in}\mu_k, & \Omega_k = \Omega_{in}. \end{cases} \tag{21}$$

Here, the parameter Ω_k is a switching variable in (20) and (21) and takes two possible values Ω_{out} and Ω_{in}, with the initial value $\Omega_{-1} = \Omega_{out}$. Ω_k is updated by

$$\Omega_k := \begin{cases} \Omega_{out}, & |q_k| > \ell_{out}\mu_k, \\ \Omega_{in}, & |q_k| < \ell_{in}\mu_k, \\ \Omega_{k-1}, & |q_k| \in [\ell_{in}\mu_k, \ell_{out}\mu_k]. \end{cases}$$

with $q_k := \mu_k q(\frac{x_k}{\mu_k})$.

Denote by $k_{2i+1}T$ the time instant at which the plant switches from the zoom-out stage to the zoom-in stage, and $k_{2i}T$ the time instant at which the plant switches from the zoom-in stage to the zoom-out stage, respectively. It is obvious that

$$\Omega_k = \Omega_{out} \quad if \quad k \in [k_{2i}, k_{2i+1} - 1]$$
$$\Omega_k = \Omega_{in} \quad if \quad k \in [k_{2i+1}, k_{2i+2} - 1]$$

for $i = 0, 1, 2, \cdots, P$, with either finite $P \in \mathbb{N}$ or $P = +\infty$.

In the sequel, we will discuss conditions in the zoom-out and zoom-in stages individually.

Zoom-out stage

During the zoom-out stage, the feedback control (20) is zero. Since $f(x)$ is locally Lipschitz, the upper bound of the state x is given by the following inequalities.

Proposition 1. For all $k \in [k_{2i}, k_{2i+1} - 1]$ we have

$$||x_{k+1}|| \leqslant e^{LT}||x_k|| + c_1||d_k|| \tag{22}$$

and

$$||\xi_{k+1}|| \leqslant \frac{e^{LT}}{\Omega_{out}}||\xi_k|| + c_1\frac{||d_k||}{\mu_k},$$

where $c_1 := e^{LT}T||D||$ and $\xi_k := x_k/\mu_k$. Here L is the Lipschitz constant of the nonlinear term $f(x)$.

Zoom-in stage

During the zoom-in stage, the feedback control (20) is Kq_k. Bounds on the state x can be derived, as stated in the following lemma:

Lemma 3. For all $k \in [k_{2i+1}, k_{2i+2} - 1]$ we have

$$||x_{k+1}|| \leqslant c_2||x_k|| + c_3||e_k|| + c_4||d_k||$$

and

$$||\xi_{k+1}|| \leqslant \frac{c_2}{\Omega_{in}}||\xi_k|| + \frac{c_3}{\Omega_{in}}||\nu_k|| + \frac{c_4}{\Omega_{in}}||\zeta_k||,$$

with

$$c_4 := \Big|\Big| \textstyle\int_0^T e^{Ar}Ddr\Big|\Big| + (L + ||A||)Te^{LT}\max_{t\in[0,T]}\Big|\Big|\displaystyle\int_0^t e^{Ar}Ddr\Big|\Big|e^{-Lt},$$
$$e_k := q_k - x_k,$$
$$\nu_k := q(\xi_k) - \xi_k,$$
$$\zeta_k := d_k/\mu_k,$$

and c_2, c_3 are given by (8) and (10), respectively.

Denote $\nu_{[0,+\infty)} := \sup_{j\in[0,+\infty)} ||\nu_j||$ and $\zeta_{[0,+\infty)} := \sup_{j\in[0,+\infty)} ||\zeta_j||$. For $c_2 < \Omega_{in}$, we claim that

$$||\xi_k|| \leqslant e^{-\lambda_1 k}||\xi_0|| + \alpha_1\nu_{[0,+\infty)} + \alpha_2\zeta_{[0,+\infty)} \quad \forall\ k \geqslant 0$$

with $\lambda_1 = -\log(c_2/\Omega_{in}), \alpha_1 = c_3/(\Omega_{in} - c_2)$, $\alpha_2 = c_4/(\Omega_{in} - c_2)$.

