On the observability of time discrete systems with conservation laws.

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Abstract. We consider various time discretizations of $\dot{z} = Az$, where A is a skew-adjoint operator, which we observe through an operator B. More precisely, we assume that the pair (A, B) is exactly observable in the continuous level, and we derive uniform observability inequality for some discretizations provided we filter the initial data. The method we use is mainly based on the resolvent estimate given in [2], which turns out to be equivalent to the exact observability property of the pair (A, B). We present some applications of our results to time discrete and fully discrete approximations schemes.

Contents

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1. Introduction.

Let X be a Hilbert space endowed with the norm $\|\cdot\|_X$ and let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator with compact resolvent. Let us consider the following abstract system:

$$\dot{z}(t) = Az(t), \quad z(0) = z_0, \tag{1.1}$$

Here and henceforth, a dot denotes differentiation with respect to the time t. The element $z_0 \in X$ is called the *initial state*, $z = z(t)$ is the state. Such systems are often used as models of vibrating systems (e.g., the wave equation), electromagnetic phenomena (Maxwell's equations) or in quantum mechanics (Schrödinger's equation).

Assume that Y is another Hilbert space equipped with the norm $\|\cdot\|_Y$. We denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y, endowed with the classical operator norm. Let $B \in \mathcal{L}(\mathcal{D}(A), Y)$ be an observation operator and define the output function

$$y(t) = Bz(t). \tag{1.2}$$

In order to give a sense to (1.2), we make the assumption that B is an admissible observation operator in the following sense (see [13]):

Definition 1.1. The operator B is an admissible observation operator for system (1.1)-(1.2) if for every $T > 0$ there exists a constant $K_T > 0$ such that

$$\int_0^T \|y(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2 \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{1.3}$$

Note that if B is *bounded* in X, i.e. if it can be extended such that $B \in \mathcal{L}(X, Y)$, then B is obviously an admissible observation operator.

Definition 1.2. System (1.1)-(1.2) is exactly observable in time T if there exists $k_T > 0$ such that

$$k_T \|z_0\|_X^2 \leq \int_0^T \|y(t)\|_Y^2 \, dt \qquad \forall\, z_0 \in \mathcal{D}(A). \tag{1.4}$$

System (1.1)-(1.2) is said to be exactly observable if it is exactly observable in some time $T > 0$.

Throughout this paper, we assume that system (1.1)-(1.2) is exactly observable. This question arises naturally when dealing with controllability and stabilizations properties of linear system, see for instance the textbook [6].

It was proved in [2] and [8] that system (1.1)-(1.2) is exactly observable if and only if the following assertion holds:

$$\begin{cases} \text{There exist constants } \ M, m > 0 \ \text{ such that} \\ M^2 \|(i\omega I - A)z\|^2 + m^2 \|Bz\|_Y^2 \geq \|z\|^2, \qquad \forall \ \omega \in \mathbb{R}, \ \ z \in \mathcal{D}(A). \end{cases} \tag{1.5}$$

This spectral condition can be viewed as a Hautus-type test, and generalized in some sense the classical Kalman rank condition, see for instance [12].

Moreover, since A is skew-adjoint with compact resolvent, its spectrum is given by $\sigma(A) = \{i\mu_j : \ j \in \Lambda\}$ with $\Lambda = \mathbb{Z}^*$ or $\mathbb{N}^*$ and where $(\mu_j)_{j\in\Lambda}$ is a sequence of real numbers. Set $(\Phi_j)_{j\in\Lambda}$ an orthonormal basis of eigenvectors of A associated to the eigenvalues $(i\mu_j)_{j\in\Lambda}$, that is:

$$A\Phi_j = i\mu_j \Phi_j. \tag{1.6}$$

In this paper, we are interested in the time semi-discretization of system (1.1)-(1.2). We are thus replacing the continuous dynamics (1.1)-(1.2) by time-discrete ones. In the sequel, we propose to analyze some of them in a very general setting through their observability properties.

To begin with, we present a natural discretization of the continuous system. For any $\triangle t > 0$, we denote by z^k and y^k respectively the approximations of the solution z and the output function y of system (1.1)–(1.2) at time $t_k = k\triangle t$ for $k \in \mathbb{Z}$. We then introduce the following *implicit midpoint* time discretization of system (1.1):

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\triangle t} = A\Big(\dfrac{z^{k+1} + z^k}{2}\Big), \qquad \text{in } \ X, \qquad k \in \mathbb{Z}, \\ z^0 \text{ given.} \end{cases} \tag{1.7}$$

Consequently, the output function of (1.7) is given by

$$y^k = Bz^k, \qquad k \in \mathbb{Z}. \tag{1.8}$$

Taking into account that the spectrum of A is purely imaginary, it is easy to show that $\left\|z^k\right\|_X$ is conserved in the discrete time variable $k \in \mathbb{Z}$, i.e. $\left\|z^k\right\|_X = \left\|z^0\right\|_X$. Consequently the scheme under consideration is stable and therefore convergent, since its consistency is obvious.

The observability problem for system (1.7) is formulated as follows:
Under some assumptions of the initial data z^0, to find a positive constant $\tilde{k}_T$, independent of $\triangle t$, such that the solutions z^k of system (1.7) *satisfy:*

$$\tilde{k}_T \left\|z^0\right\|_X^2 \leq \triangle t \sum_{k=0}^{T/\triangle t} \left\|Bz^k\right\|_Y^2. \tag{1.9}$$

Clearly, (1.9) is a discrete version of (1.4).

Note that this type of observability inequality appears naturally when dealing with stabilization and controllability problems, see for instance [6], [12] and [17]. For the discretizations processes, it is important that these inequalities hold uniformly with respect to the discretization parameter(s) (here $\triangle t$ only) to recover uniform stabilization properties or convergence of discrete controls to the continuous ones. We refer to the review article [17] and the references therein for more precise statements.

In the sequel, we are interested in describing which assumptions are needed for inequality (1.9) to hold uniformly. We will see that filtering the initial datum is one possible and relevant assumption. More precisely, if for any $s > 0$, recalling (1.6), we define

$$\mathcal{C}_s = \text{span}\,\{\Phi_j : \text{the corresponding } i\mu_j \ \text{ satisfies } |\mu_j| \leq s\}, \tag{1.10}$$

then we will prove that for many time discretizations of (1.1)-(1.2) inequality (1.9) holds uniformly in the class $\mathcal{C}_{\delta/\triangle t}$ for any $\delta > 0$.

One of the interesting feature of our result is that it allows us to derive uniform observability inequalities for fully discrete schemes as well under some uniform assumptions on the *space* semi-discrete approximation schemes of (1.1)-(1.2), see Section 5.

The outline of this paper is stated as follows.

In Section 2, we show the uniform observability property (1.9) for system (1.7)-(1.8), assuming that the initial datum is taken in an appropriate filtered space, namely $\mathcal{C}_{\delta/\triangle t}$ for any $\delta > 0$. Our proof is mainly based on the resolvent estimate (1.5), combined with standard Fourier arguments.

We then generalize to more general approximation schemes. In Section 3, we are considering conservative approximation schemes which preserve the eigenvectors in such a way that there exists a "nice" relation (3.3) between the continuous and discrete dynamics. Again we will prove that a uniform observability inequality holds provided that the initial datum is filtered at order $1/\triangle t$. We present an application to the Gauss Implicit method (see [3]).

In Section 4, we are interested in second order time discrete systems. Of course, this can be seen as a one order time discrete equation and then the results of the previous sections provide observability results for some schemes. However, there are other classical discretizations which do not conserve the eigenvectors, for instance the Newmark discretization schemes. However, we can overcome this difficulty and prove that a uniform observability inequality holds as well provided the initial datum belongs to $\mathcal{C}_{\delta/\triangle t}$ for any $\delta > 0$.

Finally, in Section 5, we give an application of our main results to the fully discrete schemes, by proposing a general approach to prove observability estimates uniform in both $\triangle t$ and the mesh size.

We end the paper by stating some further comments and open problems.

2. Study of a natural discretization.

This section is dedicated to the study of system (1.7)-(1.8). Let us first introduce some notations and definitions.

The Hilbert space $\mathcal{D}(A)$ is endowed with the norm of the graph of A:

$$\|z\|_1^2 = \|z\|_X^2 + \|Az\|_X^2 \,.$$

It follows that $B \in \mathfrak{L}(\mathcal{D}(A), Y)$ implies

$$\|Bz\|_Y \le C_B \frac{\delta}{\triangle t} \|z\|_X, \qquad z \in \mathcal{C}_{\delta/\triangle t}, \tag{2.1}$$

where C_B is a positive constant independent of $\triangle t$.

We are now in position to claim the following theorem based on the resolvent estimate (1.5):

Theorem 2.1. *Assume that* (A, B) *satisfy* (1.5) *and that* $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.

Then, for any $\delta > 0$, *there exists* $T_\delta, \triangle t_0 > 0$ *such that for any* $T > T_\delta$ *and* $\triangle t \in (0, \triangle t_0)$, *there exists a positive constant* $k_{T,\delta}$, *independent of* $\triangle t$, *such that*

$$k_{T,\delta} \left\|z^0\right\|_X^2 \le \triangle t \sum_{k=1}^{T/\triangle t} \left\|Bz^k\right\|_Y^2, \qquad \forall\, z^0 \in \mathcal{C}_{\delta/\triangle t}. \tag{2.2}$$

Moreover,

$$T_\delta \le \pi \Big(1 + \frac{\delta^2}{4}\Big)^{1/2} \Big(\frac{1}{8}\delta^2 m^2 C_B^2 + M^2(1 + \delta + \frac{\delta^2}{4})\Big)^{1/2}, \tag{2.3}$$

where C_B *is as in* (2.1).

Remark 2.2. If we are filtering at a scale smaller than $\triangle t$, for instance in the class $\mathcal{C}_{\delta/(\triangle t)^\alpha}$, with $\alpha < 1$, then it underlies that δ in (2.3) vanishes as $\triangle t$ tends to zero. Consequently, the optimal time T^* satisfies

$$T^* \le T_0 \le \pi M,$$

which coincides with the estimate obtained by the resolvent estimate (1.5) in the continuous level (see [8]). Note that however, even in the continuous level, this does not provide the optimal time in general.

Before entering into the proof, let us first present some very elementary results and notations.

Any $z^0 \in X$ can be expanded on the eigenvectors basis (1.6) as:

$$z^0 = \sum_j a_j \Phi_j. \tag{2.4}$$

We state the following property on the solution z^k of (1.7):

Proposition 2.3. *Let* z^0 *be the form in* (2.4). *Then the solution* z^k *of* (1.7) *can be computed as*

$$z^k = \sum_j a_j \exp(i\lambda_j k \triangle t)\Phi_j, \qquad \text{where} \quad \lambda_j = \frac{2}{\triangle t} \arctan\Big(\frac{\mu_j \triangle t}{2}\Big). \tag{2.5}$$

Proof. Combining (1.6) and (1.7) one finds that

$$z^k = \sum_j a_j \Big(\frac{2 + i\mu_j \triangle t}{2 - i\mu_j \triangle t}\Big)^k \Phi_j.$$

But
$$\frac{2+i\mu_j\triangle t}{2-i\mu_j\triangle t}=\frac{\sqrt{4+(\mu_j\triangle t)^2}e^{i\arctan(\mu_j\triangle t/2)}}{\sqrt{4+(\mu_j\triangle t)^2}e^{-i\arctan(\mu_j\triangle t/2)}}=e^{2i\arctan(\mu_j\triangle t/2)}.$$
Hence we deduce (2.5).

Now we introduce the discrete Fourier transform at scale $\triangle t$, which is one of the main ingredients used in the proof of Theorem 2.1.

Definition 2.4. Given any sequence $(u^k)\in l^2(\triangle t\mathbb{Z})$, we define its Fourier transform as:
$$\hat{u}(\tau)=\triangle t\sum_{k\in\mathbb{Z}}u^k\exp(-i\tau k\triangle t),\quad \tau\triangle t\in(-\pi,\pi]. \tag{2.6}$$
For any function $v\in L^2(-\pi/\triangle t,\pi/\triangle t)$,
$$\tilde{v}^k=\frac{1}{2\pi}\int_{-\pi/\triangle t}^{\pi/\triangle t}v(\tau)\exp(i\tau k\triangle t)\,d\tau,\quad k\in\mathbb{Z}. \tag{2.7}$$

According to Definition 2.4, the discrete Fourier transform at scale $\triangle t$ satisfies the following property:
$$\tilde{\hat{u}}=u,\quad \hat{\tilde{v}}=v, \tag{2.8}$$
and the Parseval's identity
$$\frac{1}{2\pi}\int_{-\pi/\triangle t}^{\pi/\triangle t}|\hat{u}(\tau)|^2\,d\tau=\triangle t\sum_{k\in\mathbb{Z}}|u^k|^2. \tag{2.9}$$
These properties will be used in the sequel.

Proof of Theorem 2.1. The proof is splitting into three parts.

Step 1**: Estimates in the class** $\mathcal{C}_{\delta/\triangle t}$. Let us take $z^0\in\mathcal{C}_{\delta/\triangle t}$ as in (2.4). Then the solution computed in time $k\triangle t$ in (2.5) has the same norm as the inital datum.
$$\left\|z^k\right\|_X^2=\sum_{|\mu_j|\leq\delta/\triangle t}|a_j|^2=\left\|z^0\right\|_X^2,\quad \forall\ k. \tag{2.10}$$
Further, since
$$\begin{aligned}\frac{z^k+z^{k+1}}{2} &= \frac{1}{2}\sum_{|\mu_j|\leq\delta/\triangle t}a_je^{ik\triangle t\lambda_j}\Big(1+\frac{2+i\mu_j\triangle t}{2-i\mu_j\triangle t}\Big)\Phi_j\\ &= \sum_{|\mu_j|\leq\delta/\triangle t}a_je^{ik\triangle t\lambda_j}\Big(\frac{2}{2-i\mu_j\triangle t}\Big)\Phi_j,\end{aligned}$$
we get that for any k,
$$\left\|\frac{z^k+z^{k+1}}{2}\right\|_X^2=\sum_{|\mu_j|\leq\delta/\triangle t}|a_j|^2\left|\frac{2}{2-i\mu_j\triangle t}\right|^2\geq\frac{1}{1+\left(\frac{\delta}{2}\right)^2}\left\|z^0\right\|_X^2. \tag{2.11}$$

Another important estimate in the class $\mathcal{C}_{\delta/\triangle t}$ is the following one :

$$\triangle t \left\| A\psi \right\|_X \leq \delta \left\| \psi \right\|_X , \qquad \forall \psi \in \mathcal{C}_{\delta/\triangle t} \ . \tag{2.12}$$

Step 2: The resolvent estimate. Set $\chi \in H^1(\mathbb{R})$. We denote by $\chi^k = \chi(k\triangle t)$. Let $g^k = \chi^k z^k$, and

$$f^k = \frac{g^{k+1} - g^k}{\triangle t} - A\Big(\frac{g^{k+1} + g^k}{2}\Big). \tag{2.13}$$

One can easily check that

$$\begin{aligned} f^k \quad &= \quad \frac{\chi^{k+1} - \chi^k}{\triangle t}\frac{z^{k+1} + z^k}{2} + \frac{\chi^{k+1} + \chi^k}{2}\frac{z^{k+1} - z^k}{\triangle t} \\ &\quad -A\Big(\frac{\chi^{k+1} + \chi^k}{2}\frac{z^{k+1} + z^k}{2} + \frac{\chi^{k+1} - \chi^k}{2}\frac{z^{k+1} - z^k}{2}\Big) \\ &= \quad \frac{\chi^{k+1} - \chi^k}{\triangle t}\Big(\frac{z^k + z^{k+1}}{2} - \frac{\triangle t}{4}A(z^{k+1} - z^k)\Big). \end{aligned} \tag{2.14}$$

Especially, recalling (2.10) and (2.12), (2.14) implies

$$\left\| f^k \right\|_X^2 \leq \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t}\Big)^2 \left\| z^0 \right\|_X^2 \Big(2 + \frac{\delta^2}{2}\Big).$$

In particular, $f^k \in l^2(\triangle t\mathbb{Z}; X)$.

