Inverse problems for the Schrödinger equation via Carleman estimate

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The fourth order Schrödinger equation arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics. (Karpman 96, Karpman&Shagalov 00, Pausader 07, 09)

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The inverse problem of the second order Schrödinger equations has already been intensely studied.

(Baudouin&Puel 02,Mandache 02, Eskin 03, Mercado et al 08, Bellassoued&Choulli 10, Yuan&Yamamoto 10, Ignat et al 12)

However, for the higher order equations, due to the increased complexity, there are few papers investigating the stability of the inverse problems via Carleman estimates.

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One of the main techniques is the Carleman estimate, which is also a powerful tool for the controllability and observability problems of PDEs.

Model

We consider the following fourth order Schrödinger equation in $\Omega = (0,1)$:

$$
(*)\begin{cases}
iu_t + u_{xxxx} + pu = 0, & (t,x) \in (0,T) \times \Omega \\
u(t,0) = u(t,1) = 0, & t \in (0,T) \\
u_x(t,0) = u_x(t,1) = 0, & t \in (0,T) \\
u(0,x) = u_0(x), & x \in \Omega.
\end{cases}
$$

Aim

Find the information of $p = p(x)$ in the whole Ω by the information at $x = 1$ only: Can we estimate $\|q - p\|_{L^2(\Omega)}$, by a suitable norm of the derivatives of $u(q) - u(p)$ at the end point $x = 1$ (or, at $x = 0$) during the time interval $(0, T)$?

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This is an inverse problem!

Answer

The answer is YES!

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- $u_0(x) \in \mathbb{R}$ or $iu_0(x) \in \mathbb{R}$ *a.e.* in Ω, $|u_0(x)| \geq r > 0$ *a.e.* in Ω.
- $u^p \in W^{1,2}(0, T; W^{3,\infty}(\Omega))$,

  $\partial_x^2(u^p - u^q)(t, 1) \in H^1(0, T)$,

  $\partial_x^3(u^p - u^q)(t, 1) \in H^1(0, T)$.

Lipschitz Stability

Theorem (C. Zheng (MCRF2015))

Suppose that p, q, u_0 satisfy A1. Then for any $m \geq 0$, exists

$C = C(m, \|u^p\|_{H^1(0,T;L^\infty(\Omega))}, r) > 0$ *s.t.*

$$\|p - q\|^2_{L^2(\Omega)} \leq C \sum_{n=2,3} \|\partial_x^n (u^p - u^q)(\cdot, 1)\|^2_{H^1(0,T)}.$$

Carleman Estimate is a weighted inequality of the solution. It is complicated but very powerful.

Weight function

$$\theta = e^{l(t,x)}, \quad l(t,x) = \lambda \frac{e^{3\mu\psi(x)} - e^{5\mu\|\psi\|_\infty}}{t(T-t)}, \qquad \varphi(t,x) = \frac{e^{3\mu\psi(x)}}{t(T-t)}$$

with $\psi(x) = (x - x_0)^2, \quad x_0 < 0$. λ, μ are large positive constants.

The set

$$\mathcal{Z} = \{ u \in L^2(0,T; H^3(\Omega) \cap H_0^2(\Omega)),$$
$$Pu = iu_t + u_{xxxx} \in L^2(Q),$$
$$u_{xx}(\cdot, 1) \in L^2(0,T), \;\; u_{xxx}(\cdot, 1) \in L^2(0,T)\}.$$

Carleman estimate

Theorem (C. Zheng (MCRF2015))

Exist $\mu_0 > 1$ *and* $C > 0$ *s. t.* $\forall \mu \geq \mu_0$, *one can find a* λ_0 *s. t. for all* $\lambda > \lambda_0 = \lambda(\mu, T)$, $(M = \lambda\mu\varphi)$

$$\int_Q \mu\theta^2 \left(M^7|u|^2 + M^5|u_x|^2 + M^3|u_{xx}|^2 + M|u_{xxx}|^2 \right) dxdt$$

$$\leq C \left(\int_Q |\theta P u|^2 dxdt + \int_0^T \theta^2 (M^3|u_{xx}|^2 + M|u_{xxx}|^2)(t, 1) dt \right)$$

holds true for all $u \in \mathcal{Z}$, *where the constants* μ_0 *and* C *only depend on* x_0.

