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INVERSE PROBLEMS FOR THE FOURTH ORDER SCHRÖDINGER EQUATION ON A FINITE DOMAIN

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Abstract. In this paper we establish a global Carleman estimate for the fourth order Schrödinger equation with potential posed on a $1-d$ finite domain. The Carleman estimate is used to prove the Lipschitz stability for an inverse problem consisting in recovering a stationary potential in the Schrödinger equation from boundary measurements.

1. **Introduction.** The fourth order Schrödinger equation arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references. For instance, its general nonlinear form is given in [12, 13] to take into account the role of small fourth order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The well-posedness and existence of the solutions has been shown (for instance, see [10, 18, 20]) by means of the energy method and harmonic analysis. In this paper, we are interested in the inverse problem for the fourth order Schrödingier equation posed on a finite interval.

To be more precise, we consider the following fourth order Schrödinger equation in $\Omega = (0,1)$:

$$\begin{cases} iu_t + u_{xxxx} + pu = 0, & (t,x) \in (0,T) \times \Omega \\ u(t,0) = u(t,1) = 0, u_x(t,0) = u_x(t,1) = 0, & t \in (0,T) \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{1}$$

For any initial data $u_0 \in H^3(\Omega) \cap H_0^2(\Omega)$ and $p \in L^2(\Omega)$, there exists a unique solution of (1) $u \in C^1([0,T]; L^2(\Omega)) \cap C([0,T]; H^3(\Omega) \cap H_0^2(\Omega))$(see, for instance, [19]).

The purpose of this paper is to determine the potential $p = p(x), x \in \Omega$ by means of the boundary measurements. The problem we are interested can be stated as follows: is it possible to estimate $\|q-p\|_{L^2(\Omega)}$, or better, a stronger norm of $q-p$, by a suitable norm of the derivatives of $u(q) - u(p)$ at the end point $x = 1$ (or, at $x = 0$) during the time interval $(0,T)$?

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Recently, the inverse problem of the Schrödinger equations has been intensely studied (see [1, 2, 7, 11, 16, 17, 22] and the references therein). One of the main techniques is the Carleman estimate ([1, 11, 14, 17, 22]), which is also a powerful tool for the controllability and observability problems of PDEs.

However, for the higher order equations, due to the increased complexity, there are few papers investigating the stability of the inverse problems via Carleman estimates. In [23], Zhang solves the exact controllability of semilinear plate equations via a Carleman estimate of the second order Schrödinger operator. Zhou ([25]) considers the observability results of the fourth order parabolic equation and Fu ([9]) derives the sharp observability inequality for the plate equation. In both papers, they show the Carleman estimates for the corresponding fourth order operators for $1-d$ cases, respectively.

To our knowledge, the result of determination of a time-independent potential for the fourth order Schrödinger equation from the boundary measurements on the endpoint is new. Furthermore, our work in this paper is the first one dealing with the Carleman estimate of the fourth order Schrödinger equation.

To begin with, we introduce a suitable weight function:

$$\psi(x) = (x-x_0)^2, \qquad x_0 < 0. \tag{2}$$

Let $\lambda \gg 1$ be a sufficiently large positive constant depending on Ω. For $t \in (0,T)$ and following [8], we introduce the functions

$$\theta = e^l, \qquad \varphi(t,x) = \frac{e^{3\mu\psi(x)}}{t(T-t)} \qquad \text{and} \qquad l(t,x) = \lambda \frac{e^{3\mu\psi(x)} - e^{5\mu\|\psi\|_\infty}}{t(T-t)} \tag{3}$$

with a positive constant μ. Denote by

$$Pu = iu_t + u_{xxxx}, \qquad Q = (0,T)\times\Omega \quad \text{and} \quad \int_Q (\cdot)dxdt = \int_0^T \int_\Omega (\cdot)dxdt.$$

We also introduce the set

$$\begin{aligned} \mathcal{Z} = \{\, & u \in L^2(0,T; H^3(\Omega)\cap H_0^2(\Omega)), Pu \in L^2(Q), \\ & u_{xx}(\cdot,1) \in L^2(0,T),\ \ u_{xxx}(\cdot,1) \in L^2(0,T)\}. \end{aligned}$$

The first main result is the following global Carleman estimate for system (1):

Theorem 1.1. *There exist two constants $\mu_0 > 1$ and $C > 0$ such that for all $\mu \ge \mu_0$, one can find a λ_0 such that for all $\lambda > \lambda_0 = \lambda(\mu,T)$,*

$$\begin{aligned} &\int_Q \left(\lambda^7\mu^8\varphi^7\theta^2|u|^2 + \lambda^5\mu^6\varphi^5\theta^2|u_x|^2 + \lambda^3\mu^4\varphi^3\theta^2|u_{xx}|^2 + \lambda\mu^2\varphi\theta^2|u_{xxx}|^2\right)dxdt \\ &\le C\Bigg(\int_Q |\theta Pu|^2 dxdt \\ &\qquad + \lambda^3\mu^3\int_0^T (\varphi^3\theta^2|u_{xx}|^2)(t,1)dt + \lambda\mu\int_0^T (\varphi\theta^2|u_{xxx}|^2)(t,1)dt\Bigg) \end{aligned} \tag{4}$$

holds true for all $u \in \mathcal{Z}$, where the constants μ_0 and C only depend on x_0.