Taking the Fourier transform of (2.13), for all $\tau \in (-\pi/\triangle t, \pi/\triangle t)$, we get

$$\begin{aligned} \hat{f}(\tau) &= \triangle t \sum_{k\in\mathbb{Z}} f^k \exp(-ik\triangle t\tau) \\ &= \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{g^{k+1} - g^k}{\triangle t} - A\Big(\frac{g^{k+1} + g^k}{2}\Big)\Big) \exp(-ik\triangle t\tau) \\ &= \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\exp(i\triangle t\tau) - 1}{\triangle t} - A\Big(\frac{\exp(i\triangle t\tau) + 1}{2}\Big)\Big) g^k \exp(-ik\triangle t\tau) \\ &= \Big(i\frac{2}{\triangle t}\tan(\frac{\tau\triangle t}{2})I - A\Big)\hat{g}(\tau)\exp\Big(i\frac{\tau\triangle t}{2}\Big)\cos\Big(\frac{\tau\triangle t}{2}\Big). \end{aligned}$$

We claim the following Lemma:

Lemma 2.5. *The solution* (z^k) *in* (2.5) *satisfy*

$$\begin{aligned} 2m^2\triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k + \chi^{k+1}}{2}\Big)^2 \left\| B\Big(\frac{z^k + z^{k+1}}{2}\Big) \right\|_Y^2 \\ \geq \left\| z^0 \right\|_X^2 \left[a_1\triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k + \chi^{k+1}}{2}\Big)^2 - a_2\triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t}\Big)^2 \right], \end{aligned} \tag{2.15}$$

with

$$\begin{aligned} a_1 &= \Big(1-\frac{1}{\beta}\Big)\Big(\frac{1}{1+\frac{\delta^2}{4}}\Big), \\ a_2 &= \Big(\frac{\beta-1}{4}\Big)(\triangle t)^2 + \frac{1}{8}\delta^2 m^2 C_B^2 + M^2\Big[1+\alpha+\frac{\delta^2}{4}\Big(1+\frac{1}{\alpha}\Big)\Big], \end{aligned} \tag{2.16}$$

for any $\alpha > 0$ *and* $\beta > 1$.

Proof. Applying the resolvent estimate (1.5) to the function

$$z(\tau) = \hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos(\frac{\tau\triangle t}{2}),$$

integrating on τ from $-\pi/\triangle t$ to $\pi/\triangle t$, it holds

$$M^2\int_{-\pi/\triangle t}^{\pi/\triangle t}\Big\|\hat{f}(\tau)\Big\|_X^2\,d\tau + m^2\int_{-\pi/\triangle t}^{\pi/\triangle t}\|Bz(\tau)\|_Y^2\,d\tau \geq \int_{-\pi/\triangle t}^{\pi/\triangle t}\|z(\tau)\|_X^2\,d\tau. \tag{2.17}$$

Applying Parseval's identity (2.7) to (2.17), and noticing that

$$\tilde{z} = \frac{g^k + g^{k+1}}{2},$$

we get

$$M^2\triangle t\sum_{k\in\mathbb{Z}}\big\|f^k\big\|_X^2 + m^2\triangle t\sum_{k\in\mathbb{Z}}\Big\|B\Big(\frac{g^k+g^{k+1}}{2}\Big)\Big\|_Y^2 \geq \triangle t\sum_{k\in\mathbb{Z}}\Big\|\frac{g^k+g^{k+1}}{2}\Big\|_X^2. \tag{2.18}$$

Now we estimate the three terms in (2.18). First of all, since f^k satisfies (2.14), we get for any positive α

$$\big\|f^k\big\|_X^2 \leq \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\big\|z^0\big\|_X^2\Big((1+\alpha)+\frac{\delta^2}{4}\Big(1+\frac{1}{\alpha}\Big)\Big), \tag{2.19}$$

where we used the inequality

$$\|a+b\|^2 \leq \|a\|^2\Big(1+\alpha\Big) + \|b\|^2\Big(1+\frac{1}{\alpha}\Big).$$

Second, since

$$\frac{g^{k+1}+g^k}{2} = \frac{\chi^{k+1}+\chi^k}{2}\frac{z^{k+1}+z^k}{2} + \frac{\triangle t}{2}\frac{\chi^{k+1}-\chi^k}{\triangle t}\frac{z^{k+1}-z^k}{2}, \tag{2.20}$$

we deduce that

$$\begin{aligned} \Big\|B\Big(\frac{g^{k+1}+g^k}{2}\Big)\Big\|_Y^2 &\leq 2\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\Big\|B\Big(\frac{z^{k+1}+z^k}{2}\Big)\Big\|_Y^2 \\ &\qquad + \frac{(\triangle t)^2}{2}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\Big\|B\Big(\frac{z^{k+1}-z^k}{2}\Big)\Big\|_Y^2 \\ &\leq 2\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\Big\|B\Big(\frac{z^{k+1}+z^k}{2}\Big)\Big\|_Y^2 + \frac{1}{8}\delta^2 C_B^2\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\big\|z^0\big\|_X^2. \end{aligned} \tag{2.21}$$

In (2.21) we use the fact that (recalling (2.1) and (2.12))

$$\left\| B\Big(\frac{z^{k+1}-z^k}{2}\Big)\right\|_Y = \left\| \triangle t BA\Big(\frac{z^{k+1}+z^k}{4}\Big)\right\|_Y \leq \frac{\delta C_B}{2\triangle t}\left\|z^0\right\|_X.$$

Finally, for any $\beta > 1$, recalling (2.10), (2.11) and (2.20), we compute

$$\begin{aligned}
\left\|\frac{g^{k+1}+g^k}{2}\right\|_X^2 &\geq \Big(1-\frac{1}{\beta}\Big)\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\left\|\frac{z^{k+1}+z^k}{2}\right\|_X^2 \\
&\quad -\Big((\beta-1)\frac{\triangle t}{2}\Big)^2\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\left\|\frac{z^{k+1}-z^k}{2}\right\|_X^2 \\
&\geq \Big(1-\frac{1}{\beta}\Big)\frac{1}{1+(\frac{\delta}{2})^2}\Big(\frac{\chi^{k+1}+\chi^k}{2}\Big)^2\left\|z^0\right\|_X^2 \\
&\quad -\Big((\beta-1)\frac{\triangle t}{2}\Big)^2\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2\left\|z^0\right\|_X^2,
\end{aligned} \tag{2.22}$$

where we used

$$\|a+b\|^2 \geq \Big(1-\frac{1}{\beta}\Big)\|a\|^2 - \Big(\beta-1\Big)\|b\|^2.$$

Applying (2.19), (2.21) and (2.22) to (2.18), we complete the proof of Lemma 2.5.

Step 3: To the observability estimate. This part is aimed to derive from Lemma 2.5 the observability estimate stated in Theorem 2.1 and to provide some estimates on the optimal time T_δ.

First of all, let us recall the following classical Lemma on Rieman sums:

Lemma 2.6. *Let $\chi(t) = \phi(t/T)$ with $\phi \in H^2 \cap H_0^1(0,1)$, extended by zero outside $(0,T)$. Recalling that $\chi^k = \chi(k\triangle t)$, the following estimates hold:*

$$\begin{aligned}
&\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 \geq T\|\phi\|_{L^2(0,1)}^2 - 2T\triangle t\|\phi\|_{L^2(0,1)}\left\|\dot\phi\right\|_{L^2(0,1)}; \\
&\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 \leq \frac{1}{T}\left\|\dot\phi\right\|_{L^2(0,1)}^2 + \frac{2}{T}\triangle t\left\|\dot\phi\right\|_{L^2(0,1)}\left\|\ddot\phi\right\|_{L^2(0,1)}.
\end{aligned} \tag{2.23}$$

Sketch of the proof. It is easy to show that for all $f(t) \in C^1(0,T)$, it holds

$$\begin{aligned}
\Big|\int_0^T f(t)dt - \triangle t\sum_{k=0}^{T/\triangle t} f(k\triangle t)\Big| &\leq \sum_{k=0}^{T/\triangle t}\iint_{[k\triangle t,(k+1)\triangle t]^2}|\dot f(s)|\,ds\,dt \\
&\leq \triangle t\int_0^T|\dot f|\,dt.
\end{aligned} \tag{2.24}$$

Replacing f by ϕ^2 we get the first line of (2.23). Similarly, the second line of (2.23) holds by replacing f by $|\dot\phi|^2$.

Taking Lemma 2.5 and 2.6 into account, the coefficient of $\left\|z^0\right\|_X^2$ in (2.15) tends to

$$k_{T,\delta,\alpha,\beta,\phi} = \Big(1-\frac{1}{\beta}\Big)\Big(\frac{1}{1+\frac{\delta^2}{4}}\Big)T\left\|\phi\right\|^2_{L^2(0,1)} - \Big(\frac{1}{8}\delta^2 m^2 C_B^2 + M^2\Big[1+\alpha+\frac{\delta^2}{4}\Big(1+\frac{1}{\alpha}\Big)\Big]\Big)\frac{1}{T}\left\|\dot{\phi}\right\|^2_{L^2(0,1)},$$

when $\triangle t \to 0$. Note that $k_{T,\delta,\alpha,\beta,\phi}$ is an increasing function of T. Let us define $T_{\delta,\alpha,\beta,\phi}$ as the unique positive solution of

$$k_{T,\delta,\alpha,\beta,\phi} = 0.$$

Then, for any time $T > T_{\delta,\alpha,\beta,\phi}$, choosing a positive $k_{T,\delta}$ such that

$$0 < k_{T,\delta} < k_{T,\delta,\alpha,\beta,\phi},$$

there exists $(\triangle t)_0 > 0$ such that for any $\triangle t < (\triangle t)_0$, (2.2) holds.

Hence the optimal time T_δ of Theorem (2.1) satisfies for any $\alpha > 0$, $\beta > 1$ and smooth function ϕ compactly supported in $[0,1]$:

$$T_\delta \le \frac{\left\|\dot{\phi}\right\|_{L^2}}{\left\|\phi\right\|_{L^2}}\Big[\frac{\beta}{\beta-1}\Big(1+\frac{\delta^2}{4}\Big)\Big]^{1/2}\Big[\frac{1}{8}\delta^2 m^2 C_B^2 + M^2\Big(1+\alpha+\frac{\delta^2}{4}(1+\frac{1}{\alpha})\Big)\Big]^{1/2}.$$

We optimize in α, β and ϕ, by taking $\alpha = \delta/2$, $\beta = \infty$ and

$$\phi(t) = \begin{cases} \sin(\pi t), & t \in (0,1) \\ 0, & \text{elsewhere}, \end{cases} \tag{2.25}$$

which is well-known to minimize the ratio

$$\frac{\left\|\dot{\phi}\right\|_{L^2}}{\left\|\phi\right\|_{L^2}}.$$

For this choice of ϕ, this quantity equals π, and thus we recover the estimate (2.3).

This theorem already has many applications. Indeed, this roughly says that for any continuous conservative system observable in finite time, there exists a time semi-discretization which uniformly preserves the observability property in finite time, provided the initial datum is filtered at a scale $1/\triangle t$. We will explain later using formally some microlocal tools why this scale seems to be the right one.

3. More General Systems.

In this section, we deal with more general discretization processes. We assume that the semi-discrete scheme has the form

$$z^{k+1} = \mathbb{T}_{\triangle t} z^k, \quad z^0 = z_0, \tag{3.1}$$

where $\mathbb{T}_{\triangle t}$ is a linear operator which has the same eigenvectors as the operator A. We also assume that the scheme is conservative. This implies that there exist real numbers $\lambda_{j,\triangle t}$ such that

$$\mathbb{T}_{\triangle t}\phi_j = \exp(i\lambda_{j,\triangle t}\triangle t)\phi_j. \tag{3.2}$$

Moreover, we assume that there is an explicit relation between $\lambda_{j,\triangle t}$ and μ_j under the following form:

$$\lambda_{j,\triangle t} = \frac{1}{\triangle t}\, h(\mu_j\triangle t), \tag{3.3}$$

where h is a smooth strictly increasing function satisfying:

$$|h(x)| \leq \pi. \tag{3.4}$$

This condition appears naturally in the application. Roughly speaking, it says that we cannot hope to measure frequencies higher than $\pi/\triangle t$ with a mesh-size $\triangle t$. In general, we also have that

$$\frac{h(x)}{x} \rightarrow_{x\rightarrow 0} 1.$$

Indeed, this property is equivalent to say that at low frequencies, the solution of (3.1) behaves similarly as the solution of (1.1). Note that the discretization analyzed in Section 2 satisfies all these properties.

Theorem 3.1. *Under assumptions* (3.2), (3.3) *and* (3.4), *for any* $\delta > 0$, *there exists a time* T_δ *such that for all* $T > T_\delta$, *there exists a constant* $k_{T,\delta} > 0$ *such that for all* $\triangle t$ *small enough, any solution of* (3.1) *with initial value* $z^0 \in \mathcal{C}_{\delta/\triangle t}$ *satisfies*

$$k_{T,\delta}\left\|z^0\right\|_X^2 \leq \triangle t \sum_{k=0}^{T/\triangle t}\left\|B\Big(\frac{z^k+z^{k+1}}{2}\Big)\right\|_Y^2. \tag{3.5}$$

Besides, we have the following estimate on T_δ*:*

$$\begin{aligned} T_\delta^2 < 2\pi^2\Big[&2\tan^2\Big(\frac{h(\delta)}{2}\Big) + C_B^2\delta^2 m^2\Big(1+\tan^2\Big(\frac{h(\delta)}{2}\Big)\Big) \\ &+ 2M^2\Big(\inf_{|\omega|\leq\delta}\Big\{\Big|\Big(1+\tan^2\Big(\frac{h(\delta)}{2}\Big)\Big)h'(\omega)\Big|\Big\}\Big)^{-2}\Big(1+\tan^4\Big(\frac{h(\delta)}{2}\Big)\Big)\Big]. \end{aligned} \tag{3.6}$$

Proof. The main idea is to use the previous result. First, we introduce an operator $A_{\triangle t}$ such that the solution of (3.1) coincides with the solution of the linear system

$$\frac{z^{k+1}-z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^k+z^{k+1}}{2}\Big), \quad z^0 = z_0. \tag{3.7}$$

This can be done with the definition

$$A_{\triangle t}\phi_j = k_{\triangle t}(\mu_j)\phi_j, \tag{3.8}$$

where

$$k_{\triangle t}(\omega) = \frac{2}{\triangle t}\tan\Big(\frac{h(\omega\triangle t)}{2}\Big). \tag{3.9}$$

Indeed, if

$$z_0 = \sum a_j \phi_j,$$

then the solution of(3.1) writes as

$$z^k = \sum a_j \phi_j \exp(i\lambda_j k\triangle t) = \sum a_j \phi_j \exp(ih(\mu_j \triangle t)k)$$

and the definition of $A_{\triangle t}$ follows naturally.

Then the result would be straightforward if we could prove the resolvent estimate for $A_{\triangle t}$. We will see in the sequel that a weak form of the resolvent estimate holds, and that this is actually sufficient to get the desired observability inequality. Note that $A_{\triangle t}$ coincides with A in the special case presented before in Section 2.