Now, we can design the parameters of the quantizer. In our design, we require M and Δ to satisfy

$$M > (3 + \alpha_1)\Delta. \tag{23}$$

Let $\Omega_{out}, \Omega_{in}$ be positive numbers satisfying the inequalities

$$e^{LT} < \Omega_{out},$$
$$c_2 < \Omega_{in} < 1,$$

along with parameters ℓ_{in} and ℓ_{out} given by

$$\ell_{in} := \Delta_M - \Delta,$$
$$\ell_{out} := M - \Delta,$$

where Δ_M will be determined by the following trivial proposition.

Proposition 2. For $c_2 < \Omega_{in}$, let M and Δ satisfy $M > (3+\alpha_1)\Delta$. Then, there exist two constants $\Delta_M, \Delta_d \in (0,+\infty)$, with $\Delta_M - \Delta > 0$, such that whenever $||\xi_0|| \leqslant \Delta_M$, $\nu_{[0,+\infty)} \leqslant \Delta$, and $\zeta_{[0,+\infty)} \leqslant \Delta_d$, we have $||q(\xi_k)|| \leqslant M - \Delta$ and $||\xi_k|| \leqslant M$ for all $k \geqslant 0$.

Furthermore, by the design procedure for the quantizer in the previous section, the ISS of the system (1) is determined by the following two facts:

1. The zoom-out can be trigged only for finitely many time steps;
2. Both the state x and the zoom variable μ are bounded during the whole process.

In fact, by the following Lemma, we show that the number of the zoom-out steps is finite. Moreover, the state x is bounded during the zoom-out interval.

Lemma 4. There exist $\rho_1, \rho_2, \varphi_1, \varphi_2 \in \mathcal{K}_\infty$ such that for any $i \in \mathbb{N}, x_{k_{2i}} \in \mathbb{R}^n, \mu_{k_{2i}} > 0$, and $d \in \mathbb{R}^s$ we have

$$k_{2i+1} - k_{2i} \leqslant 1 + \varphi_1(||x_{k_{2i}}||) + \varphi_2(d_{[k_{2i},k_{2i+1}-1]}) \tag{24}$$

and

$$||x_k|| \leqslant \rho_1(||x_{k_{2i}}||) + \rho_2(d_{[k_{2i},k_{2i+1}-1]}) \quad \forall k \in [k_{2i}, k_{2i+1}]. \tag{25}$$

Proof. By Proposition 1, it is easy to show that

1. If $k_{2i+1} - k_{2i} = 1$, we have

$$||x_k|| \leqslant e^{LT}||x_{k_{2i}}|| + c_1||d_{k_{2i}}||, \quad \forall\, k \in [k_{2i}, k_{2i+1}]. \tag{26}$$

2. If $k_{2i+1} - k_{2i} > 1$, we have

$$||\xi_{k_{2i}+1}|| \leqslant \frac{e^{LT}||x_{k_{2i}}|| + c_1||d_{k_{2i}}||}{\Omega_{out}(\mu_k+1)} \leqslant e^{LT}||x_{k_{2i}}|| + c_1||d_{k_{2i}}|| \tag{27}$$

and

$$||\xi_{k+1}|| \leqslant \frac{e^{LT}}{\Omega_{out}}||\xi_k|| + c_1\frac{||d_k||}{\mu_k}, \quad \forall\, k \in [k_{2i}+1, k_{2i+1}-1].$$

Hence, for any $k \in [k_{2i}+1, k_{2i+1}]$, it holds

$$||\xi_k|| \leqslant \left(\frac{e^{LT}}{\Omega_{out}}\right)^{k-k_{2i}-1}||\xi_{k_{2i}+1}|| + c_1\sum_{j=k_{2i}+1}^{k-1}\left(\frac{e^{LT}}{\Omega_{out}}\right)^{k-j-1}\frac{||d_j||}{\mu_j}.$$