Remark 1. Note that for simplicity, we give the exact form of the function $\psi(x)$ in (2). In fact, the statement holds true for any function satisfying

$$\psi \in C^4(\bar\Omega), \quad \psi > 0, \quad \psi_x \neq 0 \quad \text{in} \quad \bar\Omega \quad \text{with} \quad \bar\Omega = [0,1], \qquad \psi_x(0) > 0, \psi_x(1) > 0.$$

It is worth to mention that, by taking $x_0 > 1$, one could switch the observation data in (4) to the left end-point $x = 0$.

Remark 2. Note that λ_0 has the order of T^2 with respect to the time T, as we will show in the proof.

Remark 3. [24] shows an observability inequality which estimates initial data by the measurement of Δu for a Schrödinger equation without the potential q on $\Gamma_0 = \{x \in \partial\Omega; (x - x_0) \cdot \nu(x) \geq 0\}$ using a multiplier identity and Holmgren's uniqueness theorem. Observability inequalities are technically related to our inverse problem (see [21]). However, the approach in [24] can not be applied to our problem, even though less observability data are considered.

Remark 4. Note that the Carleman estimate (4) also can be applied to the controllability problems. In fact, one can derive the exact controllability of a controlled fourth order semi-linear Schrödinger equations, with controls applied at the boundary point $x = 1$ by following the standard procedure (see, for instance, [25]). Two controls will be needed on the boundary, due to the fact that there exist two boundary terms on the right hand side of (4) and each of them corresponds to a control.

In what follows, we shall denote by u^p the solution of the system (1) associated with the potential p. Following the standard procedure from the Carleman estimate to the inverse problem (see, for instance, [17]), we answer the previous question with the following Theorem:

Theorem 1.2. *Suppose that $p \in L^\infty(\Omega)$, $u_0 \in L^\infty(\Omega)$ and $r > 0$ are such that*

- *$u_0(x) \in \mathbb{R}$ or $iu_0(x) \in \mathbb{R}$ a.e. in Ω,*
- *$|u_0(x)| \geq r > 0$ a.e. in Ω, and*
- *$u^p \in W^{1,2}(0, T; W^{3,\infty}(\Omega))$.*

Then, for any $m = \|q\|_{L^\infty(\Omega)} \geq 0$, exists a constant $C = C(m, \|u^p\|_{H^1(0,T;L^\infty(\Omega))}, r)$ > 0 such that for any $p \in L^\infty(\Omega)$ satisfying

$$u^p_{xx}(t,1) - u^q_{xx}(t,1) \in H^1(0,T) \qquad \text{and} \qquad u^p_{xxx}(t,1) - u^q_{xxx}(t,1) \in H^1(0,T), \quad (5)$$

we have that

$$\begin{aligned} \|p - q\|^2_{L^2(\Omega)} \leq & C \left\| u^p_{xx}(\cdot,1) - u^q_{xx}(\cdot,1) \right\|^2_{H^1(0,T)} \\ & + C \left\| u^p_{xxx}(\cdot,1) - u^q_{xxx}(\cdot,1) \right\|^2_{H^1(0,T)}. \end{aligned} \quad (6)$$

Remark 5. By the classical regularity results for fourth order Schrödinger equations (see [5, Chapter 2] for example), we know that the q which fulfills (5) and (6) does exist.

The rest of the paper is organized as follows. In Section 2, we state a weighted point wise inequality for the fourth order Schrödinger operator. In Section 3, we establish a global Carleman estimate for a fourth order Schrödinger equation with a potential. The proof of Theorem 1.2 is given in Section 4. Finally we list several comments and some open problems for the future work.

2. **A weighted point-wise estimate for the fourth order operator.**

In this section, we shall establish a weighted identity for 1-d Schrödinger operator, which will pay an important role in the proof of the Carleman estimate (4).

Theorem 2.1. *Assume that u is sufficiently smooth and $\Psi \in C^2(\mathbb{R}^n;\mathbb{R})$. Set $v=\theta u$ where θ is given by (3). Then*

$$\begin{aligned}
&|\theta Pu|^2 - A_x - B_t - \tilde{a}_0\theta(Pu\bar{v}+\overline{Pu}v) - 6l_{xx}\theta(Pu\bar{v}_{xx}+\overline{Pu}v_{xx})\\
=&|I_1|^2+|I_2|^2+D(iv\bar{v}_x+\overline{iv}v_x)+6il_tl_{xx}(v\bar{v}_{xx}-\bar{v}v_{xx})+4il_{tx}(v_{xx}\bar{v}_x-\bar{v}_{xx}v_x)\\
&+16l_{xx}|v_{xxx}|^2+(24l_x^2l_{xx}-24l_xl_{xxx}+48l_{xx}^2-20l_{xxxx})|v_{xx}|^2\\
&\left\{\begin{array}{c} -(a_1a_2)_x-2(a_0-\Psi)a_2-4C_{41,xx}+3C_{24,x}\\ -a_{1,xxx}-\dfrac{3}{2}(a_{3,x}a_1)_x-3a_{3,x}a_0+2a_2\tilde{a}_0\end{array}\right\}|v_x|^2\\
&\left\{\begin{array}{c} +2(a_0-\Psi)\Psi-(a_1\Psi)_x-2a_0\tilde{a}_0-C_{24,xxx}+[(a_0-\Psi)a_2]_{xx}\\ -(a_1\tilde{a}_0)_x+(\dfrac{3}{2}a_{3,x}a_0-a_2\tilde{a}_0)_{xx}+C_{41,xxxx}+l_{tt}\end{array}\right\}|v|^2,
\end{aligned}\tag{7}$$