Step 1: A weak form of the resolvent estimate. By hypothesis,

$$M^2 \left\|(A - i\omega)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \quad z \in \mathcal{D}(A),\ \omega \in \mathbb{R}; \tag{3.10}$$

For

$$z = \sum_{j\ s.t.\ |\mu_j| \leq \delta/\triangle t} a_j \phi_j,$$

one can easily check that

$$\|(A - i\omega_1)z\|_X^2 = \sum |a_j|^2 \Big(\mu_j - \omega_1\Big)^2$$

and

$$\|(A_{\triangle t} - i\omega_2)z\|_X^2 = \sum |a_j|^2 \Big(k_{\triangle t}(\mu_j) - \omega_2\Big)^2.$$

Especially, for any $\omega_1 \in \mathbb{R}$, if

$$\omega_2 = k_{\triangle t}(\omega_1),$$

this last estimate takes the form

$$\|(A_{\triangle t} - i\omega_2)z\|_X^2 = \sum |a_j|^2 \Big(k_{\triangle t}(\mu_j) - k_{\triangle t}(\omega_1)\Big)^2$$

with $k_{\triangle t}$ as in (3.9). It follows that

$$\|(A_{\triangle t} - ik_{\triangle t}(\omega_1))z\|_X^2 \geq \Big(\inf_{|\omega| \leq \sup\{|\omega_1|, \delta/\triangle t\}} \Big\{|k'_{\triangle t}(\omega)|\Big\}\Big)^2 \sum |a_j|^2 \Big(\mu_j - \omega_1\Big)^2.$$

But

$$k'_{\triangle t}(\omega) = \Big(1 + \tan^2\Big(\frac{h(\omega\triangle t)}{2}\Big)\Big) h'(\omega\triangle t).$$

Hence, for any positive ϵ, setting

$$\alpha_{\triangle t,\epsilon} = k_{\triangle t}(\delta + \epsilon), \quad C_{\delta,\epsilon} = \Big(\inf\{k'_{\triangle t}(\omega) \mid |\omega| \leq \delta + \epsilon\}\Big)^{-1}, \tag{3.11}$$

we get the following weak resolvent estimate:

$$\begin{aligned} C_{\delta,\epsilon}^2 M^2 \left\|\Big(A_{\triangle t} - i\omega\Big)z\right\|_X^2 + m^2 \left\|Bz\right\|_Y^2 \geq \left\|z\right\|_X^2, \\ z \in \mathcal{C}_{\delta/\triangle t}, \quad \omega \in [-\alpha_{\triangle t,\epsilon}, \alpha_{\triangle t,\epsilon}]. \end{aligned} \tag{3.12}$$

Our purpose is now to show that this is enough to get the observability estimate.

Step 2: From the resolvent estimate (3.12) **to the Observability inequality.** We copy the previous proof. Set χ a smooth function satisfying:

$$\text{Supp}(\chi) \subset [0,T], \quad \|\chi\|_{L^\infty} \leq 1.$$

Note $\chi^k = \chi(k\triangle t)$. Let $g^k = \chi^k z^k$ and

$$f^k = \frac{g^{k+1} - g^k}{\triangle t} - A_{\triangle t}\Big(\frac{g^k + g^{k+1}}{2}\Big). \tag{3.13}$$

As before, using the discrete Fourier transform, we get

$$\hat{f}(\tau) = \Big(\frac{2i}{\triangle t}\tan\Big(\frac{\tau\triangle t}{2}\Big) - A_{\triangle t}\Big)\hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos\Big(\frac{\tau\triangle t}{2}\Big). \tag{3.14}$$

So, for any τ such that

$$|\tau| \leq \frac{h(\delta+\epsilon)}{\triangle t},$$

one can apply (3.12) :

$$C_{\delta,\epsilon}^2 M^2 \left\|\hat{f}(\tau)\right\|_X^2 + m^2 \left\|B\hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos\Big(\frac{\tau\triangle t}{2}\Big)\right\|_Y^2 \geq \left\|\hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos\Big(\frac{\tau\triangle t}{2}\Big)\right\|_X^2. \tag{3.15}$$

Integrating in $\tau \in (-\pi/\triangle t, \pi/\triangle t)$ and using Parseval's identity, we get

$$C_{\delta,\epsilon}^2 M^2 \triangle t \sum \|f^k\|_X^2 + m^2\triangle t \sum \left\|B\Big(\frac{g^k+g^{k+1}}{2}\Big)\right\|_Y^2 \geq \triangle t \sum \left\|\frac{g^k+g^{k+1}}{2}\right\|_X^2 - \frac{1}{2\pi}\int_{|\tau|>h(\delta+\epsilon)/\triangle t} \left\|\hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos\Big(\frac{\tau\triangle t}{2}\Big)\right\|_X^2 d\tau. \tag{3.16}$$

A decay estimate on $\hat{g}$. To use estimate (3.16), we have to study in detail $\hat{g}(\tau)\exp(i\frac{\tau\triangle t}{2})\cos\Big(\frac{\tau\triangle t}{2}\Big)$, whose inverse Fourier transform is

$$\frac{g^k+g^{k+1}}{2} = \frac{\chi^k+\chi^{k+1}}{2}\,\frac{z^k+z^{k+1}}{2} + \frac{(\triangle t)^2}{4}\,\frac{\chi^{k+1}-\chi^k}{\triangle t}\,\frac{z^{k+1}-z^k}{\triangle t}. \tag{3.17}$$

We recall the obvious relations:

$$\left\|\frac{z^k+z^{k+1}}{2}\right\|_X = \left\|\frac{z^0+z^1}{2}\right\|_X, \quad \left\|\frac{z^{k+1}-z^k}{\triangle t}\right\|_X = \left\|\frac{z^0+z^1}{2}\right\|_X$$

Setting

$$\begin{aligned} \frac{z^k+z^{k+1}}{2} &= \sum a_j\phi_j \exp(i\lambda_j(k+1/2)\triangle t)\cos\Big(\frac{\lambda_j\triangle t}{2}\Big), \\ b^k &= \frac{\chi^k+\chi^{k+1}}{2}, \quad c^k = \frac{z^k+z^{k+1}}{2}\, b^k, \end{aligned}$$

one can check that

$$\hat{c}(\tau) = \sum_j a_j \phi_j \exp(i\lambda_j \triangle t/2) \cos\Big(\frac{\lambda_j \triangle t}{2}\Big) \hat{b}(\tau - \lambda_j).$$

Besides,

$$|\hat{b}(\tau)| \leq \frac{(\triangle t)^2}{4\sin^2\left(\frac{\tau\triangle t}{2}\right)} \triangle t \sum_j |\Delta_{\triangle t} b^k| \leq \frac{(\triangle t)^2}{4\sin^2\left(\frac{\tau\triangle t}{2}\right)} |\mathrm{Supp}(\chi)| \left\|\partial_{tt}^2 \chi\right\|_{L^\infty}$$

It follows that for $\tau > h(\delta+\epsilon)/\triangle t$, setting $\beta_\epsilon = h(\delta+\epsilon) - h(\delta)$,

$$\|\hat{c}(\tau)\|_X^2 \leq \left\|\frac{z^0+z^1}{2}\right\|_X^2 \left(\frac{(\triangle t)^2}{4\sin^2\left(\frac{\beta_\epsilon}{2}\right)} |\mathrm{Supp}(\chi)| \left\|\partial_{tt}^2 \chi\right\|_{L^\infty}\right)^2. \tag{3.18}$$

Similarly, one can obtain that, if c_2 denotes the second term in (3.17), that is

$$c_2^k = \frac{(\triangle t)^2}{4} \frac{\chi^{k+1}-\chi^k}{\triangle t} \frac{z^{k+1}-z^k}{\triangle t},$$

then its Fourier transform satisfies for $\tau > h(\delta+\epsilon)$,

$$\|\hat{c}_2(\tau)\|_X^2 \leq \Big(\frac{\triangle t}{2}\Big)^4 \left\|\frac{z^1-z^0}{\triangle t}\right\|_X^2 \left(\frac{(\triangle t)^2}{4\sin^2\left(\frac{\beta_\epsilon}{2}\right)} |\mathrm{Supp}(\chi)| \left\|\partial_{ttt}^3 \chi\right\|_{L^\infty}\right)^2.$$

It follows that the righthand side in (3.16) (RHS in short) can be bounded from below:

$$\begin{aligned} RHS \geq & \left(\frac{\triangle t}{2} \sum \Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 \right. \\ & \left. - \frac{1}{\triangle t}\Big(\frac{(\triangle t)^2}{4\sin^2\left(\frac{\beta_\epsilon}{2}\right)} |\mathrm{Supp}(\chi)| \left\|\partial_{tt}^2 \chi\right\|_{L^\infty}\Big)^2\right) \left\|z^0+z^1\right\|_X^2 \\ & - \left(2(\triangle t)^3 \sum \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 \right. \\ & \left. - \Big(\frac{\triangle t}{2}\Big)^4 \Big(\frac{(\triangle t)^2}{4\sin^2\left(\frac{\beta}{2}\right)} |\mathrm{Supp}(\chi)| \left\|\partial_{ttt}^3 \chi\right\|_{L^\infty}\Big)^2\right) \left\|\frac{z^1-z^0}{\triangle t}\right\|_X^2. \end{aligned} \tag{3.19}$$

According to this inequality and copying the proof of the previous section, we obtain the desired observability inequality. For completeness, we sketch the proof.

To the observability estimate. Using

$$f^k = \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)\Big(\frac{z^k+z^{k+1}}{2} - \Big(\frac{\triangle t}{2}\Big)^2 A_{\triangle t}\Big(\frac{z^{k+1}-z^k}{\triangle t}\Big)\Big),$$

one can check that

$$\triangle t \sum \|f^k\|_X^2 \leq 2\triangle t \sum \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 \left(\left\|\frac{z^0+z^1}{2}\right\|_X^2 + \tan\Big(\frac{h(\delta)}{2}\Big)^2 \left\|\frac{z^1-z^0}{2}\right\|_X^2 \right).$$

Besides,

$$\triangle t \sum \left\| B\Big(\frac{g^k+g^{k+1}}{2}\Big)\right\|_Y^2 \leq 2\triangle t \sum \Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 \left\| B\Big(\frac{z^k+z^{k+1}}{2}\Big)\right\|_Y^2 + 2\triangle t \sum \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 C_B \delta^2 \|z^0\|_X^2 .$$

Using the explicit expansion of z, one can check that

$$\left\|\frac{z^1-z^0}{\triangle t}\right\|_X^2 \leq \tan^2\Big(\frac{h(\delta)}{2}\Big) \left\|\frac{z^0+z^1}{2}\right\|_X^2 ,$$
$$\left\|\frac{z^0+z^1}{2}\right\|_X^2 \geq \cos^2\Big(\frac{h(\delta)}{2}\Big) \|z^0\|_X^2 .$$

It follows that we get

$$\begin{aligned} \triangle t \sum \Big(\frac{\chi^k+\chi^{k+1}}{2}\Big)^2 \left\| B\Big(\frac{z^k+z^{k+1}}{2}\Big)\right\|_Y^2 &\geq C_1 \left\|\frac{z^0+z^1}{2}\right\|_X^2 \\ &\geq C_1 \cos^2\Big(\frac{h(\delta)}{2}\Big) \|z^0\|_X^2 , \end{aligned}$$

where C_1 depends on $\chi, \triangle t$ and δ in such a way that for any δ, for any smooth function ϕ with compact support, taking $\chi_T(t) = \phi(t/T)$ makes C_1 blows up when $T \to \infty$ uniformly in $\triangle t$. More precisely, one can compute its limit when $\triangle t \to 0$:

$$C_1(\chi, \delta, \epsilon, \triangle t) \to_{\triangle t \to 0} \frac{1}{2}\int |\chi(t)|^2 \, dt - \Big(\int |\chi'(t)|^2 \, dt\Big)\Big[2\tan^2\Big(\frac{h(\delta)}{2}\Big) + C_B^2 \delta^2 m^2 \Big(1+\tan^2\Big(\frac{h(\delta)}{2}\Big)\Big) + 2M^2 C_{\delta,\epsilon}^2 \Big(1+\tan^4\Big(\frac{h(\delta)}{2}\Big)\Big)\Big].$$

Note that ϵ does not appear as a significant parameter. This parameter actually enters only in the convergence speed of C_1 when $\triangle t \to 0$.
Then we can conclude from (3.20) by using smooth approximations of ϕ defined in (2.25) and the continuity of the function h' in δ that there exists a time

$$T_\delta^2 \leq 2\pi^2 \left[2\tan^2\Big(\frac{h(\delta)}{2}\Big) + C_B^2\delta^2 m^2\Big(1+\tan^2\Big(\frac{h(\delta)}{2}\Big)\Big) + 2M^2 C_{\delta,\epsilon=0}^2 \Big(1+\tan^4\Big(\frac{h(\delta)}{2}\Big)\Big)\right],$$

such that the observability inequality holds for any $T > T_\delta$ for small enough $\triangle t$.

Application. Let us present an application of the previous result to the so-called fourth order Gauss method discretization of equation (1.1).

To begin with, we introduce the following discrete system:

$$\begin{cases} \kappa_i = A\Big(z^k + \triangle t \sum_{j=1}^{2} \alpha_{ij}\kappa_i\Big), \qquad i = 1,2, \\ z^{k+1} = z^k + \dfrac{\triangle t}{2}(\kappa_1 + \kappa_2), \qquad (\alpha_{ij}) = \begin{pmatrix} \frac14 & \frac14 - \frac{\sqrt3}{6} \\ \frac14 + \frac{\sqrt3}{6} & \frac14 \end{pmatrix}. \\ z^0 \in \mathcal{C}_{\delta/\triangle t} \text{ given}, \end{cases} \tag{3.20}$$

The case where $\mu_j \triangle t \geq 2\sqrt{3}$ leads to instability [3]-[4], and then we will assume from the beginning that

$$\delta < 2\sqrt{3}.$$

From the previous section, we could use Theorem 3.1 as soon as the eigenvectors of the continuous system (1.1) are eigenvectors for the semi-dicrete system (3.20) as well. But if $z^0 = \Phi_j$, an easy computation shows that

$$z^1 = \exp(i\ell_j \triangle t) z^0,$$

where

$$\ell_j = \frac{2}{\triangle t}\arctan\Big(\frac{\mu_j \triangle t}{2 - (\mu_j\triangle t)^2/6}\Big). \tag{3.21}$$

In other words,

$$\ell_j \triangle t = h(\mu_j \triangle t),$$

where

$$h(x) = 2\arctan\Big(\frac{x}{2 - x^2/6}\Big).$$

Then, a simple application of Theorem 3.1 gives :

Theorem 3.2. *For any $\delta \in (0, 2\sqrt{3})$, there exists a time $T_\delta > 0$ such that for any $T > T_\delta$, there exists $(\triangle t)_0$ such that for all $\triangle t < (\triangle t)_0$, there exists a constant $k_{T,\delta} > 0$, independent of $\triangle t$, such that the solutions of the system* (3.20) *satisfy*

$$k_{T,\delta} \left\|z^0\right\|_X^2 \leq \triangle t \sum_{k=0}^{T/\triangle t} \left\|Bz^k\right\|_Y^2, \quad \forall\, z^0 \in \mathcal{C}_{\delta/\triangle t}. \tag{3.22}$$

Note that Theorem 3.1 also provides an estimate on T_δ by using (3.6). Indeed, one can easily compute the function $k_{\triangle t}$ in (3.9) and its derivative :

$$k_{\triangle t}(\omega) = \frac{\omega}{1 - \frac{1}{12}\,(\omega\triangle t)^2}.$$

4. Second order systems

In this section we investigate the second order evolution equations occurring in the study of vibrating systems. More precisely, let H be a Hilbert space endowed with the norm $\|\cdot\|$ and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint positive operator with compact resolvent. Consider the initial value problem:

$$\ddot{u}(t) + A_0 u(t) = 0, \tag{4.1}$$

$$u(0) = u_0, \qquad \dot{u}(0) = v_0, \tag{4.2}$$

which can be seen as a generic model for the free vibrations of elastic structures such as strings, beams, membranes, plates or three-dimensional elastic bodies.