Combining this inequality with $\mu_j \geqslant \sum_{s=1}^{j-k_{2i}}\Omega_{out}^s$, we obtain

$$\frac{||d_j||}{\mu_j} \leqslant \frac{||d_j||}{\sum_{s=1}^{j-k_{2i}}\Omega_{out}^s}.$$

By choosing $e^{LT} < \Omega_{out}$, we conclude that $||\xi_k|| \leqslant \ell_{in}$ and $k_{2i+1} - k_{2i} - 1$ are bounded, by (27) and the fact that $\lim\limits_{j\to\infty}\dfrac{||d_j||}{\mu_j} = 0$. Hence, one can find a continuous, nondecreasing and bounded function $\widetilde{\varphi}$ such that

$$k_{2i+1} - k_{2i} - 1 \leqslant \widetilde{\varphi}(\xi_{k_{2i}+1}, d_{[k_{2i}+1,k_{2i+1}-1]}) \leqslant \varphi(||x_{k_{2i}}||, d_{[k_{2i},k_{2i+1}-1]}).$$

Since $\varphi(0,0)=0$, one can find φ_1,φ_2, (24) holds. Moreover, since

$$\begin{aligned}||x_k|| &\leqslant e^{LT(k-k_{2i}-1)}||x_{k_{2i}+1}||+c_1\sum_{j=k_{2i}+1}^{k-1}e^{LT(k-1-j)}||d_j||\\ &\leqslant e^{LT(k_{2i+1}-k_{2i}-1)}||x_{k_{2i}+1}||+\frac{e^{LT(k_{2i+1}-k_{2i}-2)}-1}{e^{LT}-1}c_1 d_{[k_{2i}+1,k_{2i+1}-1]},\end{aligned}$$

for all $k\in[k_{2i}+1,k_{2i+1}]$, there exist $\widetilde{\rho}_1,\widetilde{\rho}_1$ such that

$$\begin{aligned}||x_k|| &\leqslant \left(e^{LT}\right)^{\varphi_1(||x_{k_{2i}+1}||)+\varphi_2(d_{[k_{2i}+1,k_{2i+1}-1]})}||x_{k_{2i}+1}||\\ &\quad+\frac{\left(e^{LT}\right)^{\varphi_1(||x_{k_{2i}+1}||)+\varphi_2(d_{[k_{2i}+1,k_{2i+1}-1]})-2}-1}{e^{LT}-1}c_1 d_{[k_{2i}+1,k_{2i+1}-1]}\\ &\leqslant \widetilde{\rho}_1(||x_{k_{2i}+1}||)+\widetilde{\rho}_1(d_{[k_{2i}+1,k_{2i+1}-1]}).\end{aligned} \tag{28}$$

Finally, by combining (26) and (28) we arrive at (25). □

Now, it is ready to prove Theorem 2.

Proof of Theorem 2: By Lemma 4, we only need to verify that: 1) x is bounded during the zoom-in intervals; 2) μ is bounded during the zoom-out interval and the zoom-in interval . The first fact is a direct consequence of Lemmas 7 and 8 in Appendix, while the second fact follows from Lemmas 6 and 9. With these two facts, the rest of the proof is almost identical to that of Theorem 2 in [13]. Finally, according to the relationship between R and M/Δ in (19), (9) follows immediately due to (23). □

5 Examples

In this section, two examples are provided to illustrate our main results for the continuous-time and sampled-data feedback control, respectively.

Example 1: Consider the system

$$\begin{cases}\dot{x}_1=-2x_1+x_2,\\ \dot{x}_2=2x_1\sin x_1+2x_2+u+100d.\end{cases} \tag{29}$$

where $u\in\mathbb{R}$ is the input and $d\in\mathbb{R}$ is the bounded disturbance whose bound is unknown. It is easy to compute that the Lipschitz constant $L=4$. We select the feedback matrix $K=(-1,-4)$. Hence, $N=1$.