where

$$\left\{\begin{array}{ll}
a_0=l_x^4-6l_x^2l_{xx}+3l_{xx}^2+4l_xl_{xxx}-l_{xxxx}, & a_2=6(l_x^2-l_{xx}),\\
a_1=-4(l_x^3-3l_xl_{xx}+l_{xxx}), & a_3=-4l_x,\\
\tilde{a}_0=l_x^4-\Psi-2l_xl_{xxx}-3l_{xx}^2, & \\
C_{24}=4l_x(l_x^4-2\Psi-2l_xl_{xxx}-3l_{xx}^2), & \\
C_{41}=-6l_x^2l_{xx}+6l_xl_{xxx}+6l_{xx}^2-l_{xxxx}, &
\end{array}\right.\tag{8}$$

and

$$I_1=iv_t+\Psi v+a_2v_{xx}+v_{xxxx},\qquad I_2=-il_tv+(a_0-\Psi)v+a_1v_x+a_3v_{xxx}.\tag{9}$$

Moreover, we have

$$\begin{aligned}
A=&ia_3(v_t\bar{v}_{xx}-\bar{v}_tv_{xx})-\frac{i}{2}a_3(v_{xt}\bar{v}_x-\bar{v}_{xt}v_x)+\frac{i}{2}a_{3,x}(v_t\bar{v}_x-\bar{v}_tv_x)\\
&+\frac{3}{2}a_{3,x}(v_{xxx}\bar{v}_{xx}+\bar{v}_{xxx}v_{xx})+a_1(v_{xxx}\bar{v}_x+\bar{v}_{xxx}v_x)+C_{41}(v_{xxx}\bar{v}+\bar{v}_{xxx}v)\\
&+il_t(v_{xxx}\bar{v}-\bar{v}_{xxx}v)-il_t(v_{xx}\bar{v}_x-\bar{v}_{xx}v_x)+(C_{24}-C_{41,x})(v_{xx}\bar{v}+\bar{v}_{xx}v)\\
&-il_{tx}(v_{xx}\bar{v}-\bar{v}_{xx}v)+i(l_{txx}+a_2l_t)(v_x\bar{v}-\bar{v}_xv)+\frac{i}{4}(2a_1-a_{3,xx})(v_t\bar{v}-\bar{v}_tv)\\
&+[(a_0-\Psi)a_2-C_{24,x}-a_{1,x}-C_{41,xx}+\frac{3}{2}a_{3,x}a_0-a_2\tilde{a}_0](v_x\bar{v}+\bar{v}_xv)\\
&+a_3|v_{xxx}|^2+(a_2a_3-\frac{3}{2}a_{3,x}a_3-\frac{3}{2}a_{3,xx}-a_1)|v_{xx}|^2\\
&+(a_1a_2+a_{1,xx}-C_{24}-2C_{41,x}+\frac{3}{2}a_{3,x}a_1)|v_x|^2\\
&+\left(a_1(\Psi+\tilde{a}_0)+[(a_0-\Psi)a_2]_x-(\frac{3}{2}a_{3,x}a_0-a_2\tilde{a}_0-C_{24})_{xx}-C_{41,xxx}\right)|v|^2,\\
B=&-l_t|v|^2-\frac{i}{2}a_3(v_x\bar{v}_{xx}-\bar{v}_xv_{xx})+\frac{i}{4}(2a_1-a_{3,xx})(v\bar{v}_x-\bar{v}v_x),\\
D=&2(6l_x^2l_{xt}+6l_xl_{xx}l_t-6l_{xx}l_{xt}-3l_xl_{xxt}-3l_tl_{xxx}+l_{xxxt}).
\end{aligned}$$

Remark 6. The key ideas of the proof is as the follows: we separate θPu into the even-order part I_1 and the odd-order part I_2 as in (9) and compute the real part of the product of $I_1\bar{I}_2$. Identity (7) is a result by collecting those like terms in the multiplication. Note that the extra function Ψ plays a crucial role for adjusting the coefficient of the like terms, as we will see in Section 3.