The energy of (4.1)

$$E(t) = \|\dot{u}(t)\|^2 + \left\| A_0^{\frac{1}{2}} u(t) \right\|^2, \tag{4.3}$$

is constant in time.

The output function $y(t)$ is stated by

$$y(t) = B_1 u(t), \qquad \text{or} \qquad y(t) = B_2 \dot{u}(t). \tag{4.4}$$

In the sequel we will focus on the second case, that is $y(t) = B_2 \dot{u}(t)$, but our proof works as well for $y(t) = B_1 u(t)$.

To underline the link with the conservative system (1.1)-(1.2), we now transform the second order model (4.1)–(4.4) into a first order one (1.1)–(1.2).

Setting $v = \dot{u}$, equation (4.1) can be rewritten as

$$\dot{z} = Az, \quad z = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}, \tag{4.5}$$

for which the energy space is defined by

$$X = \mathcal{D}(A_0^{\frac{1}{2}}) \times H$$

and the domain of A satisfies

$$\mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}(A_0^{\frac{1}{2}}).$$

These two vector spaces are Hilbert spaces for the following norms:

$$\|z\|_X^2 = \left\| A_0^{\frac{1}{2}} u \right\|^2 + \|v\|^2, \qquad \|z\|_{\mathcal{D}(A)}^2 = \|A_0 u\|^2 + \left\| A_0^{\frac{1}{2}} v \right\|^2$$

It is easy to check that A is a skew adjoint operator with compact resolvent and its spectrum is explicitly given by the spectrum of A_0. Indeed, if we denote by a sequence of non-negative numbers μ_j^2 the eigenvalues of A_0, i.e.,

$$A_0 \phi_j = \mu_j^2 \phi_j, \qquad j \in \mathbb{N}^*,$$

with corresponding eigenvectors (ϕ_j), then the eigenvalues of A are $\pm i\mu_j$, with corresponding eigenvectors

$$\Phi_{\pm j} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm \frac{1}{i\mu_j} \phi_j \\ \phi_j \end{pmatrix}.$$

Consequently, the admissible observation operator B is given by $B = (0, B_2)$, which is assumed to belong to $\mathcal{L}(\mathcal{D}(A_0^{\frac{1}{2}}), Y)$. Therefore there exists a constant C_B such that

$$\|B_2 v\|_Y \leq C_B \left\| A_0^{\frac{1}{2}} v \right\|, \qquad v \in \mathcal{D}(A_0^{\frac{1}{2}}). \tag{4.6}$$

In the sequel, we assume that the system (4.1)–(4.4) is observable.

Now we present the time discrete schemes we are interested in. For any $\triangle t > 0$, we consider the following time Newmark approximation scheme of the system (4.1)–(4.2):

$$\begin{cases} \dfrac{u^{k+1} + u^{k-1} - 2u^k}{(\triangle t)^2} + A_0\Big(\beta u^{k+1} + (1-2\beta)u^k + \beta u^{k-1}\Big) = 0, \\ \Big(\dfrac{u^0 + u^1}{2}, \dfrac{u^1 - u^0}{\triangle t}\Big) = (u_0, v_0) \in \mathcal{D}(A_0^{\frac{1}{2}}) \times H, \end{cases} \tag{4.7}$$

where β is a positive number.

The energy of (4.7) given by

$$E^k = \frac{1}{2}\left\| A_0^{\frac{1}{2}}\Big(\frac{u^k + u^{k+1}}{2}\Big)\right\|^2 + \frac{1}{2}\left\|\frac{u^{k+1} - u^k}{\triangle t}\right\|^2 + (4\beta - 1)\frac{(\triangle t)^2}{8}\left\| A_0^{\frac{1}{2}}\Big(\frac{u^{k+1} - u^k}{\triangle t}\Big)\right\|^2, \ k \in \mathbb{Z}, \tag{4.8}$$

which is a discrete counterpart of the continuous energy (4.3), is constant. In view of (4.8), we will assume in the sequel that $\beta \geq 1/4$, for which system (4.7) is unconditionally stable.

We assume that the output is given by the following discretization of (4.4)

$$y^k = B_2\Big(\frac{u^{k+1} - u^k}{\triangle t}\Big). \tag{4.9}$$

Let us explain why the study of system (4.7)-(4.9) cannot be tackled from Theorems 2.1 and 3.1. For this, we express this system as a first order one. Setting

$$z^k = \begin{pmatrix} \dfrac{u^k + u^{k+1}}{2} \\ \dfrac{u^{k+1} - u^k}{\triangle t} \end{pmatrix}, \qquad A_{\triangle t} = \begin{pmatrix} 0 & I \\ -A_0\Big(I + (\triangle t)^2(\beta - \frac{1}{4})A_0\Big)^{-1} & 0 \end{pmatrix},$$

system (4.7) can be rewritten as

$$\frac{z^{k+1} - z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^k + z^{k+1}}{2}\Big), \quad z^0 = \begin{pmatrix} u^0 \\ v_0 \end{pmatrix}. \tag{4.10}$$

One can check that the eigenvectors of the operator $A_{\triangle t}$ are

$$\Phi_{\pm j, \triangle t} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm\frac{1}{i\lambda_{j,\triangle t}}\phi_j \\ \phi_j \end{pmatrix}, \quad \lambda_{j,\triangle t} = \mu_j \frac{1}{\sqrt{1 + (\beta - 1/4)(\triangle t)^2 \mu_j^2}}. \tag{4.11}$$

It follows that this approximation scheme does not conserve the eigenvectors of the continuous system, except in the case where $\beta = 1/4$, which actually corresponds to the natural midpoint discretization already seen in Section 2. This is particularly obvious when looking at system (4.7) as (4.10).

As before, for any $s > 0$, we define as in (1.10)

$$\mathcal{C}_s = \text{span}\,\{\Phi_j : \text{the corresponding } i\mu_j \text{ satisfies } |\mu_j| \leq s\}, \tag{4.12}$$

Note that for any $s > 0$, the vector space $\mathcal{C}_s$ coincides with the space

$$\tilde{\mathcal{C}}_{s,\triangle t} = \text{span}\,\{\Phi_{j,\triangle t} : \text{the corresponding } \lambda_{j,\triangle t} = \mu_j \, \frac{1}{\sqrt{1+(\beta-1/4)(\triangle t)^2\mu_j^2}} \text{ satisfies } |\mu_j| \leq s\}.$$

In other words, this space is stable under the actions of the discrete semi-groups generated by $A_{\triangle t}$. It follows that taking the initial value in that space makes sense.

Once again, we claim that the semi-discrete system (4.7)-(4.9) is observable when filtering in the class $\mathcal{C}_{\delta/\triangle t}$:

Theorem 4.1. *Let $\beta \geq 1/4$ and $\delta > 0$. There exists a time T_δ such that for all $T > T_\delta$, there exists a positive constant $k_{T,\delta}$, such that for $\triangle t$ small enough, the solution of* (4.7) *with initial datum $(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}$ satisfies*

$$k_{T,\delta} E^0 \leq \triangle t \sum_{k\triangle t \in (0,T)} \left\| B_2\Big(\frac{u^{k+1}-u^k}{\triangle t}\Big)\right\|_Y^2. \tag{4.13}$$

Remark 4.2. The assumption that $B = (0\ ; B_2)$ is done only to simplify the presentation, the theorem is still true when dealing with an observable admissible operator $B = (B_1\ ; 0)$, i.e. with the output function $y(t) = B_1 u(t)$, see Theorem 4.6 below. However, we are not able to handle the case where the observable operator $B = (B_1\ ; B_2)$ acts on the two components u and $\dot{u}$, that is $y(t) = B_1 u(t) + B_2 \dot{u}(t)$, except when $\beta = 1/4$, where Theorem 2.1 applies. This probably comes from technical reason.

Before entering into the proof, we give some preliminaries properties on the solutions of the Newmark approximation scheme (4.7).

Lemma 4.3. *The solution of* (4.7) *can be expanded as*

$$u^k = \sum_{j\in\mathcal{Z}^*} a_j \exp(i\omega_j k\triangle t)\phi_j, \tag{4.14}$$

where

$$\omega_j = \frac{2}{\triangle t}\arctan\left(\frac{\triangle t\mu_j}{2\sqrt{1+(\beta-1/4)\triangle t^2\mu_j^2}}\right), \tag{4.15}$$

and the coefficients $(a_j)_{j\in\mathcal{Z}^*}$ are determined by the initial data.

Proof. It is sufficient to look for solutions of the form $u^k = \exp(i\omega_j k\triangle t)\phi_j$. System (4.7) is then equivalent to

$$\Big[\frac{e^{i\omega_j(k+1)\triangle t} + e^{i\omega_j(k-1)\triangle t} - 2e^{i\omega_j k\triangle t}}{(\triangle t)^2}\Big]\phi_j \\ + A_0\Big[\beta e^{i\omega_j(k+1)\triangle t} + (1-2\beta)e^{i\omega_j k\triangle t} + \beta e^{i\omega_j(k-1)\triangle t}\Big]\phi_j = 0.$$

This gives that

$$-\frac{4}{(\triangle t)^2}\sin^2\Big(\frac{\omega_j\triangle t}{2}\Big) + \mu_j^2\Big(1 - 4\beta\sin^2\Big(\frac{\omega_j\triangle t}{2}\Big)\Big) = 0.$$

This leads to

$$\sin^2\Big(\frac{\omega_j\triangle t}{2}\Big)(1+\beta\mu_j^2(\triangle t)^2) = \frac{\mu_j^2(\triangle t)^2}{4}.$$

Thus we deduce that

$$\tan^2\Big(\frac{\omega_j\triangle t}{2}\Big) = \frac{\mu_j^2(\triangle t)^2}{4}\frac{1}{1+(\beta-1/4)\mu_j^2(\triangle t)^2}$$

by which we arrive at (4.15).

Besides, using the orthogonality of the eigenvectors, one can compute the energy in terms of the coefficients (a_j)

$$E^k = \sum_{j>0}\Big(|a_j|^2 + |a_{-j}|^2\Big)\mu_j^2\frac{1+(\beta-1/4)\mu_j^2(\triangle t)^2}{1+\beta\mu_j^2(\triangle t)^2}.$$

This implies in particular that if the initial data is taken in $\mathcal{C}_{\delta/\triangle t}$, the energy of system (4.7) is equivalent to

$$\bar{E} = \sum_{j>0}\Big(|a_j|^2 + |a_{-j}|^2\Big)\mu_j^2. \tag{4.16}$$

Indeed, in this case, we get :

$$\frac{1}{1+\beta\delta^2}\bar{E} \le \bar{E^k} \le \bar{E}.$$

Proof of Theorem 4.1. Set $\delta > 0$, $(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}$, and let u^k be the solution of (4.7).

By assumption, the continuous system (A, B) is exactly observable, where B has the particular form $B = (0\,; B_2)$. Hence, from (1.5) the resolvent estimate holds true and takes the following form :

$$M^2\Big(\Big\|A_0^{\frac{1}{2}}(i\omega z_1 - z_2)\Big\|^2 + \|A_0 z_1 + i\omega z_2\|^2\Big) + m^2\|B_2 z_2\|_Y^2 \ge \Big\|A_0^{\frac{1}{2}}z_1\Big\|^2 + \|z_2\|^2, \\ \forall\omega\in\mathbb{R},\quad z = \begin{pmatrix} z_1 \\ z_2\end{pmatrix} \in \mathcal{D}(A), \tag{4.17}$$

for some constants $M, m > 0$.

We follow the proof given in Theorem 2.1. We split the proof into three parts.

Step 1: Fourier Transform estimates. Here we introduce the functions we will use through the proof and we study their Fourier transform.

Set $\chi \in \mathcal{D}(\mathbb{R})$. We denote $\chi^k = \chi(k\triangle t)$. Let $g^k = \chi^k u^k$ and

$$f^k = \frac{g^{k+1} + g^{k-1} - 2g^k}{(\triangle t)^2} + A_0\Big(\beta g^{k+1} + (1-2\beta)g^k + \beta g^{k-1}\Big).$$

Taking (4.7) into account, we get that

$$\begin{aligned} f^k = & \frac{1}{\triangle t^2}\Big(\chi^{k+1} + \chi^{k-1} - 2\chi^k\Big)\Big(u^k + \beta(\triangle t)^2 A_0 u^k\Big) \\ & + \frac{\chi^{k+1} - \chi^k}{\triangle t}\Big(\frac{u^{k+1} - u^k}{\triangle t} + \beta(\triangle t)^2 A_0\Big(\frac{u^{k+1} - u^k}{\triangle t}\Big)\Big) \\ & + \frac{\chi^k - \chi^{k-1}}{\triangle t}\Big(\frac{u^k - u^{k-1}}{\triangle t} + \beta(\triangle t)^2 A_0\Big(\frac{u^k - u^{k-1}}{\triangle t}\Big)\Big). \quad (4.18) \end{aligned}$$

In particular, both f and g are in $l^2(\triangle t\mathbb{Z}, H)$ and then their Fourier transform is well-defined and belong to $L^2(-\frac{\pi}{\triangle t}, \frac{\pi}{\triangle t})$. From the definition of f^k we deduce

$$\hat{f}(\tau) = \Big(-\frac{4}{(\triangle t)^2}\frac{\sin^2\left(\frac{\tau\triangle t}{2}\right)}{1 - 4\beta\sin^2\left(\frac{\tau\triangle t}{2}\right)} + A_0\Big)\hat{g}(\tau)\Big(1 - 4\beta\sin^2\left(\frac{\tau\triangle t}{2}\right)\Big).$$

Note that the symbol of the operator switches from hyperbolic type to elliptic one when the sign of the term

$$1 - 4\beta\sin^2\left(\frac{\tau\triangle t}{2}\right)$$

switches from positive to negative. Hence, we shall be careful in using the resolvent estimate (4.17), which gives relevant estimates only in the hyperbolic range.

However, we claim that the frequencies involved in our problem actually are in this range, that is

$$\{\tau \ s.t.\ 1 - 4\beta\sin^2\left(\frac{\tau\triangle t}{2}\right) > 0\}.$$

More precisely, similarly as in the proof given in the previous section, the Fourier transform of g is mainly concentrated in $(-\delta/\triangle t, \delta/\triangle t)$.