In order to illustrate Theorem 1, we simulated the above control system (29) with the algorithm parameters $M=3,\Delta=0.2,T_{in}=T_{out}=0.2$, $T_c=0.01$, $\Omega_{in}=0.85,\Omega_{out}=3$, $\mu_0=10$. The behavior (on a log scale) of the state $x(t)$ and the quantizer's range $M\mu(t)$ under the initial condition $x_0=(0,10000)^T$ and $d(t)$ randomly distributed on the interval $[0,0.01]$ is shown in Figure 1. As expected, after an initial overshoot, the state settles below a bound.

Example 2: Consider the system

$$\begin{cases}\dot{x}_1=\sin x_1+\dfrac{1}{5}x_1\sin x_2+u_1(t)+10^3d(t),\\ \dot{x}_2=x_2+\dfrac{1}{10}x_2\sin x_1+u_2(t)+10^4d(t).\end{cases} \tag{30}$$

where $(u_1,u_2)^T\in\mathbb{R}^2$ is the input and $d\in\mathbb{R}$ is the completely unknown bounded disturbance. Obviously $L=\sqrt{2}$. Select parameters $T=0.2,A=I,K=-5I$. In order to illustrate Theorem 2, we simulate the control system (30) with the quantizer parameters $M=50,\Delta=0.2,\Omega_{in}=0.8,\Omega_{out}=3,\mu_0=10$. Figure 2 shows the behavior (on a log scale) of the state $x(t)$ for the initial condition $x_0=(600,0)^T$ and $||d(t)||\leqslant 0.3$.

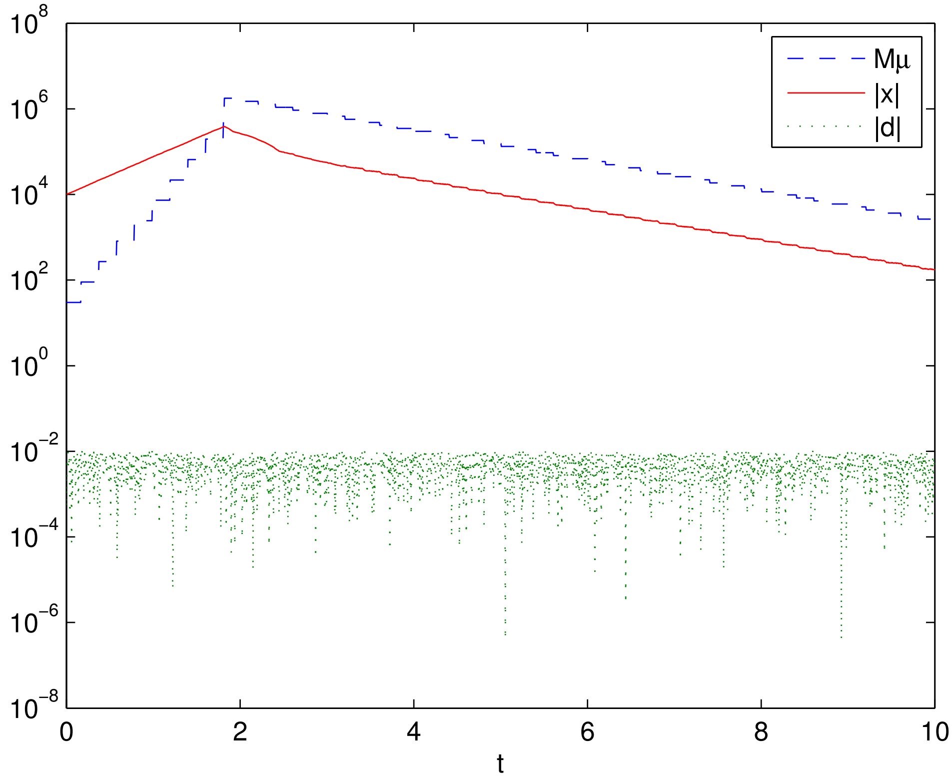

Figure 1 Simulation results for Example 1.