Proof. We may assume that u is sufficiently smooth. Since $v=\theta u$ and notice the definitions of $a_i, i=0,1,2,3$ in (8), it is esay to get

$$\theta Pu = iv_t - il_t v + v_{xxxx} + a_0 v + a_1 v_x + a_2 v_{xx} + a_3 v_{xxx}.$$

We divide Pu into I_1 and I_2 as in (9). Multiplying θPu by its conjugate we have

$$|\theta Pu|^2 = |I_1|^2 + |I_2|^2 + (I_1\bar{I}_2 + \bar{I}_1 I_2) = |I_1|^2 + |I_2|^2 + \sum_{i,j=1}^{4} I_{ij}, \tag{10}$$

where I_{ij} denotes the sum of the i-th term of I_1 times the j-th term of $\bar{I}_2$ in $I_1\bar{I}_2$ and its conjugate part in $\bar{I}_1 I_2$.

The computations will be treated in the following two parts.

Part I: We compute $I_{1j}, j=1,2,3,4$. We first have

$$I_{11} = -l_t(v_t\bar{v} + \bar{v}_t v) = -(l_t|v|^2)_t + l_{tt}|v|^2.$$

On the other hand, it is easy to get that

$$\begin{aligned} I_{13} = ia_1(v_t\bar{v}_x - \bar{v}_t v_x) =& \frac{i}{2}\{-a_{1,x}(v_t\bar{v} - \bar{v}_t v) + [a_1(v\bar{v}_x - \bar{v}v_x)]_t \\ &+ [a_1(v_t\bar{v} - \bar{v}_t v)]_x - a_{1,t}(v\bar{v}_x - \bar{v}v_x)\}. \end{aligned} \tag{11}$$

Moreover,

$$\begin{aligned} I_{14} =& ia_3(v_t\bar{v}_{xxx} - \bar{v}_t v_{xxx}) \\ =& -\frac{3i}{2}a_{3,x}(v_t\bar{v}_{xx} - \bar{v}_t v_{xx}) - \frac{i}{2}a_{3,xx}(v_t\bar{v}_x - \bar{v}_t v_x) \\ &- \frac{i}{2}[a_3(v_x\bar{v}_{xx} - \bar{v}_x v_{xx})]_t + \frac{i}{2}a_{3,t}(v_x\bar{v}_{xx} - \bar{v}_x v_{xx}) \\ &+ \frac{i}{2}\{2a_3(v_t\bar{v}_{xx} - \bar{v}_t v_{xx}) - a_3(v_{xt}\bar{v}_x - \bar{v}_{xt}v_x) + a_{3,x}(v_t\bar{v}_x - \bar{v}_t v_x)\}_x. \end{aligned} \tag{12}$$

By replacing a_1 in (11) by $a_{3,xx}$, substituting it into the last term of (12), we have

$$\begin{aligned} I_{14} =& -\frac{3i}{2}a_{3,x}(v_t\bar{v}_{xx} - \bar{v}_t v_{xx}) + \frac{i}{4}a_{3,xxx}(v_t\bar{v} - \bar{v}_t v) \\ &- \frac{i}{4}\{[a_{3,xx}(v\bar{v}_x - \bar{v}v_x)]_t + [a_{3,xx}(v_t\bar{v} - \bar{v}_t v)]_x - a_{3,xxt}(v\bar{v}_x - \bar{v}v_x)\} \\ &- \frac{i}{2}[a_3(v_x\bar{v}_{xx} - \bar{v}_x v_{xx})]_t + \frac{i}{2}a_{3,t}(v_x\bar{v}_{xx} - \bar{v}_x v_{xx}) \\ &+ \frac{i}{2}\{2a_3(v_t\bar{v}_{xx} - \bar{v}_t v_{xx}) - a_3(v_{xt}\bar{v}_x - \bar{v}_{xt}v_x) + a_{3,x}(v_t\bar{v}_x - \bar{v}_t v_x)\}_x. \end{aligned} \tag{13}$$

Set

$$\tilde{a}_0 = a_0 - \Psi - \frac{1}{2}a_{1,x} - \frac{1}{4}a_{3,xxx}.$$

Obviously, it is the coefficient of the term $i(v_t\bar{v} - \bar{v}_t v)$ in $\sum_{j=1}^{4} I_{1j}$. Taking the exact form of a_0, a_1, a_3 in (8) into account, one can verifty that $\tilde{a}_0$ is exactly the one in (8). Furthermore,

$$\begin{aligned} &\tilde{a}_0 i(v_t\bar{v} - \bar{v}_t v) = \tilde{a}_0\theta(Pu\bar{v} + \overline{Pu}v) - 2a_0\tilde{a}_0|v|^2 - a_1\tilde{a}_0(v_x\bar{v} + \bar{v}_x v) \\ &- a_2\tilde{a}_0(v_{xx}\bar{v} + \bar{v}_{xx}v) - a_3\tilde{a}_0(v_{xxx}\bar{v} + \bar{v}_{xxx}v) - \tilde{a}_0(v_{xxxx}\bar{v} + \bar{v}_{xxxx}v). \end{aligned} \tag{14}$$