Lemma 4.4. *For any $\epsilon > 0$, set*

$$\alpha(x) = 2\arcsin\Big(\frac{x}{2\sqrt{1+\beta x^2}}\Big), \qquad \alpha_\epsilon = \alpha(\delta+\epsilon), \qquad \gamma_\epsilon = \alpha(\delta+\epsilon) - \alpha(\delta). \quad (4.19)$$

Then, recalling the definition of $\bar{E}$ *in* (4.16), *we have the tho following estimates:*

$$\frac{1}{2\pi}\int_{-\pi/\triangle t}^{\pi/\triangle t}\left\|A_0^{\frac12}\hat{g}(\tau)\right\|^2 d\tau \le \frac{1}{2\pi}\int_{-\alpha_\epsilon/\triangle t}^{\alpha_\epsilon/\triangle t}\left\|A_0^{\frac12}\hat{g}(\tau)\right\|^2 d\tau + \frac{\alpha_\epsilon}{\pi\triangle t}\Big(\frac{\triangle t}{\sin(\frac{\gamma_\varepsilon}{2})}\Big)^4|\mathrm{Supp}(\chi)|^2\Big|\frac{d^2\chi}{dt^2}\Big|_\infty^2\bar{E}. \quad (4.20)$$

$$\frac{1}{2\pi}\int_{-\pi/\triangle t}^{\pi/\triangle t}\left\|\frac{2}{\triangle t}\sin\Big(\frac{\tau\triangle t}{2}\Big)\hat{g}(\tau)\right\|^2 d\tau \le \frac{1}{2\pi}\int_{-\alpha_\varepsilon/\triangle t}^{\alpha_\epsilon/\triangle t}\left\|\frac{2}{\triangle t}\sin\Big(\frac{\tau\triangle t}{2}\Big)\hat{g}(\tau)\right\|^2 d\tau + \frac{\alpha_\epsilon}{\pi\triangle t}\Big(\frac{\triangle t}{\sin(\frac{\gamma_\varepsilon}{2})}\Big)^2|\mathrm{Supp}(\chi)|^2\Big|\frac{d^2\chi}{dt^2}\Big|_\infty^2\bar{E}. \quad (4.21)$$

Proof. Since the function u^k is in $l^\infty(\triangle t\mathbb{Z}, H)$, it belongs to $\mathcal{S}'(\triangle t\mathbb{Z}, H)$, and then its Fourier transform makes sense. Actually, using (4.14),

$$\hat{u}(\tau) = \sum_{k\ s.t. |\mu_k|<\delta/\triangle t} a_k\delta_{\omega_k}(\tau)\phi_k,$$

and so

$$\mathrm{Supp}(\hat{u}) \subset \Big[-\frac{2}{\triangle t}\arcsin\Big(\frac{\delta}{2\sqrt{1+\beta\delta^2}}\Big), \frac{2}{\triangle t}\arcsin\Big(\frac{\delta}{2\sqrt{1+\beta\delta^2}}\Big)\Big].$$

In particular, we can estimate $\hat{g}$ by using

$$\hat{g}(\tau) = \int_{-\pi/\triangle t}^{\pi/\triangle t}\hat{\chi}(\tau-\eta)\hat{u}(\eta)d\eta = \sum_{k\ s.t. |\mu_k|<\delta/\triangle t} a_k\phi_k\hat{\chi}(\tau-\omega_k).$$

Recalling the definitions α_ϵ and γ_ϵ in (4.19), taking into account the above two formulas, for $|\tau| \ge \alpha_\varepsilon/\triangle t$, we get

$$\begin{aligned}\left\|A_0^{\frac12}\hat{g}(\tau)\right\|^2 &\le \sum_{k\ s.t. |\mu_k|<\delta/\triangle t}|a_k|^2|\mu_k|^2|\hat{\chi}(\tau-\omega_k)|^2\\ &\le \sup_{|\xi|>\gamma_\varepsilon/\triangle t}\Big\{|\hat{\chi}(\xi)|^2\Big\}\Big(\sum_{k\ s.t. |\mu_k|<\delta/\triangle t}|a_k|^2|\mu_k|^2\Big), \quad (4.22)\end{aligned}$$

On the other hand, since χ is smooth and compactly supported, one can use as before (see (3.18)) that

$$|\hat{\chi}(\tau)| \le \Big(\frac{\triangle t}{\sin(\frac{\tau\triangle t}{2})}\Big)^2|\mathrm{Supp}(\chi)|\Big|\frac{d^2\chi}{dt^2}\Big|_\infty. \quad (4.23)$$

Integrating (4.22) on τ in the interval $|\tau| > \alpha_\epsilon/\triangle t$, taking into account (4.23), we attain

$$\int_{|\tau|>\alpha_\epsilon/\triangle t}\left\|A_0^{\frac12}\hat{g}(\tau)\right\|^2 d\tau \le \frac{2\alpha_\epsilon}{\triangle t}\Big(\frac{\triangle t}{\sin(\frac{\gamma_\varepsilon}{2})}\Big)^4|\mathrm{Supp}(\chi)|^2\Big|\frac{d^2\chi}{dt^2}\Big|_\infty^2\bar{E},$$

which implies (4.20). Inequality (4.21) can be derived similarly and is left to the reader.

Step 2: The resolvent estimate. Now we apply the resolvent estimate (4.17) to the desired function $\hat{f}(\tau)$. For τ such that

$$|\tau| \leq \frac{\alpha_\varepsilon}{\triangle t},$$

which satisfies

$$1 \geq 1 - 4\beta \sin^2\Big(\frac{\tau \triangle t}{2}\Big) \geq \frac{1}{1+\beta(\delta+\varepsilon)^2}, \tag{4.24}$$

taking

$$\begin{aligned} \omega(\tau) &= \frac{2}{\triangle t} \frac{\sin\Big(\frac{\tau\triangle t}{2}\Big)}{\sqrt{1-4\beta\sin^2\Big(\frac{\tau\triangle t}{2}\Big)}} \\ z_1(\tau) &= \hat{g}(\tau)\Big(1-4\beta\sin^2\Big(\frac{\tau\triangle t}{2}\Big)\Big) \\ z_2(\tau) &= i\omega(\tau)z_1(\tau) \end{aligned}$$

into (4.17), it gives

$$M^2\left\|\hat{f}(\tau)\right\|^2 + m^2\left\|i\omega(\tau)B_2 z_1(\tau)\right\|_Y^2 \geq \left\|A_0^{\frac{1}{2}} z_1(\tau)\right\|^2 + \left\|i\omega(\tau)z_1(\tau)\right\|^2. \tag{4.25}$$

Using (4.24) and integrating in τ on $[-\pi/\triangle t, \pi/\triangle t]$, we obtained that:

$$\begin{aligned} M^2 \int_{-\pi/\triangle t}^{\pi/\triangle t} \left\|\hat{f}(\tau)\right\|^2 d\tau + m^2 \int_{-\pi/\triangle t}^{\pi/\triangle t} \left\|B_2\Big(\exp(i\frac{\tau\triangle t}{2})\frac{2}{\triangle t}\sin\Big(\frac{\tau\triangle t}{2}\Big)\hat{g}(\tau)\Big)\right\|_Y^2 d\tau \\ \geq \Big(\frac{1}{1+\beta(\delta+\varepsilon)^2}\Big)^2 \int_{-\alpha_\varepsilon/\triangle t}^{\alpha_\varepsilon/\triangle t} \left\|A_0^{\frac{1}{2}}\hat{g}(\tau)\right\|^2 d\tau \\ + \frac{1}{1+\beta(\delta+\varepsilon)^2} \int_{-\alpha_\varepsilon/\triangle t}^{\alpha_\varepsilon/\triangle t} \left\|\exp(i\frac{\tau\triangle t}{2})\frac{2}{\triangle t}\sin\Big(\frac{\tau\triangle t}{2}\Big)\hat{g}(\tau)\right\|^2 d\tau. \end{aligned}$$

Using the estimates (4.20) and (4.21), Parseval's identity leads to

$$\begin{aligned} &M^2\triangle t\sum_{k\in\mathbb{Z}}\left\|f^k\right\|^2 + m^2\triangle t\sum_{k\in\mathbb{Z}}\left\|B_2\Big(\frac{g^{k+1}-g^k}{\triangle t}\Big)\right\|_Y^2 \\ &\geq \Big(\frac{1}{1+\beta(\delta+\varepsilon)^2}\Big)^2\Big(\triangle t\sum_{k\in\mathbb{Z}}\left\|A_0^{\frac{1}{2}}g^k\right\|^2 - \frac{(\triangle t)^3}{\sin^4(\gamma_\varepsilon/2)}|\mathrm{Supp}(\chi)|^2\left|\frac{d^2\chi}{dt^2}\right|_\infty^2\bar{E}\Big) \\ &+\frac{1}{1+\beta(\delta+\varepsilon)^2}\Big(\triangle t\sum_{k\in\mathbb{Z}}\left\|\frac{g^{k+1}-g^k}{\triangle t}\right\|^2 - \frac{\triangle t}{\sin^2(\gamma_\varepsilon/2)}|\mathrm{Supp}(\chi)|^2\left|\frac{d^2\chi}{dt^2}\right|_\infty^2\bar{E}\Big). \end{aligned} \tag{4.26}$$

Step 3: To the observability inequality. The last part is very close to the second step presented in Theorem 3.1. For completeness, we present however the main steps.

For the first term of (4.26), using the explicit expression of f^k given in (4.18) and the following remark

$$z = \sum_{j \ s.t \ |\mu_j| < \delta/\triangle t} a_j \phi_j \Longrightarrow \left\| \Big(I + \beta (\triangle t)^2 A_0 \Big) z \right\|^2 \leq (1 + \beta\delta)^2 \left\| z \right\|^2,$$

we easily get that

$$\begin{aligned} \triangle t \sum_{k \in \mathbb{Z}} \left\| f^k \right\|^2 \leq 6(1+\beta\delta^2) \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t} \Big)^2 \left\| \frac{u^{k+1} - u^k}{\triangle t} \right\|^2 \\ + 3(1+\beta\delta^2) \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1} - 2\chi^k + \chi^{k-1}}{\triangle t^2} \Big)^2 \left\| u^k \right\|^2. \quad (4.27) \end{aligned}$$

For the second term of (4.26), we get

$$\begin{aligned} \triangle t \sum_{k\in\mathbb{Z}} \left\| B_2 \Big(\frac{g^{k+1} - g^k}{\triangle t} \Big) \right\|_Y^2 \leq 2 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k + \chi^{k+1}}{2} \Big)^2 \left\| B_2 \Big(\frac{u^{k+1} - u^k}{\triangle t} \Big) \right\|_Y^2 \\ + 2\triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t} \Big)^2 \left\| B_2 \Big(\frac{u^{k+1} - u^k}{2} \Big) \right\|_Y^2 \\ \leq 2 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k + \chi^{k+1}}{2} \Big)^2 \left\| B_2 \Big(\frac{u^{k+1} - u^k}{\triangle t} \Big) \right\|_Y^2 \\ + 2C_B^2 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t} \Big)^2 \left\| A_0^{\frac{1}{2}} (u^{k+1} + u^k) \right\|^2. \quad (4.28) \end{aligned}$$

Besides,

$$\triangle t \sum_{k\in\mathbb{Z}} \left\| A_0^{\frac{1}{2}} g^k \right\|^2 = \triangle t \sum_{k\in\mathbb{Z}} |\chi^k|^2 \left\| A_0^{\frac{1}{2}} u^k \right\|^2 \quad (4.29)$$

and

$$\begin{aligned} \triangle t \sum_{k\in\mathbb{Z}} \left\| \frac{g^{k+1} - g^k}{\triangle t} \right\|^2 \geq \frac{1}{2} \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k + \chi^{k+1}}{2} \Big)^2 \left\| \frac{u^{k+1} - u^k}{\triangle t} \right\|^2 \\ - 2\triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1} - \chi^k}{\triangle t} \Big)^2 \left\| \frac{u^{k+1} + u^k}{2} \right\|^2. \quad (4.30) \end{aligned}$$

We also remark that

$$
\begin{aligned}
&E^k \leq \frac{1}{4}\Big(\left\| A_0^{\frac{1}{2}} u^k \right\|^2 + \left\| A_0^{\frac{1}{2}} u^{k+1} \right\|^2 \Big) + \frac{1}{2} \left\| \frac{u^{k+1}-u^k}{\triangle t} \right\|^2 \Big(1 + (\beta - \frac{1}{4})\delta^2 \Big),\\
&\left\| \frac{u^k+u^{k+1}}{2} \right\|^2 \leq \left\| A_0^{-\frac{1}{2}} \right\|^2 E^0, \qquad \left\| \frac{u^{k+1}-u^k}{\triangle t} \right\|^2 \leq E^0,\\
&\left\| u^k \right\|^2 \leq \left\| A_0^{-\frac{1}{2}} \right\|^2 \sum_{k\in\mathbb{Z}} \mu_k^2 \Big(|a_k|^2 + |a_{-k}|^2 \Big) \leq \left\| A_0^{-\frac{1}{2}} \right\|^2 (1+\beta\delta^2) E^0,
\end{aligned}
\tag{4.31}
$$

wher $\left\| A_0^{-\frac{1}{2}} \right\|$ stands for the operator norm of $A_0^{-\frac{1}{2}}$ from H into itself.

Putting (4.27), (4.28), (4.29) and (4.30) in (4.26) and using (4.31), we obtain:

$$
\begin{aligned}
2m^2 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k+\chi^{k+1}}{2} \Big)^2 & \left\| B_2 \Big(\frac{u^{k+1}-u^k}{\triangle t} \Big) \right\|_Y^2 \\
\geq & \Big(\frac{1}{1+\beta(\delta+\epsilon)^2} \Big)^2 \triangle t \sum_{k\in\mathbb{Z}} |\chi^k|^2 \left\| A_0^{\frac{1}{2}} u^k \right\|^2 \\
& + \frac{1}{2(1+\beta(\delta+\epsilon)^2)} \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^k+\chi^{k+1}}{2} \Big)^2 \left\| \frac{u^{k+1}-u^k}{\triangle t} \right\|^2 \\
& - \frac{2}{1+\beta(\delta+\varepsilon)^2} \left\| A_0^{-\frac{1}{2}} \right\|^2 E^0 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t} \Big)^2 \\
& - 4C_B^2 E^0 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t} \Big)^2 - 6(1+\beta\delta^2) E^0 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1}-\chi^k}{\triangle t} \Big)^2 \\
& - 3(1+\beta\delta^2)^2 \left\| A_0^{-\frac{1}{2}} \right\|^2 E^0 \triangle t \sum_{k\in\mathbb{Z}} \Big(\frac{\chi^{k+1}-2\chi^k+\chi^{k-1}}{(\triangle t)^2} \Big)^2 \\
& - \frac{\triangle t}{\sin^2(\frac{\gamma_\varepsilon}{2})} (1+\beta\delta^2) E^0 \Big(1 + \Big(\frac{\triangle t}{\sin(\frac{\gamma_\epsilon}{2})} \Big)^2 \Big) |\mathrm{Supp}(\chi)|^2 \Big| \frac{d^2\chi}{dt^2} \Big|_\infty^2 .
\end{aligned}
\tag{4.32}
$$

To conclude the proof, we now specify the function χ. We look for

$$\chi_T(t) = \phi(\frac{t}{T}),$$

where ϕ is a nonnegative smooth function satisfying:

$$\mathrm{Supp}(\phi) \subset (0,4), \qquad \phi(t) = 1 \qquad \forall t \in (1,3).$$

With this choice,

$$|\mathrm{Supp}(\chi)| \Big| \frac{d^2\chi}{dt^2} \Big|_\infty \leq 4T \frac{1}{T^2} \Big| \frac{d^2\phi}{dt^2} \Big|_\infty ,$$

and so it is bounded uniformly by C_ϕ/T when $T \to \infty$. Putting this into (4.32) we have

$$\begin{aligned} 2m^2\triangle t \sum_{k\triangle t\in(0,4T)} \left\| B_2\Big(\frac{u^{k+1}-u^k}{\triangle t}\Big)\right\|_Y^2 &\geq \Big(\frac{1}{1+\beta(\delta+\varepsilon)^2}\Big)^2 2TE^0 \\ &- \triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 E^0\Big(\frac{2}{1+\beta(\delta+\varepsilon)^2}\left\|A_0^{-\frac12}\right\|^2 + 4C_B^2 + 6(1+\beta\delta^2)\Big) \\ &- 3(1+\beta\delta^2)^2\left\|A_0^{-\frac12}\right\|^2 E^0\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-2\chi^k+\chi^{k-1}}{\triangle t^2}\Big)^2 \\ &- \frac{\triangle t}{\sin^2(\frac{\gamma_\varepsilon}{2})}\frac{C_\phi}{T}^2(1+\beta\delta^2)E^0\Big(1+\Big(\frac{\triangle t}{\sin(\frac{\gamma_\varepsilon}{2})}\Big)^2\Big). \end{aligned} \tag{4.33}$$

But

$$\begin{aligned} &\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-\chi^k}{\triangle t}\Big)^2 \to_{\triangle t\to 0} \frac{1}{T}\left\|\dot\phi\right\|_{L^2}^2 \\ &\triangle t\sum_{k\in\mathbb{Z}}\Big(\frac{\chi^{k+1}-2\chi^k+\chi^{k-1}}{\triangle t^2}\Big)^2 \to_{\triangle t\to 0} \frac{1}{T^3}\left\|\ddot\phi\right\|_{L^2}^2. \end{aligned} \tag{4.34}$$

Hence choosing T large enough, the right hand side of (4.33) is positive. More precisely, if $T > 4T'_{\delta,\varepsilon}$ for some $\varepsilon > 0$, where $T'_{\delta,\varepsilon}$ is the solution of

$$\begin{aligned} \Big(\frac{1}{1+\beta(\delta+\varepsilon)^2}\Big)^2 4T - \frac{1}{T}\left\|\dot\phi\right\|_{L^2}^2\Big(\frac{2}{1+\beta(\delta+\varepsilon)^2}\left\|A_0^{-\frac12}\right\|^2 + 4C_B^2 + 6(1+\beta\delta^2)\Big) \\ - 3(1+\beta\delta^2)^2\left\|A_0^{-\frac12}\right\|^2\frac{1}{T^3}\left\|\ddot\phi\right\|_{L^2}^2 = 0, \end{aligned} \tag{4.35}$$

for $\triangle t$ small enough, there exists $k_{T,\delta}$ such that

$$k_{T,\delta}E^0 \leq \triangle t\sum_{k\triangle t\in(0,T)}\left\|B_0\Big(\frac{u^{k+1}-u^k}{\triangle t}\Big)\right\|_Y^2. \tag{4.36}$$

One can then optimize in ε and prove that the optimal time T_δ satisfies

$$T_\delta \leq 4T'_{\delta,\varepsilon=0}. \tag{4.37}$$

Remark 4.5. Note that we did not optimize the time estimate (4.37) only for simplicity. However, one can try to optimize each Cauchy-Schwarz inequality, but this would make the proof harder to read. Besides, the final result we would obtain on the time estimate would have the same magnitude, and would not be sharp. One could try, as an exercise, to slightly improve this time estimate as we did in Section 2.