6 Conclusion

In a networked control system, quantization complexity is of essential importance in resource management. It has been recognized that there is certain fundamental limit on quantization rate below which stabilization of an unstable system by quantized feedback cannot be achieved. In this paper, a lower bound on quantization rate R has been derived to ensure the input-to-state stabilizability for a class of nonlinear systems under external disturbances whose bounds are unknown. Constructive design procedures for both the quantizer and feedback controller are developed for continuous-time and sampled-data feedback systems. Theoretical extension of the methodology of this paper to broader classes of nonlinear systems and practical implementation to networked control systems are worth further attention.

Conflict of interest The authors declare that they have no conflict of interest.

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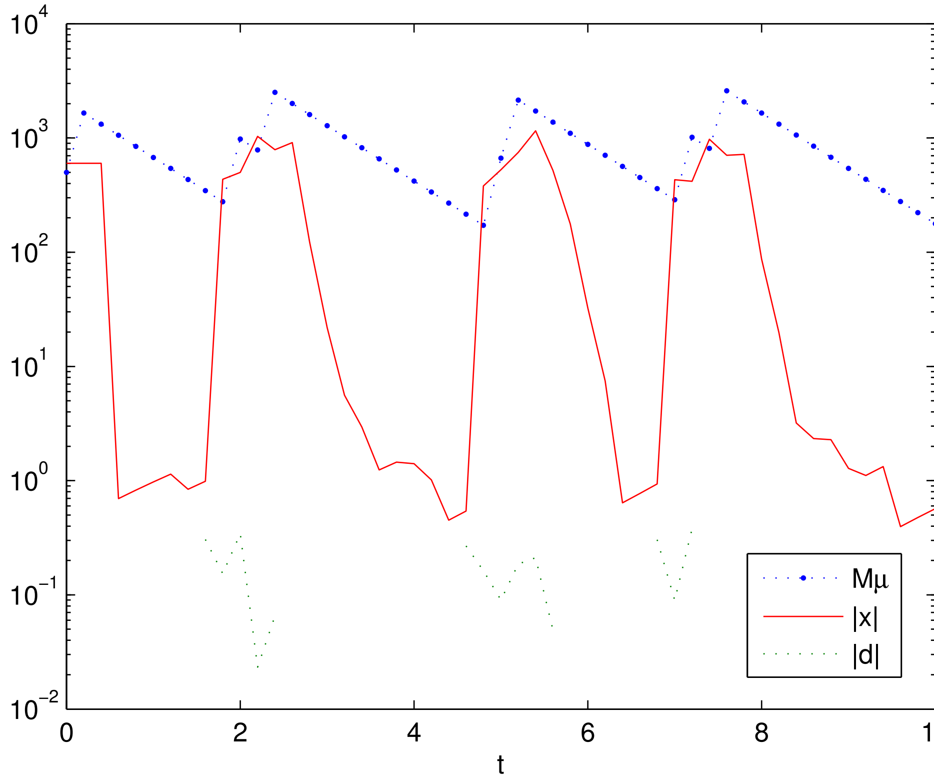

Figure 2 Simulation results for Example 2.

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Appendix A Quantizer for Theorem 1

In general, we need three parameters M, Δ and Δ_0 for the quantizer: M is the quantization range, Δ the quantization error, and Δ_0 the minimal quantization resolution. More precisely, we choose $M > \Delta > 0$ and $\Delta_0 > 0$ such that

$$||z|| \leqslant M \Rightarrow ||q(z) - z|| \leqslant \Delta, \tag{A1}$$

$$||z|| > M \Rightarrow ||q(z)|| > M - \Delta, \tag{A2}$$

$$||z|| \leqslant \Delta_0 \Rightarrow q(z) = 0, \tag{A3}$$

$$M > 5\Delta + \frac{2||BK||}{N}\Delta. \tag{A4}$$

The first condition (A1) ensures that the quantization error is bounded by Δ when the quantizer is not saturated. Saturation of the quantizer is indicated by (A2). (A3) defines the minimum resolution of the quantizer so that the measurement is zero when the signal magnitude is below this resolution.