Meanwhile, for the first term of I_{14}, recalling that $a_3 = -4l_x$, we have

$$\begin{aligned}
&-\frac{3i}{2}a_{3,x}(v_t\bar{v}_{xx} - \bar{v}_t v_{xx})\\
=&6l_{xx}\theta(Pu\bar{v}_{xx} + \overline{Pu}v_{xx}) + 6il_{xx}l_t(v\bar{v}_{xx} - \bar{v}_{xx})\\
&-6l_{xx}a_0(v_{xx}\bar{v} + \bar{v}_{xx}v) - 6l_{xx}a_1(v_{xx}\bar{v}_x + \bar{v}_{xx}v_x) - 12l_{xx}a_2|v_{xx}|^2\\
&-6l_{xx}a_3(v_{xxx}\bar{v}_{xx} + \bar{v}_{xxx}v_{xx}) - 6l_{xx}(v_{xxxx}\bar{v}_{xx} + \bar{v}_{xxxx}v_{xx}).
\end{aligned} \tag{15}$$

Summing up $I_{12} = i(a_0 - \Psi)(v_t\bar{v} - \bar{v}_t v)$, I_{13} as (11) and I_{14} as (13), taking (14) and (15) into accout, we arrive at

$$\begin{aligned}
\sum_{j=2,3,4} I_{1j} =&\{\cdot\}_x + \{\cdot\}_t + \tilde{a}_0\theta(Pu\bar{v} + \overline{Pu}v) + 6l_{xx}\theta(Pu\bar{v}_{xx} + \overline{Pu}v_{xx})\\
&+\frac{3}{2}a_{3,x}(v_{xxxx}\bar{v}_{xx} + \bar{v}_{xxxx}v_{xx}) - \tilde{a}_0(v_{xxxx}\bar{v} + \bar{v}_{xxxx}v)\\
&+\frac{3}{2}a_{3,x}a_3(v_{xxx}\bar{v}_{xx} + \bar{v}_{xxx}v_{xx}) - a_3\tilde{a}_0(v_{xxx}\bar{v} + \bar{v}_{xxx}v)\\
&+\frac{3}{2}a_{3,x}a_1(v_{xx}\bar{v}_x + \bar{v}_{xx}v_x) + (\frac{3}{2}a_{3,x}a_0 - a_2\tilde{a}_0)(v_{xx}\bar{v} + \bar{v}_{xx}v)\\
&+\frac{i}{2}a_{3,t}(v_x\bar{v}_{xx} - \bar{v}_x v_{xx}) + \frac{3}{2}il_t a_{3,x}(v_{xx}\bar{v} - \bar{v}_{xx}v) + 3a_{3,x}a_2|v_{xx}|^2\\
&+\frac{i}{4}(a_{3,xxt} - 2a_{1,t})(v\bar{v}_x - \bar{v}v_x) + a_1\tilde{a}_0(v_x\bar{v} + \bar{v}_x v) - 2a_0\tilde{a}_0|v|^2,
\end{aligned} \tag{16}$$

with

$$\{\cdot\}_x = \begin{pmatrix} ia_3(v_t\bar{v}_{xx} - \bar{v}_t v_{xx}) - \frac{i}{2}a_3(v_{xt}\bar{v}_x - \bar{v}_{xt}v_x)\\ +\frac{i}{2}a_{3,x}(v_t\bar{v}_x - \bar{v}_t v_x) + \frac{i}{4}(2a_1 - a_{3,xx})(v_t\bar{v} - \bar{v}_t v) \end{pmatrix}_x$$

and

$$\{\cdot\}_t = \left(-\frac{i}{2}a_3(v_x\bar{v}_{xx} - \bar{v}_x v_{xx}) + \frac{i}{4}(2a_1 - a_{3,xx})(v\bar{v}_x - \bar{v}v_x)\right)_t.$$

Part II: We compute the rest of I_{ij}, with some extra terms coming from (16).

Set $C_{24} = a_3\Psi - a_3\tilde{a}_0$, which is the same notation as in (8). We have the following identity:

$$\begin{aligned}
&I_{24} - a_3\tilde{a}_0(v_{xxx}\bar{v} + \bar{v}_{xxx}v) = C_{24}(v_{xxx}\bar{v} + \bar{v}_{xxx}v)\\
=&\begin{pmatrix} C_{24}(v_{xx}\bar{v} + \bar{v}_{xx}v) - C_{24}|v_x|^2\\ -C_{24,x}(v_x\bar{v} + \bar{v}_x v) + C_{24,xx}|v|^2 \end{pmatrix}_x + 3C_{24,x}|v_x|^2 - C_{24,xxx}|v|^2.
\end{aligned}$$

Consequently, it holds

$$\begin{aligned}
&\sum_{j=1}^{4} I_{2j} - a_3\tilde{a}_0(v_{xxx}\bar{v} + \bar{v}_{xxx}v)\\
=&0 + 2(a_0 - \Psi)\Psi v\bar{v} + a_1\Psi(v_x\bar{v} + \bar{v}_x v) + C_{24}(v_{xxx}\bar{v} + \bar{v}_{xxx}v)\\
=&\{C_{24}(v_{xx}\bar{v} + \bar{v}_{xx}v) - C_{24,x}(v_x\bar{v} + \bar{v}_x v) - C_{24}|v_x|^2 + (a_1\Psi + C_{24,xx})|v|^2\}_x\\
&+ 3C_{24,x}|v_x|^2 + \{2(a_0 - \Psi)\Psi - (a_1\Psi)_x - C_{24,xxx}\}|v|^2.
\end{aligned}$$