For sake of completeness, we also give the complete assumptions when dealing with an observation operator

$$y(t) = B_1u(t) \quad \text{and} \quad y^k = B_1u^k. \tag{4.38}$$

corresponding respectively to the continuous and the discrete outputs.

Theorem 4.6. *Assume that* $B_1 \in \mathfrak{L}(\mathcal{D}(A_0), Y)$*, and that the system* (4.1)-(4.2) *is both admissible and observable with respect to the output* (4.38)*.*

Set $\delta > 0$*. Then there exists* $T_\delta > 0$ *such that for any* $T > T_\delta$*, there exists a constant* $k_{T,\delta} > 0$ *such that for all* $\triangle t$ *small enough, for any initial data in* $\mathcal{C}_{\delta/\triangle t}$*, the solution of the system* (4.7) *satisfies*

$$k_{T,\delta} E^0 \leq \triangle t \sum_{k\triangle t \in (0,T)} \left\| B_1 u^k \right\|_Y^2 . \tag{4.39}$$

Sketch of the proof. The proof is very similar to the previous one. Let us then only point out the main differences.

Inequality (4.17) becomes

$$M^2 \Big(\left\| A_0^{\frac{1}{2}} (i\omega z_1 - z_2) \right\|^2 + \left\| A_0 z_1 + i\omega z_2 \right\|^2 \Big) + m^2 \left\| B_1 z_1 \right\|_Y^2 \geq \left\| A_0^{\frac{1}{2}} z_1 \right\|^2 + \left\| z_2 \right\|^2 ,$$
$$\forall \omega \in \mathbb{R}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathcal{D}(A), \tag{4.40}$$

In the second step, with the same choice of z_1, z_2, ω, we obtain that for $\tau \leq \alpha_\epsilon / \triangle t$,

$$m^2 \left\| B_1 \hat{g}(\tau) \Big(1 - 4\beta \sin^2 \Big(\frac{\tau \triangle t}{2} \Big) \Big) \right\|_Y^2 \leq m^2 \left\| B_1 \hat{g}(\tau) \right\|_Y^2 .$$

And then, integrating this in $\tau \in (-\pi/\triangle t, \pi/\triangle t)$ and using Parseval's identity, this provides

$$m^2 \triangle t \sum |\chi^k|^2 \left\| B_1 u^k \right\|_Y^2$$

into (4.26). The rest of the proof is the same.

Applications. There are plenty of applications of the previous result. For instance, we present an application to the boundary observability of the wave equation.

Consider a nonempty open bounded domain $\Omega \in \mathbb{R}^d$ and Γ_0 be an open subset of $\partial\Omega$. We consider the following initial boundary value problem:

$$\begin{cases} u_{tt} - \Delta_x u = 0, & x \in \Omega, \quad t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0, \quad u_t(x,0) = v_0, & x \in \Omega \end{cases} \tag{4.41}$$

with the output

$$y(t) = \frac{\partial u}{\partial \nu} \Big|_{\Gamma_0} . \tag{4.42}$$

This system is conservative and the energy of (4.41)

$$E(t) = \frac{1}{2} \int_\Omega \Big[|u_t(t,x)|^2 + |\nabla u(t,x)|^2 \Big] dx, \tag{4.43}$$

remains constant, i.e.

$$E(t) = E(0), \qquad \forall\, t \in [0,T]. \tag{4.44}$$

The boundary observability for the system is described as: *For some constant* $C = C(T, \Omega, \Gamma_0) > 0$*, solutions of* (4.41) *satisfy*

$$E(0) \leq C \int_0^T \int_{\Gamma_0} \Big| \frac{\partial u}{\partial \nu} \Big|^2 d\Gamma_0 dt, \qquad \forall \ (u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega). \tag{4.45}$$

Note that this inequality holds true for some triple (T, Ω, Γ_0) satisfying *Geometric Optics Condition* introduced in [1], which asserts that all rays of Geometric Optics in Ω touches the sub-boundary Γ_0 in a time T. In this case, (4.45) is established by means of micro-local analysis tools (see [1]). From now, we assume this condition to hold.

We then introduce the following time semi-discretization of (4.41):

$$\begin{cases} \dfrac{u^{k+1} + u^{k-1} - 2u^k}{(\triangle t)^2} = \Delta_x \Big(\beta u^{k+1} + (1-2\beta) u^k + \beta u^{k-1} \Big), x \in \Omega, k \in \mathbb{Z} \\ u^k = 0, \qquad\qquad\qquad x \in \partial\Omega, k \in \mathbb{Z} \\ \Big(\dfrac{u^0 + u^1}{2}, \dfrac{u^1 - u^0}{\triangle t} \Big) = (u_0; \ v_0) \ \in H_0^1(\Omega) \times L^2(\Omega), \end{cases} \tag{4.46}$$

where β is a given parameter satisfying $\beta \geq \frac{1}{4}$.

The output functions y^k are given by

$$y^k = \frac{\partial u^k}{\partial \nu} \Big|_{\Gamma_0}. \tag{4.47}$$

System (4.41)–(4.42) (or system (4.46)–(4.47)) can be written in form (4.1)–(4.2) (or (4.7)) with observable (4.38) if we introduce the following notations:

$$H = L^2(\Omega), \ \ \mathcal{D}(A_0) = H^2(\Omega) \cap H_0^1(\Omega), \ \ Y = L^2(\Gamma_0),$$
$$A_0 \varphi = -\Delta_x \varphi \quad \forall \varphi \in \mathcal{D}(A_0),$$
$$B_1 \varphi = \frac{\partial \varphi}{\partial \nu} \Big|_{\Gamma_0}, \varphi \in \mathcal{D}(A_0),$$

One can easily check that A_0 is self-adjoint in H, positive and boundedly invertible and

$$\mathcal{D}(A_0^{\frac{1}{2}}) = H_0^1(\Omega), \qquad \mathcal{D}(A_0^{\frac{1}{2}})^* = H^{-1}(\Omega).$$

Proposition 4.7. *With the above notation,* $B_1 \in \mathfrak{L}(\mathcal{D}(A_0), Y)$ *is an admissible observation operator, i.e. for all* $T > 0$ *there exists a constant* $K_T > 0$*, independent of* $\triangle t$*, such that:*

If u *satisfies* (4.41) *then*

$$\int_0^T \int_{\Gamma_0} \Big| \frac{\partial \varphi}{\partial \nu} \Big|^2 d\Gamma_0 dt \leq K_T \Big(\|u_0\|^2_{H_0^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} \Big)$$

for all $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$.

The above proposition is classical (see, for instance, p. 44 of [6]), so we skip the proof.

Hence we are in the position to give the following theorem:

Theorem 4.8. *Set* $\beta \geq 1/4$. *For any* $\delta > 0$, *system* (4.46) *is uniformly observable with*

$$(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}.$$

More precisely, there exists T_δ, *such that for any* $T > T_\delta$, *there exists a positive constant* $k_{T,\delta}$ *independent of* $\triangle t$, *such that the solutions of system* (4.46) *satisfy*

$$k_{T,\delta}\Big(\|\nabla u_0\|^2 + \|v_0\|^2 \Big) \leq \triangle t \sum_{k=0}^{T/\triangle t} \int_{\Gamma_0} \Big| \frac{\partial \varphi^k}{\partial \nu} \Big|^2 d\Gamma_0, \tag{4.48}$$

for any $(u_0, v_0) \in \mathcal{C}_{\delta/\triangle t}$.

Proof. The scheme proposed here comes from a Newmark discretization. Hence, it is a direct consequence of Theorem 4.6, once we have checked that the in the class $\mathcal{C}_{\delta/\triangle t}$, the energy E^0 is equivalent to the left hand side of (4.48):

$$\frac{1}{2}\Big(\|\nabla u_0\|^2 + \|v_0\|^2 \Big) \leq E^0 \leq \frac{1}{2}\Big(1 + \Big(\beta - \frac{1}{4}\Big)\delta^2\Big)\Big(\|\nabla u_0\|^2 + \|v_0\|^2 \Big).$$

On this particular example, one can try to use microlocal tools to explain the propagation of the waves in the semi-discrete level. Recall that for the wave equation we may use the Geometric Control Condition [1], which says, roughly speaking, that the optimal time of control is given by the time needed for all the rays to meet Γ_0. When discretizing in time the wave equation as in (4.7), the symbol of the operator becomes (*see for instance* [7])

$$\frac{4}{\triangle t^2} \sin^2\Big(\frac{\tau \triangle t}{2}\Big) - \Big|\xi\Big|^2 \Big(1 - 4\beta \sin^2\Big(\frac{\tau \triangle t}{2}\Big)\Big).$$

As said before, this symbol is not hyperbolic in the whole range of frequency when $\beta \geq 1/4$. Hence we consider only the case where $\beta = 1/4$, that is when the symbol is

$$\frac{4}{\triangle t^2} \tan^2\Big(\frac{\tau \triangle t}{2}\Big) - \Big|\xi\Big|^2.$$

In this case, one can reasonably assume that the optimal time is given by the time needed by all the rays to meet Γ_0 as in the continuous case. But the bicharacteristic derived from this hamiltonian are straight lines as in the continuous case, except that their velocity is not 1 anymore. Indeed, one can prove that along the rays corresponding to $\Big|\xi\Big| < \delta/\triangle t$,

$$\Big|\frac{dx}{dt}\Big| = \frac{1}{1 + \tan^2(\frac{\tau \triangle t}{2})} \geq \frac{1}{1 + \frac{\delta^2}{4}}.$$

It follows that we expect that the optimal observability time T_δ^* in the class $\mathcal{C}_{\delta/\triangle t}$ is

$$T_\delta^* = (1 + \frac{\delta^2}{4})T_0^*,$$

where T_0 is the optimal observability time for the continuous system. According to this, the estimate T_δ given in (2.3) has the good growth in δ.

5. Fully discrete schemes.

In this section, we analyze the fully discrete approximation schemes of

$$\dot{z} = Az, \quad z(0) = z^0 \in X, \tag{5.1}$$

where A is a skew adjoint operator in X with compact resolvent. The observation is given by

$$y = Bz(t), \tag{5.2}$$

where $B \in \mathfrak{L}(\mathcal{D}(A), Y)$.
Besides, we assume that this system is exactly observable.

We assume that we have a sequence (A_h, B_h) of operators such that:

1. The limit operators (A_0, B_0) coincide with (A, B).
2. The operators A_h are skew adjoint in a sequence of Hilbert spaces X_h and with compact resolvent.
3. The operators B_h are in $\mathfrak{L}(\mathcal{D}(A_h), Y_h)$, where Y_h are Hilbert spaces, and their operator norms from $\mathcal{D}(A_h)$ to Y_h are uniformly bounded by C_B.
4. The sequence of operators (A_h, B_h) is uniformly admissible, that is for any T, there exists a constant K_T such that for any h small enough,
$$\int_0^T \left\|B_h \exp(tA_h)z^0\right\|_{Y_h}^2 \, dt \le K_T \left\|z^0\right\|_{X_h}^2, \tag{5.3}$$
where $\exp(tA_h)$ stands for the semi group associated to the equation
$$\dot{z} = A_h z, \quad z(0) = z^0 \in X_h.$$
5. The sequence of operators (A_h, B_h) is uniformly observable, that is there exists a time T^* such that for any $T > T^*$, there exists a constant k_T such that for any h small enough,
$$k_T \left\|z^0\right\|_{X_h}^2 \le \int_0^T \|B_h \exp(tA_h)z_0\|_{Y_h}^2 \, dt. \tag{5.4}$$

The parameter h refers to the mesh size of the space semi-discrete approximation.

Remark 5.1. We might think that the pair (A_h, B_h) converge to (A, B) when $h \to 0$ and that the Hilbert spaces $(X_h, \mathcal{D}(A_h))$ converge to $(X, \mathcal{D}(A))$. This will be the case in the applications, in the sense of the gamma convergence.

Lemma 5.2. *There exist two positive constants M and m such that for all $h > 0$, for all $\omega \in \mathbb{R}$,*

$$M^2 \|(A_h - i\omega)z\|_{X_h}^2 + m^2 \|B_h z\|_{Y_h}^2 \geq \|z\|_{X_h}^2, \quad z \in \mathcal{D}(A_h). \tag{5.5}$$

Proof. It is sufficient to remark that the constants entering in (5.5) depends only on T^*, K_T and k_T, see for instance [12]. Let us sketch the proof for completeness.

Fix $h > 0$ and $\omega \in \mathbb{R}$. Set $z^0 \in \mathcal{D}(A_h)$, and

$$z(t) = \exp(tA_h)z^0, \qquad v(t) = z(t) - e^{i\omega t}z^0, \qquad f = (A_h - i\omega)z^0.$$

Then

$$\dot{z} = \exp(tA_h)A_h z^0 = \exp(tA_h)(i\omega z^0 + f) = i\omega z(t) + \exp(tA_h)z^0.$$

Hence we deduce that

$$\dot{v} = i\omega v + \exp(tA_h)f,$$

and thus

$$v(t) = \int_0^t e^{i\omega(t-s)} \exp(sA_h)f \, ds.$$

But $z(t) = v(t) + e^{i\omega t}z^0$, and then

$$\int_0^T \|B_h z(t)\|_{Y_h}^2 \, dt \leq 2T \left\|B_h z^0\right\|_{Y_h}^2 + 2\int_0^T \left\|\int_0^t e^{i\omega(t-s)} B_h \exp(sA_h)f \, ds\right\|_{Y_h}^2 \, dt.$$

From Cauchy Schwarz's formula, the second term can be handled as follows:

$$\begin{aligned}\int_0^T \left\|\int_0^t e^{i\omega(t-s)} B_h \exp(sA_h)f \, ds\right\|_{Y_h}^2 \, dt &\leq \int_0^T t \int_0^t \|B_h \exp(sA_h)f\|_{Y_h}^2 \, ds \, dt \\ &\leq \frac{T^2}{2}\int_0^T \|B_h \exp(sA_h)f\|_{Y_h}^2 \, ds.\end{aligned}$$

Then, choosing $T > T^*$, and using (5.3)–(5.4), we get

$$k_T \left\|z^0\right\|_{X_h}^2 \leq 2T \left\|B_h z^0\right\|_{Y_h}^2 + \frac{T^2}{2}K_T \|f\|_{X_h}^2,$$

from we deduce (5.5).