Let $T_{in}, T_c, T_{out}, \Omega_{in}, \Omega_{out}$ be some positive numbers satisfying $T_{in} \leqslant T_{out}$, $T_c < \frac{1}{2}T_{out}$, $\Omega_{in} < 1$ with

$$\Omega_{in}(M - 2\Delta) - 3\Delta > \frac{2||BK||}{N}\Delta.$$

Moreover, $T_{out} < \log \Omega_{out}/L$ with

$$\Omega_{out} > \frac{M}{M - 2\Delta}.$$

Note that T_{in} is the unit of time after the last zoom-in or zoom-out before executing another zoom-in, T_{out} is the unit of time after a zoom-out before executing another zoom-out, Ω_{in} is the zoom-in factor and Ω_{out} is the zoom-out factor, respectively.

Define

$$\ell_{in} := \Omega_{in}(M - 2\Delta) - 2\Delta, \ell_{out} := M - \Delta.$$

In the control strategy to be developed below, all system variables will be continuous from the right by construction. Variables which are not mentioned remain constants in the following algorithm.

We use the one-parameter family of quantizers

$$q_\mu(x) := \mu q\left(\frac{x}{\mu}\right) \quad \mu > 0.$$

Here μ, called "zoom" variable, is an adjustable scaling parameter with initial value μ_0. It is known to both the sender and receiver and updated at discrete instants of time by the following algorithm.

Algorithm A1 Updating μ

if $\mathit{capture}^- =$ "*no*" **then**
  if $\tau_{out}^- = T_{out}$ **then**
    $\mu \Leftarrow \Omega_{out}\mu^-$;
    $\tau_{out} \Leftarrow 0$;
  end if
  if $||q_{\mu^-}(x)|| \leqslant \ell_{out}\mu^-$ and $\tau_{out}^- \in [T_c, T_{out} - T_c]$ **then**
    $\mu \Leftarrow \Omega_{out}\mu^-$;
    capture $\Leftarrow$ "*yes*";
  end if
else {$\mathit{capture}^- =$ "*yes*"}
  if $||q_{\mu^-}(x)|| \geqslant \ell_{out}\mu^-$ **then**
    $\mu \Leftarrow \Omega_{out}\mu^-$;
    $\tau_{out} \Leftarrow 0$;
  end if
  if $||q_{\mu^-}(x)|| \leqslant \ell_{in}\mu^-$ and $\min\{\tau_{in}^-, \tau_{out}^-\} \geqslant T_{in}$ **then**
    $\mu \Leftarrow \Omega_{in}\mu^-$;
    $\tau_{in} \Leftarrow 0$;
  end if
end if

Based on the quantized signal, the feedback control law is given by

$$u(t) = \begin{cases} 0, & \mathit{capture} = \text{"}no\text{"}, \\ Kq_\mu(x), & \mathit{capture} = \text{"}yes\text{"}. \end{cases} \tag{A5}$$

Remark 6. The parameter "*capture*" is an auxiliary logical variable which is used to distinguish the open-loop stage and the control stage. It takes values in the set {"*yes*", "*no*"} and is initialized at "*no*". The parameters "τ_{out}" and "τ_{in}" are functions of the continuous time t, called "auxiliary reset clock variables". The clock variables are initialized at 0 and take values in the intervals $[0, T_{out}]$ and $[0, T_{in}]$, respectively. Moreover, they satisfy

$$\dot{\tau}_{out} = \begin{cases} 1, & \tau_{out} < T_{out}, \\ 0, & \tau_{out} = T_{out}. \end{cases}$$

and

$$\dot{\tau}_{in} = \begin{cases} 1, & \tau_{in} < T_{in}, \\ 0, & \tau_{in} = T_{in}. \end{cases}$$