Remarks

- One could switch the observation data from $x = 1$ to $x = 0$ by taking a different weight function.

Remarks

- The Carleman estimate can be applied to the controllability problems. In fact, one can derive the exact controllability of a controlled fourth order semi-linear Schrödinger equations, with controls applied at the boundary point $x = 1$ by following the standard HUM procedure. Two controls will be needed on the boundary, since there exist two boundary terms and each of them corresponds to a control.

Remarks

- The Carleman estimate can be applied to the controllability problems. In fact, one can derive the exact controllability of a controlled fourth order semi-linear Schrödinger equations, with controls applied at the boundary point $x = 1$ by following the standard HUM procedure. Two controls will be needed on the boundary, since there exist two boundary terms and each of them corresponds to a control. Is it sharp? Open problem

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Model Transformation

Pick p, q as two different potentials. Corresponding solution u^p and u^q. Set $y = u^p - u^q$. Then

$$
(**)\begin{cases}
iy_t + y_{xxxx} + q(x)y = (q-p)(x)u^p(t,x), & (t,x) \in Q \\
y(t,0) = y(t,1) = 0, y_x(t,0) = y_x(t,1) = 0, & t \in (0,T) \\
y(0,x) = 0, & x \in \Omega.
\end{cases}
$$

Model Transformation

Extend $(**)$ from $t \in (0, T)$ to $(-T, T)$. Change t into $t + T$.

Then $t \in (0, 2T)$. Set $z(t,x) = y_t(2T - t, x)$. Then

$$
(***) \begin{cases} z_t + iz_{xxxx} + iq(x)z = i(q-p)(x)u_t^p(t,x), & \text{in}(0,2T) \times \Omega \\ z(t,0) = z(t,1) = 0, & t \in (0,2T) \\ z_x(t,0) = z_x(t,1) = 0, & t \in (0,2T) \\ z(T,x) = -i(q-p)(x)u^p(T,x), & x \in \Omega. \end{cases}
$$

Model Transformation

Extend (∗∗) from $t \in (0, T)$ to $(-T, T)$. Change t into $t + T$.

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(***) \begin{cases} z_t + iz_{xxxx} + iq(x)z = i(q-p)(x)u_t^p(t,x), & \text{in}(0,2T)\times\Omega \\ z(t,0) = z(t,1) = 0, & t \in (0,2T) \\ z_x(t,0) = z_x(t,1) = 0, & t \in (0,2T) \\ z(T,x) = -i(q-p)(x)u^p(T,x), & x \in \Omega. \end{cases}
$$

$z(T, x) = y_t(T, x)$ comes from the first equation of (∗∗).

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Estimation on z

Set the functional J and denote

$$J = \int_0^{2T} \int_\Omega I_1 \theta \bar{z} dx dt,$$

$$P_p = \partial_t + i\partial_x^4 + ip,$$

$$I_1 = i(\theta z)_t + l_x^4 \theta z + 6(l_x^2 - l_{xx})(\theta z)_{xx} + (\theta z)_{xxxx},$$

Upper bound of J

For the functional J, we have

$$
\begin{aligned}
|J| \leq & \left(\int_0^{2T}\int_\Omega |I_1|^2 dxdt\right)^{1/2}\left(\int_0^{2T}\int_\Omega \theta^2|z|^2 dxdt\right)^{1/2} \\
\leq & \lambda^{-7/2}\mu^{-4}\int_0^{2T}\int_\Omega |I_1|^2 dxdt + \lambda^{7/2}\mu^4\int_0^{2T}\int_\Omega \theta^2|z|^2 dxdt \\
\leq & C\lambda^{-7/2}\mu^{-4}\left(\int_0^{2T}\int_\Omega |I_1|^2 dxdt + \int_0^{2T}\int_\Omega \mu M^7\theta^2|z|^2 dxdt\right).
\end{aligned}
$$

(Left hand side of the following revised Carleman Estimate)

Upper bound of J

Lemma

Each $\|p\|_{L^\infty} \leq m$, *exist* $\mu_0 \geq 1$, $\lambda_0, C > 0$, *solutions of* $(***)$ *satisfy*

$$\int_{(0,2T)\times\Omega} \left(|I_1|^2 + \mu M^7 |\theta z|^2\right) dxdt \leq C \int_{(0,2T)\times\Omega} |\theta P_p z|^2 dxdt$$
$$+ C \int_0^{2T} \theta^2 (M^3 |z_{xx}|^2 + M |z_{xxx}|^2)(t,1) dt$$