Then, by remarking that in the proofs presented above, the time observability constants of the time-discrete approximation scheme only depends on C_B, M, m, and the filtering parameter δ, we can derive observability results for fully discretized approximations schemes, for instance :

Theorem 5.3. *Consider a sequence of operators (A_h, B_h) as above, and let us consider the schemes*

$$\begin{cases} \dfrac{z^{k+1} - z^k}{\triangle t} = A_h\Big(\dfrac{z^{k+1} + z^k}{2}\Big), & \text{in } X, \quad k \in \mathbb{Z}, \\ z^0 \in X_h. \end{cases} \tag{5.6}$$

with the output function

$$y^k = B_h z^k, \qquad k \in \mathbb{Z}. \tag{5.7}$$

Define

$$\mathcal{C}_s^{(h)} = \text{span}\,\{\Phi_j^{(h)} : \textit{the corresponding } i\mu_j^{(h)};\ \textit{satisfies } |\mu_j^{(h)}| \leq s\}, \tag{5.8}$$

where $\mu_j^{(h)}$ *and* $\Phi_j^{(h)}$ *denote the eigenvalue and the corresponding eigenvector of* A_h*.*

Set $\delta > 0$*. Then there exists* T_δ *such that for all* $T > T_\delta$*, there exists a positive constant* $k_{T,\delta}$ *such that for all* $\triangle t$ *and* h *small enough, the solution of* (5.6) *satisfies*

$$k_{T,\delta} \left\| z^0 \right\|_{X_h}^2 \leq \triangle t \sum_{k \triangle t \in (0,T)} \left\| B_h z^k \right\|_{Y_h}^2. \tag{5.9}$$

This theorem does not need any new proof: it is a simple corollary of Theorem 2.1 and inequality (5.5). This provides a simple efficient way to prove uniform observability results for fully discrete schemes as we will show in the applications hereafter.

Roughly, if the sequence of *space* discrete operators (A_h, B_h) satisfies items 1 and 2, which are natural, one only needs to study the *space* semi-discrete system

$$\dot{z} = A_h z, \qquad y(t) = B_h z(t),$$

and prove (5.3) and (5.4) to obtain uniform observability estimates on the fully discrete schemes (5.6)-(5.7).

Applications. Let us consider the wave equation in a 2-d square. More precisely, let $\Omega = (0,\pi) \times (0,\pi) \subset \mathbb{R}^2$. We consider the wave equation with Dirichlet boundary conditions

$$\begin{cases} \ddot{u} - \Delta_x u = 0 & \text{in} \quad \Omega \times (0,T) \\ u = 0, & \text{on} \quad \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \dot{u}(x,0) = u_1(x) & \text{in} \quad \Omega. \end{cases} \tag{5.10}$$

In (5.10) $\dot{} = \partial/\partial t$ denotes partial derivation with respect to time and Δ_x is the Laplacian in the space variable $x = (x_1, x_2) \in \Omega$. Moreover, the energy of (5.10)

$$E(t) = \frac{1}{2} \int_\Omega \Big[|\dot{u}(x,t)|^2 + |\nabla u(x,t)|^2 \Big] dx, \qquad t \in \mathbb{R}$$

is constant.

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1, \pi) : x_1 \in (0,\pi)\} \cup \{(\pi, x_2) : x_2 \in (0,\pi)\} \stackrel{\triangle}{=} \Gamma_1 \cup \Gamma_2.$$

As in (4.42), the output function $y(t) = Bu(t)$ is given by

$$Bu = \frac{\partial u}{\partial \nu}\Big|_{\Gamma_0} = \frac{d}{dx_2} u(x_1,\pi)\Big|_{\Gamma_1} + \frac{d}{dx_1}(\pi, x_2)\Big|_{\Gamma_2}.$$

It is by now well known (*see* [6] and the previous section) that there exists $T > 0$ and $C > 0$ such that

$$E(0) \leq C \int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial \nu}\right|^2 d\Gamma_0 dt$$

holds for every finite-energy solution of (5.10).

let us consider the finite-difference semi-discretization of (5.10). The following can be found in [16]. We introduce a finite difference approximation scheme.

Given $J, K \in \mathbb{N}$ we set

$$h_1 = \frac{\pi}{J+1}, \qquad h_2 = \frac{\pi}{K+1}. \tag{5.11}$$

We denote by $u_{jk}(t)$ the approximation of the solution u of (5.10) at the point $x_{jk} = (jh_1, kh_2)$. The discrete schemes of (5.10) is as follows:

$$\begin{cases} \ddot{u}_{jk} - \dfrac{u_{j+1k} + u_{j-1k} - 2u_{jk}}{h_1^2} - \dfrac{u_{jk+1} + u_{jk-1} - 2u_{jk}}{h_2^2} = 0 \\ \qquad\qquad 0 < t < T,\ j = 1, \cdots, J;\ k = 1, \cdots, K \\ u_{jk} = 0, \quad 0 < t < T,\ j = 0, J+1;\ k = 0, K+1 \\ u_{jk}(0) = u_{jk,0}, \quad \dot{u}_{jk}(0) = u_{jk,1},\ j = 1, \cdots, J;\ k = 1, \cdots, K. \end{cases} \tag{5.12}$$

System (5.12) is a system of JK linear differential equations. Moreover, if we denote the unknown

$$U(t) = (u_{11}(t), u_{21}(t), \cdots, u_{J1}(t), \cdots, u_{1K}(t), u_{2K}(t), \cdots, u_{JK}(t))^T,$$

then system (5.12) can be rewritten in vectorial form as follows

$$\begin{cases} \ddot{U}(t) + A_h U(t) = 0, \qquad 0 < t < T. \\ U(0) = U_{h,0}, \quad \dot{U}(0) = U_{h,1}, \end{cases} \tag{5.13}$$

where $(U_{h,0}, U_{h,1}) = (u_{jk,0}, u_{jk,1})_{1 \leq j \leq J, 1 \leq k \leq K} \in \mathbb{R}^{2JK}$ are the initial datum and the corresponding solution of (5.12) is given by $(U_h, \dot{U}_h) = (u_{jk}, \dot{u}_{jk})_{1 \leq j \leq J, 1 \leq k \leq K}$. Note that the entries of A_h belonging to $\mathcal{M}_{JK}(\mathbb{R})$ may be easily deduced from (5.12).

As a discretization of the output, we chose

$$(B_h U)_j = \frac{u_{jK}}{h_2}, \quad j = 1 \cdots J, \tag{5.14}$$

$$(B_h U)_{J+k} = \frac{u_{Jk}}{h_1}, \quad k = 1, \cdots K, \tag{5.15}$$

The corresponding norm for the observation operator B_h is given by

$$\|B_h U(t)\|_{Y_h}^2 = h_1 \sum_{j=1}^{J} \left|\frac{u_{jK}(t)}{h_2}\right|^2 + h_2 \sum_{k=1}^{K} \left|\frac{u_{Jk}(t)}{h_1}\right|^2.$$

Besides, the energy of the system (5.13) is given by

$$E_h(t) = \frac{h_1 h_2}{2} \sum_{j=0}^{J} \sum_{k=0}^{K} \Big(|\dot{u}_{jk}(t)|^2 + \Big| \frac{u_{j+1k}(t) - u_{jk}(t)}{h_1} \Big|^2 + \Big| \frac{u_{jk+1}(t) - u_{jk}(t)}{h_2} \Big|^2 \Big). \tag{5.16}$$

As in the continuous case, this quantity is constant.

$$E_h(t) = E_h(0), \qquad \forall \ \ 0 < t < T.$$

In order to prove the uniform observability of (5.13), we have to filter the high frequencies. To do that we consider the eigenvalue problem associated with (5.13):

$$A_h \varphi = \lambda^2 \varphi. \tag{5.17}$$

As in the continuous case, the eigenvalues $\infty\lambda^{j,k,h_1,h_2}$ are simple and purely imaginary. Let us denote φ^{j,k,h_1,h_2} the corresponding eigenvectors.

Let us now introduce the following classes of solutions of (5.13) for any $0 < \gamma < 1$:

$$\widehat{\mathcal{C}_\gamma}(h) = \text{span}\, \{\varphi^{j,k,h_1,h_2} \text{ such that } \ \lambda^{j,k,h_1,h_2} \max(h_1, h_2) \leq 2\sqrt{\gamma}\}.$$

The following Lemma holds (*see* [16]):

Lemma 5.4. *Let $0 < \gamma < 1$. Then there exist T_γ such that for all $T > T_\gamma$ there exists $k = k_{\gamma,T} > 0$ such that*

$$k_{\gamma,T} E_h(0) \leq \int_0^T \|B_h U(t)\|_{Y_h}^2 \, dt$$

holds for every solution of (5.13) *in the class $\widehat{\mathcal{C}_\gamma}(h)$ and every h_1, h_2 small enough satisfying*

$$\sup \Big| \frac{h_1}{h_2} \Big| < \sqrt{\frac{\gamma}{4 - \gamma}}.$$

Now we present the time discrete schemes we are interested in. For any $\triangle t > 0$, we consider the following time Newmark approximation scheme of the system (5.13):

$$\begin{cases} \dfrac{U^{k+1} + U^{k-1} - 2U^k}{(\triangle t)^2} + A_h \Big(\beta U^{k+1} + (1 - 2\beta) U^k + \beta U^{k-1} \Big) = 0, \\ \Big(\dfrac{U^0 + U^1}{2}, \dfrac{U^1 - U^0}{\triangle t} \Big) = (U_{h,0}, U_{h,1}), \end{cases} \tag{5.18}$$

with $\beta \geq 1/4$.

The energy of (5.18) given by

$$E^k = \frac{1}{2}\left\| A_h^{\frac{1}{2}}\Big(\frac{U^k + U^{k+1}}{2}\Big)\right\|^2 + \frac{1}{2}\left\|\frac{U^{k+1} - U^k}{\triangle t}\right\|^2 + (4\beta - 1)\frac{(\triangle t)^2}{8}\left\| A_h^{\frac{1}{2}}\Big(\frac{U^{k+1} - U^k}{\triangle t}\Big)\right\|^2, \ k \in \mathbb{Z} \quad (5.19)$$

which is a discrete counterpart of the time continuous energy (4.3) and remains constant (see (4.8) as well).

The following theorem can be seen as a directly consequence of the Theorem 4.6:

Theorem 5.5. *Set* $0 < \gamma < 1$. *Assume that the mesh sizes* $h_1, h_2, \triangle t$ *goes to zero such that*

$$\sup\Big|\frac{h_1}{h_2}\Big| < \sqrt{\frac{\gamma}{4-\gamma}}, \qquad \frac{\max\{h_1, h_2\}}{\triangle t} \le \tau, \quad (5.20)$$

where τ *is a positive constant.*

Then, for any $0 < \delta \le 2\sqrt{\gamma}/\tau$, *there exist* $T_\delta > 0$ *such that for any* $T > T_\delta$, *there exists* $k_{T,\delta,\gamma} > 0$ *such that the observability inequality*

$$k_{T,\delta,\gamma} > 0 E^k \le \triangle t \sum_{k\triangle t \in (0,T)} \left\|B_h U^k\right\|_{Y_h}^2$$

holds for every solution of (5.18) *in the class* $\widehat{\mathcal{C}_\gamma}(h)$ *for* $h_1, h_2, \triangle t$ *small enough satisfying* (5.20)*.*

Proof. We are in the setting given before and thus Lemma 5.2 hold. Hence, to apply Theorem 4.6, we only need to verify that $\mathcal{C}_{\delta/\triangle t} \subset \widehat{\mathcal{C}_\gamma}(h)$. But

$$|\lambda| < \frac{\delta}{\triangle t} \le 2\frac{\sqrt{\gamma}}{\tau\triangle t} \le 2\frac{\sqrt{\gamma}}{\max\{h_1, h_2\}}.$$

and the proof is over.

6. Further comments and open problems

1. The resolvent estimate seems to be adapted to work on the time-discrete approximation schemes, as we have seen in this paper. However, although this method is quite robust, we are not able to deal with observability inequality with loss. Actually, this question is also open at the continuous level.

2. There are still some issues to analyze, even at the continuous level, to recover the optimal time. To our knowledge, the time obtained by the resolvent estimate is not optimal in general. Another spectral method is known, the so-called wave packet method introduced in [11]. However, the result it provides is always worse than before, especially on the time estimate. Indeed, the equivalence between the wave packet estimate given in [11] and the observability inequality (1.4) is obtained by using the equivalence on the resolvent estimate. However we

conjecture that the wave packet method as exposed in [11] shall be more robust and meaningful, but this would require the derivation of a more direct proof.

3. There are several different methods to derive uniform observability inequalities for the systems (4.46). In [15], a discrete multiplier technique is established to derive the uniform observability of the wave equations in a bounded domain. The same order of filtering parameter $\delta/(\triangle t)$ is attained but a smallness condition on δ is imposed. Theorem 2.1 generalized this result to any $\delta > 0$, as the dispersion diagram in [15] suggested.

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This appendix is aimed to develop analternate proof to Theorem 3.1 based on the wave packet method from [11] :

The exact observability property of system (1.1)-(1.2) is equivalent to a so-called wave packet estimate. Let us define for $\omega \in \mathbb{R}$ and $\varepsilon > 0$

$$J_\varepsilon(\omega) = \{m \in \Lambda \quad \text{such that} \quad |\mu_m - \omega| < \varepsilon\}.$$

$$\left\{ \begin{array}{l} \text{There exists } \ \varepsilon_0 > 0 \ \text{ and } \ \alpha > 0 \ \text{ such that for all } \ n \in \Lambda \\ \text{and for all } \ z = \displaystyle\sum_{m \in J_{\varepsilon_0}(\mu_n)} c_m \Phi_m : \qquad \|Bz\|_Y \geq \alpha \|z\|_X . \end{array} \right. \tag{.1}$$

Appendix A. Wave packet estimates : Admissibility and Observability.

A.1. Admissibility.

Let us assume that system (1.1)-(1.2) is admissible. Then there exists a positive constant K_T such that :

$$\int_0^T \|y(t)\|_Y^2 \, dt \leq K_T \|z_0\|_X^2 \qquad \forall \, z_0 \in \mathcal{D}(A). \tag{A.1}$$

The goal of this section is to prove that this property can be read on the wave packets as well.

Proposition A.1. *System* (1.1)-(1.2) *is admissible if and only if*

$$\left\{ \begin{array}{l} \textit{There exists } r > 0 \ \textit{ and } \ D > 0 \ \textit{ such that for all } \ n \in \Lambda \\ \textit{and for all } \ z = \displaystyle\sum_{l \in J_r(\mu_n)} c_l \Phi_l : \qquad \|Bz\|_Y \leq D \|z\|_X . \end{array} \right. \tag{A.2}$$

Proof. We will prove separately the two implications by Ingham's type arguments.

Assume that system (1.1)-(1.2) is admissible. We will need the existence of a time T such that there exists a function m satisfying

$$\left\{ \begin{array}{ll} m(t) \leq 0, & |t| \geq T/2, \\ m(t) \leq 1, & |t| \leq T/2, \\ \hat{m}(\tau) = 1, & |\tau| \leq 1. \end{array} \right. \tag{A.3}$$

The existence of such time T and function m is postponed.

Then, let us consider a wave packet $z_0 = \sum_{l \in J_1(\mu_n)} c_l \Phi_l$. Then the admissibility gives

$$
\begin{aligned}
K_T \|z_0\|_X^2 &\geq \int_0^T \|Bz(t)\|_Y^2 \, dt \\
&\geq \int_{\mathbb{R}} m(t - T/2) \|Bz(t)\|_Y^2 \, dt \\
&\geq \sum_{l_1, l_2 \in J_1(\mu_n)^2} \hat{m}(\mu_{l_1} - \mu_{l_2}) < a_{l_1} B\Phi_{l_1}, a_{l_2} B\Phi_{l_2} >_Y \\
&\geq \sum_{l_1, l_2 \in J_1(\mu_n)^2} < a_{l_1} B\Phi_{l_1}, a_{l_2} B\Phi_{l_2} >_Y \\
&\geq \|Bz_0\|_Y^2 .
\end{aligned}
$$

This concludes the proof of estimate (A.2).