Now we compute $I_{3j}, j = 1, 2, 3, 4$. It holds

$$I_{31} = ia_2 l_t(v_{xx}\bar{v} - \bar{v}_{xx}v) = \{ia_2 l_t(v_x\bar{v} - \bar{v}_x v)\}_x - (ia_2 l_t)_x(v_x\bar{v} - \bar{v}_x v),$$

and

$$\sum_{j=2,3,4} I_{3j} = C_x + [(a_0-\Psi)a_2]_{xx}|v|^2 - [(a_1a_2)_x + 2(a_0-\Psi)a_2]|v_x|^2 - (a_2a_3)_x|v_{xx}|^2,$$

with

$$C = (a_0-\Psi)a_2(v_x\bar v + \bar v_x v) + [(a_0-\Psi)a_2]_x|v|^2 + a_1a_2|v_x|^2 + a_2a_3|v_{xx}|^2.$$

For the term I_{41}, it holds:

$$\begin{aligned} I_{41} =& il_t(v_{xxxx}\bar v - \bar v_{xxxx}v)\\ =&[il_t(v_{xxx}\bar v - \bar v_{xxx}v - v_{xx}\bar v_x + \bar v_{xx}v_x) - il_{tx}(v_{xx}\bar v - \bar v_{xx}v)\\ &+ il_{txx}(v_x\bar v - \bar v_x v)]_x + 2il_{tx}(v_{xx}\bar v_x - \bar v_{xx}v_x) - il_{txxx}(v_x\bar v - \bar v_x v). \end{aligned}$$

I_{42} is considered with an extra term from I_{14} as follows:

$$\begin{aligned} I_{42} - \tilde a_0(v_{xxxx}\bar v + \bar v_{xxxx}v) =& C_{41}(v_{xxxx}\bar v + \bar v_{xxxx}v)\\ =& E_x + 2C_{41}|v_{xx}|^2 - 4C_{41,xx}|v_x|^2 + C_{41,xxxx}|v|^2, \end{aligned}$$

with

$$\begin{aligned} E =& C_{41}(v_{xxx}\bar v + \bar v_{xxx}v) - C_{41}(v_{xx}\bar v_x + \bar v_{xx}v_x) - C_{41,x}(v_{xx}\bar v + \bar v_{xx}v)\\ &+ C_{41,xx}(v_x\bar v + \bar v_x v) - 2C_{41,x}|v_x|^2 - C_{41,xxx}|v|^2. \end{aligned}$$

Note that it is not hard to verify that C_{41} has the form as in (8).

Finally, the last two terms I_{43} and I_{44} equal to

$$I_{43} = a_1(v_{xxxx}\bar v_x + \bar v_{xxxx}v_x) = F_x - a_{1,xxx}|v_x|^2 + 3a_{1,x}|v_{xx}|^2,$$

with

$$F = a_1(v_{xxx}\bar v_x + \bar v_{xxx}v_x) - a_{1,x}(v_{xx}\bar v_x + \bar v_{xx}v_x) + a_{1,xx}|v_x|^2 - a_1|v_{xx}|^2,$$

and

$$I_{44} = a_3(v_{xxxx}\bar v_{xxx} + \bar v_{xxxx}v_{xxx}) = (a_3|v_{xxx}|^2)_x - a_{3,x}|v_{xxx}|^2.$$

By the previous computations, combining all "$\frac{\partial}{\partial t}$-terms", all "$\frac{\partial}{\partial x}$-terms" and by (10) we arrive at the desired inequality (7). □

3. **Global Carleman estimate: Proof of Theorem 1.1.** In this section, we obtain a global Carleman estimate inequality for the Schrödinger equation (1) via the point wise inequality (7). Recalling the definitions of l and φ in (3), it is easy to check that

$$\begin{aligned} |\partial_x^n l| &\le C(\psi)\lambda\mu^n\varphi, \qquad n=1,\cdots,8,\\ |\partial_x^n l_t| &\le C(\psi)\lambda\mu^n T\varphi^2, \quad n=1,\cdots,3,\\ |l_t| &\le C\lambda T\varphi^2, \qquad |l_{tt}| \le C\lambda T^2\varphi^3. \end{aligned}$$

We now give the proof of Theorem 1.1.

Proof. The proof is divided into several steps.

Step 1. Take

$$\Psi(t,x) = l_x^4.$$

Recalling the notations in (8), it is easy to check that the term $\{\cdots\}|v|^2$ in (7) satisfies

$$\{\cdots\}|v|^2 = 16l_x^6 l_{xx}|v|^2 - D_1|v|^2, \qquad |D_1| \ge -C(\psi)\lambda^6\mu^8\varphi^6. \tag{17}$$

Similarly, we have

$$\{\cdots\}|v_x|^2 = 144 l_x^4 l_{xx}|v_x|^2 - D_2|v_x|^2, \qquad |D_2| \geq -C(\psi)\lambda^4\mu^6\varphi^4, \tag{18}$$

and

$$\{\cdots\}|v_{xx}|^2 = 24 l_x^2 l_{xx}|v_{xx}|^2 - D_3|v_{xx}|^2, \qquad |D_3| \geq -C(\psi)\lambda^2\mu^4\varphi^2. \tag{19}$$