Upper bound of J

Apply the revised Carleman estimate on the last line, we have

$$
\begin{aligned}
|J| \leq & C\lambda^{-\frac{7}{2}}\mu^{-4}\int_{\Omega} e^{2l(T,x)}|(q-p)(x)|^2 dx \\
& + C\lambda^{-\frac{1}{2}}\mu^{-1}\int_0^{2T} |z_{xx}(t,1)|^2 dt \\
& + C\lambda^{-\frac{5}{2}}\mu^{-3}\int_0^{2T} |z_{xxx}(t,1)|^2 dt.
\end{aligned} \tag{1}
$$

lower bound of J

Let $v = \theta z$, we have

$$J = \int_Q I_1 \bar{v}\, dxdt$$
$$= \int_Q i v_t \bar{v}\, dxdt \text{ Imaginary part}$$
$$+ \int_Q \{\cdot\}|v|^2 - \{\cdot\}|v_x|^2 + |v_{xx}|^2)\, dxdt. \text{ Real part}$$

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Hence,

$$\mathrm{Im}(J) = \frac{1}{2}\int_\Omega |v(T,x)|^2 dx = \frac{1}{2}\int_\Omega e^{2l(T,x)} |(q-p)(x)|^2 |u^p(T,x)|^2 dx.$$

lower bound of J

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Hence,

$$\mathrm{Im}(J) = \frac{1}{2}\int_\Omega |v(T,x)|^2 dx = \frac{1}{2}\int_\Omega e^{2l(T,x)}|(q-p)(x)|^2 |u^p(T,x)|^2 dx.$$

Since $u(T,x)$ is bounded below by $r > 0$, it follows that

$$(2) \qquad \frac{r^2}{2}\int_\Omega e^{2l(T,x)}|(q-p)(x)|^2 dx \le \mathrm{Im}(J).$$

Combine(1) and (2), we have

$$\int_\Omega e^{2l(T,x)}|(q-p)(x)|^2dx \leq C\left(\int_0^{2T} |z_{xx}(t,1)|^2 + |z_{xxx}(t,1)|^2\right)dt$$

for λ and μ large enough. Moreover, since

$$e^{2l(T,x)} \geq e^{2M} > 0, \qquad \text{with} \qquad M = \frac{\lambda}{T^2}(1-e^{5\mu\|\psi\|_\infty}),$$

we finish the proof of the Lipschitz Stability.

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(1). Set $v = \theta u$. $\theta P u = I_1 + I_2$, with

$$I_1 = i v_t + \{\cdot\} v + \{\cdot\} v_{xx} + v_{xxxx},$$

$$I_2 = -i l_t v + \{\cdot\} v + \{\cdot\} v_x + \{\cdot\} v_{xxx}.$$

Here we need the term $\{\cdot\} v$ to regulate the two terms.

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Here we need the term $\{\cdot\} v$ to regulate the two terms. (2). Compute $\theta P u \overline{\theta P u}$. Sort out to (DIFFICULT)

$$\begin{aligned} |\theta P u|^2 =& |I_1|^2 + |I_2|^2 + \{\cdot\}_x + +\{\cdot\}_t + \mathrm{Re}(\theta P u \bar{v} + \theta P u \bar{v}_{xx}) \\ &+ \mathrm{Re}(\{\cdot\} v \bar{v}_x + \{\cdot\} v \bar{v}_{xx} + \{\cdot\} v_x \bar{v}_{xx}) \\ &+ \{\cdot\}|v|^2 + \{\cdot\}|v_x|^2 + \{\cdot\}|v_{xx}|^2 + \{\cdot\}|v_{xxx}|^2. \end{aligned}$$

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(3). Integrate on $(0, T) \times \Omega$. Use the properties of the weight function θ to make all of the coefficients having positive symbols. Finally we arrive at the desired Carleman estimate.

Future works

1. Is it sharp for two observers?

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2. Higher dimensions.

Future works

1. Is it sharp for two observers?
2. Higher dimensions.
3. Discrete schemes, Convergence, Implementation, etc.

References I

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THANK YOU!