Now we assume that estimate (A.2) holds for some $r > 0$ and $D > 0$. Set $z_0 \in \mathcal{D}(A)$, and expand z_0 as

$$
z_0 = \sum_{k \in \mathbb{Z}} z_k, \qquad z_k = \sum_{l \in J_r(2kr)} c_l \Phi_l.
$$

Here again, we shall assume the existence of a time T and a function M such that

$$
\left\{
\begin{array}{ll}
M(t) \geq 0, & |t| \geq T/2, \\
M(t) \geq 1, & |t| \leq T/2, \\
\text{Supp } \hat{M} \subseteq (-2r, 2r). &
\end{array}
\right. \tag{A.4}
$$

Then

$$
\begin{aligned}
\int_0^T \|Bz(t)\|_Y^2 &\leq \int_{\mathbb{R}} M(t - T/2) \|Bz(t)\|_Y^2 \, dt \\
&\leq \sum_{k_1, k_2} \int_{\mathbb{R}} M(t - T/2) < Bz_{k_1}(t), Bz_{k_2}(t) >_Y \, dt.
\end{aligned}
$$

But these scalar products vanishes most of the time. Indeed, if $|k_1 - k_2| \geq 2$,

$$
\begin{aligned}
&\int_{\mathbb{R}} M(t - T/2) < Bz_{k_1}(t), Bz_{k_2}(t) >_Y \, dt = \\
&\qquad \sum_{(l_1, l_2) \in J_r(2k_1 r) \times J_r(2k_2 r)} \hat{M}(\mu_{l_1} - \mu_{l_2}) < a_{l_1} B\Phi_{l_1}, a_{l_2} B\Phi_{l_2} >_Y = 0
\end{aligned}
$$

from (A.4).This implies that

$$\begin{aligned}
\int_0^T \|Bz(t)\|_Y^2 &\leq \int_{\mathbb{R}} M(t-T/2) \sum_k \Big(\|Bz_k(t)\|_Y^2 + 2\Re < Bz_k(t), Bz_{k+1}(t) > \Big)\, dt \\
&\leq 3 \int_{\mathbb{R}} M(t-T/2) \sum_k \|Bz_k(t)\|_Y^2 \; dt \\
&\leq 3 \int_{\mathbb{R}} M(t-T/2) \sum_k \|Bz_k(t)\|_Y^2 \; dt \\
&\leq 3D \int_{\mathbb{R}} M(t-T/2) \sum_k \|z_k(t)\|_X^2 \; dt \\
&\leq 3D\hat{M}(0) \|z_0\|_X^2 .
\end{aligned}$$

This closes the proof, since admissibility at time T is obviously equivalent to admissibility in any time $\tilde{T}$.

It follows that Proposition A.1 holds as soon as we have proved the existence of the two functions m and M satisfying (A.3) and (A.4), which appears naturally in many Ingham's type theorem proofs, [5].

I think one possible function M can be found in Tucsnak's book. For the other one, i do not have a single clue, but i am really convinced it shall exist.

A.2. From the Wave Packet estimate (.1) to the Observability.

This subsection is devoted to prove the following proposition, which is a by-product of the analysis of [11], as Marius Tucsnak pointed it out to us.

Proposition A.2. *Assume that* (.1) *holds, that is :*

$$\left\{ \begin{array}{l} \text{There exists } \ \varepsilon > 0 \ \text{ and } \ \alpha > 0 \ \text{ such that for all } \ n \in \Lambda \\ \text{and for all } \ z = \displaystyle\sum_{m \in J_{\varepsilon_0}(\mu_n)} c_m \Phi_m : \qquad \|Bz\|_Y \geq \alpha \|z\|_X . \end{array} \right.$$

Then the observability estimate (1.4) *holds in a time* T^* *such that*

$$T^* \leq \pi \sqrt{\frac{1}{\varepsilon}^2 + \frac{K_1}{\alpha^2(1-\exp(-1))} \Big(1 + \frac{1}{\varepsilon}\Big)^2}, \tag{A.5}$$

where K_1 *is the admissibility constant corresponding to* $T = 1$ *in* (1.3).

Proof. Let $z \in \mathcal{D}(A)$, $\omega \in \mathbb{R}$. Denote by

$$V(\omega, \varepsilon) = \text{span}\Big(\Phi_j \text{ such that } |\mu_j - \omega| \leq \varepsilon\Big).$$

Write

$$z = z_1 + z_2, \quad z_1 \in V(\omega, \varepsilon), \quad z_2 \in V(\omega, \varepsilon)^*.$$

Using Cauchy Schwarz inequality, we get that for any $\eta \in (0,1)$

$$\begin{aligned} \|Bz\|_Y^2 &\geq (1-\eta)\|Bz_1\|_Y^2 - \Big(\frac{1}{\eta}-1\Big)\|Bz_2\|_Y^2 \\ &\geq (1-\eta)\alpha^2\|z_1\|_X^2 - \Big(\frac{1}{\eta}-1\Big)\|Bz_2\|_Y^2 \end{aligned}$$

Besides,

$$\begin{aligned} \|Bz_2\|_Y &\leq \left\|B(A-i\omega I)^{-1}(A-i\omega I)z_2\right\|_Y \\ &\leq \left\|B(A-i\omega I)^{-1}\right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*,Y)}\|(A-i\omega I)z_2\|_X \end{aligned}$$

We claim that the following lemma holds :

Lemma A.3. *Let us define* $K(\omega,\varepsilon)$ *as*

$$K(\omega,\varepsilon) = \left\|B(A-i\omega I)^{-1}\right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*,Y)}.$$

Then for any $\varepsilon > 0$, $K(\omega,\varepsilon)$ *is uniformly bounded in* ω, *that is*

$$K(\varepsilon) = \sup_{\omega\in\mathbb{R}} K(\omega,\varepsilon) < \infty. \tag{A.6}$$

Besides, the following estimate holds

$$K(\varepsilon) \leq \sqrt{\frac{K_1}{1-\exp(-1)}}\Big(1+\frac{1}{\epsilon}\Big), \tag{A.7}$$

where K_1 *is the admissibility constant in* (1.3).

Let us postpone the proof to the end of the section.
It follows that for all $m, M \in \mathbb{R}^2$,

$$\begin{aligned} M^2\|(A-i\omega)z\|_X^2 + m^2\|Bz\|_Y^2 &\geq m^2\alpha^2(1-\eta)\|z_1\|_X^2 \\ &\quad + \Big(M^2 - m^2\Big(\frac{1}{\eta}-1\Big)K(\varepsilon)^2\Big)\|(A-\infty\omega I)z_2\|_X^2 \\ &\geq m^2\alpha^2(1-\eta)\|z_1\|_X^2 + \varepsilon^2\Big(M^2 - m^2\Big(\frac{1}{\eta}-1\Big)K(\varepsilon)^2\Big)\|z_2\|_X^2 \end{aligned}$$

For all $\eta \in (0,1)$, one recover the resolvent estimate by solving

$$m^2\alpha^2(1-\eta) = \varepsilon^2\Big(M^2 - m^2\Big(\frac{1}{\eta}-1\Big)K(\varepsilon)^2\Big) = 1,$$

which leads to

$$m_\eta^2 = \frac{1}{\alpha^2(1-\eta)}, \quad M_\eta^2 = \frac{1}{\varepsilon^2} + \frac{1}{\alpha^2\eta}K(\varepsilon)^2. \tag{A.8}$$

Remember that from the resolvent estimate the estimate on the optimal time is given by

$$T^* \leq \pi M,$$

and since the resolvent estimate holds for all $\eta \in (0,1)$ with (M_η, m_η) as in (A.8), it follows that the optimal time satisfies

$$T^* \leq \pi \sqrt{\frac{1}{\varepsilon^2} + \frac{1}{\alpha^2} K(\varepsilon)^2}. \tag{A.9}$$

Hence the estimate (A.5) follows.

Proof of Lemma A.3. **Note that i do not claim that the estimate given in Lemma** A.3 **is sharp, and it would interesting to try to improsve this estimate !**

Let us first notice this resolvent equality.

$$\begin{aligned} (A - i\omega I) - I &= A - (1 + i\omega) I \\ (A - (1+i\omega)I)^{-1}(I - (A - i\omega I)^{-1}) &= (A - i\omega I)^{-1}. \end{aligned}$$

Hence

$$K(\omega, \varepsilon) \leq \left\| B(A - (1+i\omega)I)^{-1} \right\|_{\mathfrak{L}(X,Y)} \ \left\| (I - (A - i\omega I)^{-1}) \right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*, X)}$$

Obviously

$$\left\| (I - (A - i\omega I)^{-1}) \right\|_{\mathfrak{L}(V(\omega,\varepsilon)^*, X)} \leq 1 + \frac{1}{\varepsilon}$$

Hence we restrict ourselves to the study of

$$\left\| B(A - (1+i\omega)I)^{-1} \right\|_{\mathfrak{L}(X,Y)}.$$

Let us remark that for all $z = \sum a_j \Phi_j \in X$,

$$\begin{aligned} (A - (1+i\omega)I)^{-1} z &= \sum \frac{1}{i(\mu_j - \omega) - 1} a_j \Phi_j \\ &= \int_0^\infty \exp(-t) z(t)\ dt, \end{aligned}$$

where $z(t)$ is the solution of (1.1) with initial value z. This implies that

$$\begin{aligned} \left\| B(A - (1+i\omega)I)^{-1} z \right\|_Y^2 &= \left\| \int_0^\infty \exp(-t) Bz(t)\ dt \right\|_Y^2 \\ &\leq \Big(\int_0^\infty \exp(-t)\ dt \Big) \Big(\int_0^\infty \exp(-t) \left\| Bz(t) \right\|_Y^2\ dt \Big) \\ &\leq \int_0^\infty \exp(-t) \left\| Bz(t) \right\|_Y^2\ dt. \end{aligned}$$

But using the admissibility of the operator, we obtain

$$\begin{aligned} \int_0^\infty \exp(-t) \left\| Bz(t) \right\|_Y^2\ dt &\leq \sum_{k \in \mathbb{N}} \exp(-k) \int_k^{k+1} \left\| Bz(t) \right\|_Y^2\ dt \\ &\leq \Big(\sum_{k \in \mathbb{N}} \exp(-k) \Big) K_1 \left\| z \right\|_X^2 \\ &\leq \frac{K_1}{1 - \exp(-1)} \left\| z \right\|_X^2. \end{aligned}$$

The estimate (A.7) follows.

Appendix B. An alternate proof of Theorem 3.1.

Let us recall the setting of Section 3. We consider a scheme which can be read as

$$z^{k+1} = \mathbb{T}_{\triangle t} z^k, \quad z^0 = z_0, \tag{B.1}$$

where $\mathbb{T}_{\triangle t}$ is a linear operator which has the same eigenvectors as the operator A. We also assume that the scheme is conservative. This implies that there exist real numbers $\lambda_{j,\triangle t}$ such that

$$\mathbb{T}_{\triangle t}\phi_j = \exp(i\lambda_{j,\triangle t}\triangle t)\phi_j. \tag{B.2}$$

Moreover, we assume that there is an explicit relation between $\lambda_{j,\triangle t}$ and μ_j under the following form:

$$\lambda_{j,\triangle t} = \frac{1}{\triangle t}\, h(\mu_j \triangle t), \tag{B.3}$$

where h is a smooth strictly increasing function satisfying:

$$|h(x)| \leq \pi. \tag{B.4}$$

Again we introduce an operator $A_{\triangle t}$ such that the solution of (3.1) coincides with the solution of the linear system

$$\frac{z^{k+1} - z^k}{\triangle t} = A_{\triangle t}\Big(\frac{z^k + z^{k+1}}{2}\Big), \quad z^0 = z_0. \tag{B.5}$$

This is done by the definition

$$A_{\triangle t}\phi_j = k_{\triangle t}(\mu_j)\phi_j, \tag{B.6}$$

where

$$k_{\triangle t}(\omega) = \frac{2}{\triangle t}\tan\Big(\frac{h(\omega\triangle t)}{2}\Big). \tag{B.7}$$

Set $\delta > 0$. The main idea of this proof is to work on the continuous level, but not on the continuous system (1.1), but rather on the continuous systems

$$\dot{z} = A_{\triangle t} z, \qquad z(0) = z_0 \in \mathcal{C}_{\delta/\triangle t}. \tag{B.8}$$

with output $y(t) = Bz(t)$ as before.

Our goal is to prove a uniform resolvent estimate similar as (1.5) for the sequence of operator $(A_{\triangle t}, B)$ and then to apply Theorem (2.1) to obtain the desired observability estimate uniformly in $\triangle t$.

Note that we are very close to the mathematical setting presented in Section 5 to address the fully discrete problem. Hence we propose to obtain a uniform resolvent estimate as in Section 5 and we only to check the hypothesis of Lemma 5.5.

Lemma B.1. • *Uniform continuity. For all $\triangle t > 0$,*

$$\|B\|_{\mathfrak{L}(\mathcal{D}(A_{\triangle t})\cap \mathcal{C}_{\delta/\triangle t}, Y)} \leq C_B \sup_{|\mu_j|\leq \delta/\triangle t} \Big\{ \frac{\mu_j}{k_{\triangle t}(\mu_j \triangle t)} \Big\}. \tag{B.9}$$

- *Uniform Admissibility. System* (B.8) *is uniformly admissible.*
- *Uniform Observability. System* (B.8) *is uniformly observable.*

Proof. To prove the first item, we write, for $z \in \mathcal{C}_{\delta/\triangle t}$, that

$$\begin{aligned} \|Bz\|_Y &\leq C_B \|Az\|_X \\ &\leq C_B \left\| A A_{\triangle t}^{-1} \right\|_{\mathfrak{L}(X,X)} \|A_{\triangle t} z\|_X , \end{aligned}$$

and the result follows.

The proof of the two others items is based on Lemma B.2:

Lemma B.2. *For $\omega \in \mathbb{R}$ and $\varepsilon > 0$, set*

$$J_\varepsilon^{\triangle t}(\omega) = \{ m \in \Lambda \quad \textit{such that} \quad |\lambda_m - \omega| < \varepsilon \},$$

where (λ_m) is the sequence of eigenvalues of $A_{\triangle t}$. We have the following Lemma: Let

$$\varepsilon = \frac{1}{\sup_{\mu \triangle t \leq \delta} \{ k'_{\triangle t}(\mu) \}} \ \varepsilon_0.$$

Then $J_\varepsilon^{\triangle t}(\lambda_n) \subset J_{\varepsilon_0}(\mu_n)$ if $|\mu_n| \leq \delta/\triangle t$.

Indeed, from the admissibility of the continuous system, we deduce that (A.2) holds for (A, B). Then from Lemma B.2, we show that estimate (A.2) holds as well, and finally we conclude from Proposition A.1 and the estimates given on the admissibility constants K_T in(A.1) that this implies the uniform admissibility.

For the last item, we use Proposition A.2. First, since (A, B) is observable, (.1) hols for B on the eigenvectors of A. From Lemma B.2, it is obvious that (.1) holds uniformly in $\triangle t$ by choosing ϵ small enough. Besides, the estimates given in (A.8) prove that the resolvent estimate holds uniformly, and therefore that $(A_{\triangle t}, B)$ are uniformly observable in a finite time, on which we have the estimate (A.5).

Using Section 5 and the theorems therein provides the result. Note however that this proof is longer than the one presented in Section 3 and is based on the main result Theorem 2.1 as well. Actually, this gives a longer path than via the resolvent estimate to prove (3.12).

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