Appendix B Some Lemmas and technical proofs

Lemma 5. Assume that the number $\sqrt[n]{R}$ is an odd integer (see [12]). Then the quantization rate R, i.e., the number of the elements in $\mathcal{Q}$, satisfies

$$R \geqslant \left(\frac{M}{\Delta}\sqrt{n}\right)^n.$$

Proof. Firstly, we divide the the minimum circumscribed hypercube of the ball $\{z : ||z|| \leqslant M\}$ into R equal hypercubic boxes, numbered from 1 to R in some specific way. Secondly, for each hypbercubic box, there is an unique ball in $\mathbb{R}^n$ which is minimally circumscribed to the small box. Let $q(z)$ be the center of this ball that contains z. In case z lies on the boundary of several balls, the value of $q(z)$ can be chosen arbitrarily among the candidates. Then we obtain

$$||q(z) - z|| \leqslant \frac{\sqrt{n}M}{R^{\frac{1}{n}}},$$

which implies that

$$\frac{\sqrt{n}M}{R^{\frac{1}{n}}} \leqslant \Delta.$$

This completes the proof of Lemma 5. □

Proof of Lemma 3: We consider the following linear system

$$\begin{cases} \dot{y}(t) = Ay + Bu_k + Dd, \quad t \in I_k, \\ y_k = x_k. \end{cases}$$

It is straightforward that

$$y(t) = e^{A(t-kT)}x_k + \int_0^{t-kT} e^{Ar}Bdru_k + \int_{kT}^{t} e^{A(t-r)}Dd(r)dr$$

for $t \in I_k$. This implies that

$$\begin{aligned} ||y_{k+1}|| &\leqslant \left|\left|e^{AT}x_k + \int_0^T e^{Ar}BdrK(x_k + e_k)\right|\right| + \left|\left|\int_{kT}^{(k+1)T} e^{A((k+1)T-r)}Dd(r)dr\right|\right| \\ &\leqslant \left|\left|e^{AT} + \int_0^T e^{Ar}BdrK\right|\right| ||x_k|| + \left|\left|\int_0^T e^{Ar}BdrK\right|\right| ||e_k|| + \left|\left|\int_0^T e^{Ar}Ddr\right|\right| ||d_k||. \end{aligned}$$

Next, we consider the following nonlinear system

$$\begin{cases} \dot{\varphi}(t) = f(\varphi + y) - Ay, \quad t \in I_k, \\ \varphi_k = 0. \end{cases}$$

It is easy to prove that

$$\begin{aligned} |\varphi(t)| &\leqslant \int_{kT}^{t} ||f(\varphi(s) + y(s)) - Ay(s)||ds \\ &\leqslant \int_{kT}^{t} L||\varphi(s)|| + (L + ||A||)||y(s)||ds \\ &\leqslant e^{L(t-kT)}(L + ||A||)\int_{kT}^{t} ||y(s)||e^{L(kT-s)}ds \\ &\leqslant e^{L(t-kT)}(L + ||A||) \times \int_{kT}^{t} \left|\left|e^{A(s-kT)}x_k + \int_0^{s-kT} e^{Ar}Bdru_k + \int_{kT}^{s} e^{A(s-r)}Dd(r)dr\right|\right| e^{L(kT-s)}ds \end{aligned}$$

for all $t \in I_k$. This implies that

$$\begin{aligned}||\varphi_{k+1}|| \leqslant & e^{LT}(L+||A||)\int_{kT}^{(k+1)T}\Big\{\Big|\Big|e^{A(s-kT)}x_k+\int_0^{s-kT}e^{Ar}BdrK(x_k+e_k)\\ &+\int_{kT}^{s}e^{A(s-r)}Dd(r)dr\Big|\Big|\Big\}e^{L(kT-s)}ds\\ \leqslant & e^{LT}(L+||A||)\int_{kT}^{(k+1)T}\Big\{\Big|\Big|(e^{A(s-kT)}+\int_0^{s-kT}e^{Ar}BdrK)x_k\Big|\Big|\\ &+\Big|\Big|\int_0^{s-kT}e^{Ar}BdrKe_k\Big|\Big|+\Big|\Big|\int_{kT}^{s}e^{A(s-r)}Dd(r)dr\Big|\Big|\Big\}e^{L(kT-s)}ds\\ \leqslant & e^{LT}(L+||A||)\int_{kT}^{(k+1)T}\Big\{\Big|\Big|(e^{A(s-kT)}+\int_0^{s-kT}e^{Ar}BdrK)\Big|\Big|||x_k||\\ &+\Big|\Big|\int_0^{s-kT}e^{Ar}BKdr\Big|\Big|||e_k||+\Big|\Big|\int_0^{s-kT}e^{Ar}Ddr\Big|\Big|||d_k||\Big\}e^{L(kT-s)}ds.\end{aligned}$$

It is obvious that $||x_{k+1}|| \leqslant ||y_{k+1}|| + ||\varphi_{k+1}||$ and we complete the proof of Lemma 3. □

The following Lemma 6 indicates that the zoom variable μ is bounded at the end of each zoom-out interval.