Now we consider those hybrid terms in (7). It holds

$$D(v\bar{v}_x - \bar{v}v_x) \geq -C(\psi)\lambda^3\mu^3 T\varphi^4(|v|^2 + |v_x|^2), \tag{20}$$

$$6il_t l_{xx}(v\bar{v}_{xx} - \bar{v}v_{xx}) \geq -C(\psi)\lambda^2\mu^2 T\varphi^3(|v|^2 + |v_{xx}|^2), \tag{21}$$

$$4il_{tx}(v_{xx}\bar{v}_x - \bar{v}_{xx}v_x) \geq -C(\psi)\lambda\mu T\varphi^2(|v_x|^2 + |v_{xx}|^2), \tag{22}$$

$$\tilde{a}_0\theta(Pu\bar{v} + \overline{Pu}v) \geq -C(\psi)\lambda^4\mu^8\varphi^4|v|^2 - C(\psi)|\theta Pu|^2, \tag{23}$$

and

$$6l_{xx}\theta(Pu\bar{v}_{xx} + \overline{Pu}v_{xx}) \geq -C(\psi)\lambda^2\mu^4\varphi^2|v_{xx}|^2 - C(\psi)|\theta Pu|^2. \tag{24}$$

Taking (17)–(24) into (7), one can find a sufficiently large constant $C(\psi) > 0$, only depending on ψ, such that

$$\begin{aligned} &C(\psi)(|\theta Pu|^2 - A_x - B_t + \lambda^6\mu^8\varphi^6|v|^2 + \lambda^4\mu^6\varphi^4|v_x|^2 + \lambda^2\mu^4\varphi^2|v_{xx}|^2) \\ \geq &16 l_x^6 l_{xx}|v|^2 + 144 l_x^4 l_{xx}|v_x|^2 + 24 l_x^2 l_{xx}|v_{xx}|^2 + 16 l_{xx}|v_{xxx}|^2. \end{aligned} \tag{25}$$

Step 2. Now we integrate (25) with respect to t and x. By the definition of B and $v = \theta u$ with $\theta(0,x) = \theta(T,x) = 0$, it is obvious that

$$-\int_Q B_t dxdt = 0. \tag{26}$$

Hence, we have

$$\begin{aligned} &C(\psi)\left(\int_Q |\theta Pu|^2 dxdt - \int_Q A_x dxdt\right) \\ \geq \ &\int_Q (16 l_x^6 l_{xx} - C(\psi)\lambda^6\mu^8\varphi^6)|v|^2 dxdt + \int_Q (144 l_x^4 l_{xx} - C(\psi)\lambda^4\mu^6\varphi^4)|v_x|^2 dxdt \\ &+ \int_Q (24 l_x^2 l_{xx} - C(\psi)\lambda^2\mu^4\varphi^2)|v_{xx}|^2 dxdt + \int_Q 16 l_{xx}|v_{xxx}|^2 dxdt. \end{aligned} \tag{27}$$

Since

$$l_x = \lambda\mu\psi_x\varphi = \lambda\mu(x - x_0)\varphi, \qquad l_{xx} = \lambda\mu(4\mu(x - x_0)^2 + 2)\varphi \tag{28}$$

by (2) and $\varphi \leq \frac{T^2}{4}\varphi^2$, by choosing $\mu \geq \mu_0 \geq 1$ and $\lambda \geq \lambda_0(\mu) = C(\psi)(T + T^2)$, it holds that

$$\int_Q (16 l_x^6 l_{xx} - C(\psi)\lambda^6\mu^8\varphi^6)|v|^2 dxdt \geq 16\int_Q 2^8(x - x_0)^8\lambda^7\mu^8\varphi^7|v|^2 dxdt.$$

Similarly,

$$\int_Q (144 l_x^4 l_{xx} - C(\psi)\lambda^4\mu^6\varphi^4)|v_x|^2 dxdt \geq 144\int_Q 2^6(x - x_0)^6\lambda^5\mu^6\varphi^5|v_x|^2 dxdt,$$

$$\int_Q (24 l_x^2 l_{xx} - C(\psi)\lambda^2\mu^4\varphi^2)|v_{xx}|^2 dxdt \geq 24\int_Q 2^4(x - x_0)^4\lambda^3\mu^4\varphi^3|v_{xx}|^2 dxdt,$$

and

$$\int_Q 16 l_{xx}|v_{xxx}|^2 dxdt \geq 16\int_Q 2^2(x - x_0)^2\lambda\mu^2\varphi|v_{xxx}|^2 dxdt.$$

For the term A_x, since v, v_x, v_t and v_{tx} vanish as $x = 0, 1$ for any $t \in (0, T)$, we have

$$\begin{aligned}
-\int_Q A_x dxdt &= -\int_0^T A(t,1)dt + \int_0^T A(t,0)dt \\
&= \int_0^T \big((20l_x^3 + 12l_x l_{xx} - 10l_{xx})|v_{xx}|^2 \\
&\qquad + 4l_x|v_{xxx}|^2 + 6l_{xx}(v_{xxx}\bar{v}_{xx} + \bar{v}_{xxx}v_{xx})\big)\,|_{x=0}^{x=1}dt.
\end{aligned}$$