Lemma 6. There exists a continuous bounded function ρ_μ^{out} such that for any $\mu > 0$ we have $\rho_\mu^{out}(\mu, 0, 0) > 0$ and the following is true for all $i \in \{0, 1, \cdots, P\}$ and all $\mu_{k_{2i}} > 0, x_{k_{2i}} \in \mathbb{R}^n, d \in \mathbb{R}^s$:

$$\mu_{k_{2i+1}} \leqslant \rho_\mu^{out}(\mu_{k_{2i}}, ||x_{k_{2i}}||, d_{[k_{2i},k_{2i+1}-1]}).$$

Proof. The proof is identical to Lemma IV.6 in [13]. □

The following Lemma 7 establishes an appropriate bound on the state x during the zoom-in intervals.

Lemma 7. There exist $\lambda, \gamma \in (0, +\infty)$ such that

$$||x_k|| \leqslant e^{-\lambda(k-k_{2i+1})}(||x_{2i+1}|| + \mu_{k_{2i+1}}) + \gamma d_{[k_{2i+1},k-1]} \quad \forall k \in [k_{2i+1}, k_{2i+2}].$$

Proof. During the zoom in intervals we have by construction

$$||x_k|| \leqslant M\mu_k,$$
$$\Big|\Big|q\left(\frac{x_k}{\mu_k}\right) - \frac{x_k}{\mu_k}\Big|\Big| \leqslant \Delta.$$

Meanwhile, the x-subsystem satisfies

$$||x_{k+1}|| \leqslant c_2||x_k|| + c_3||e_k|| + c_4||d_k||$$

and the μ-subsystem evolves according to

$$\mu_{k+1} = \Omega_{in}\mu_k$$

for all $k \in [k_{2i+1}, k_{2i+2} - 1]$. This is a cascade of an ISS system and a GAS system, hence the conclusion holds. □

The following Lemma 8 establishes a different bound on the state x during the zoom-in intervals.

Lemma 8. There exists a continuous function $\rho_x^{in} : \mathbb{R}_{>0} \times \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$, with $\rho_x^{in}(\mu, 0, 0) = 0$ for all $\mu > 0$, and such that for any $s \geqslant 0, \rho_x^{in}(\cdot, \cdot, s)$ is nondecreasing in its first two arguments and for any $i \in \{0, 1, \cdots, P\}$ the following holds for all $\mu_{k_{2i+1}}, x_{k_{2i+1}}, d$:

$$||x_k|| \leqslant \rho_x^{in}(\mu_{k_{2i+1}}, ||x_{k_{2i+1}}||, d_{[k_{2i+1},k_{2i+2}-1]}) \quad \forall k \in [k_{2i+1}, k_{2i+2}].$$

Proof. The proof is almost the same as that of Lemma IV.8 in [13] with the difference that $H := c_2 + c_3 + c_3L_q$ in our case. □

The following Lemma 9 indicates that if the zoom-in interval is bounded then the state x and the zoom variable μ are bounded by the function of the disturbance d at the end of the zoom-in interval.

Lemma 9. Consider an arbitrary $i \in \{0, 1, \cdots, P\}$. If $k_{2i+2} < \infty$, then $i < P - 1$ and there exists a $\widetilde{\gamma} \in (0, +\infty)$ such that

$$\max\{||x_{k_{2i+2}}||, \mu_{k_{2i+2}}\} \leqslant \widetilde{\gamma} d_{[k_{2i+1},k_{2i+2}-1]}.$$

Proof. see Lemma IV.9 in [13]. □