Recalling l_x and l_{xx} in (28), by taking λ sufficiently large, we have

$$\int_0^T A(t,0)dt > 0,$$

and

$$-\int_0^T A(t,1)dt \le C(\psi)\int_0^T \left(\lambda^3\mu^3\varphi^3(t,1)|v_{xx}(t,1)|^2 + \lambda\mu\varphi(t,1)|v_{xxx}(t,1)|^2\right) dt.$$

Substituting the previous estimates into (27), it holds

$$\begin{aligned}
&\int_Q (\lambda^7\mu^8\varphi^7|v|^2 + \lambda^5\mu^6\varphi^5|v_x|^2 + \lambda^3\mu^4\varphi^3|v_{xx}|^2 + \lambda\mu^2\varphi|v_{xxx}|^2)dxdt \\
\le& C(\psi)\left(\int_Q |\theta P u|^2 dxdt + \int_0^T (\lambda^3\mu^3\varphi^3(t,1)|v_{xx}(t,1)|^2 + \lambda\mu\varphi(t,1)|v_{xxx}(t,1)|^2)dt\right).
\end{aligned}$$

Moreover, since $v = e^l u$, we compute

$$\begin{aligned}
v_x &= \theta(u_x + l_x u), \\
v_{xx} &= \theta(u_{xx} + 2l_x u_x + (l_x^2 + l_{xx})u), \\
v_{xxx} &= \theta(u_{xxx} + 3l_x u_{xx} + (3l_x^2 + 3l_{xx})u_x + (l_x^3 + 3l_x l_{xx} + l_{xxx})u).
\end{aligned}$$

By Young's inequality, it is not difficult to obtain

$$\begin{aligned}
&\int_Q \lambda^7\mu^8\varphi^7\theta^2|u|^2 dxdt + \int_Q \lambda^5\mu^6\varphi^5\theta^2|u_x|^2 dxdt \\
&+ \int_Q \lambda^3\mu^4\varphi^3\theta^2|u_{xx}|^2 dxdt + \int_Q \lambda\mu^2\varphi\theta^2|u_{xxx}|^2 dxdt \\
\le& C(\psi)\int_Q |\theta P u|^2 dxdt \\
&+ C(\psi)\int_0^T \left(\lambda^3\mu^3\varphi^3(t,1)\theta^2(t,1)|u_{xx}(t,1)|^2 + \lambda\mu\varphi(t,1)\theta^2(t,1)|u_{xxx}(t,1)|^2\right) dt,
\end{aligned}$$

which is exactly the statement of Theorem 1.1. □

4. **Boundary observations.** In this section, we give the proof of Theorem 1.2, which is a direct application of the Carleman inequality (4). The standard procedure can be found in [1, 17].

We first state a revised Carleman estimate:

Proposition 1. *Let $p \in L^\infty(\Omega, \mathbb{R})$. Let I_1 be defined in (9) and*

$$P_p = \partial_t + i\partial_x^4 + ip$$

and the space

$$\mathcal{Z}_p = \big\{ z \in L^2(Q); L_p z \in L^2(Q),\ z(t,0) = z(t,1) = z_x(t,0) = z_x(t,1) = 0, \\ \text{for all } t \in (0,T),\ u_{xx}(\cdot,1) \in L^2(0,T),\ u_{xxx}(\cdot,1) \in L^2(0,T) \big\}.$$

Then for any $m \geq 0$*, there exist* $\mu_0 \geq 1$*,* $\lambda_0 \geq 0$ *and* $C > 0$ *such that for each* $p \in L^\infty(\Omega)$ *with* $\|p\|_{L^\infty} \leq m$ *it holds*

$$\begin{aligned}&\int_Q \left(\lambda^7\mu^8\varphi^7\theta^2|z|^2 + |I_1|^2\right) dxdt \\ \leq & C\left(\int_Q \theta^2|P_p z|^2 dxdt + \int_0^T \left(\lambda^3\mu^3\varphi^3\theta^2|z_{xx}|^2 + \lambda\mu\varphi\theta^2|z_{xxx}|^2\right)(t,1)dt\right)\end{aligned} \tag{29}$$

for all $\lambda \geq \lambda_0$*,* $\mu \geq \mu_0$ *and* $z \in \mathcal{Z}_p$*.*

Proof. The term $|I_1|^2$ can be added by directly taking (10) into account. Moreover, the operator P can be changed to P_p since p is assumed to be uniformly bounded and the cost is a slight change of C with respect to the upper bound m. □

Now we state the proof of Th. 1.2.

Proof. We divide the proof into two parts.

Step 1.(Model transformation) Pick any p, q as in the statement of the theorem, and introduce the difference $y := u^p - u^q$ of the corresponding solutions of (1).

Then y fulfills the system

$$\begin{cases} iy_t + y_{xxxx} + q(x)y = f(x)R(t,x), & (t,x) \in Q \\ y(t,0) = y(t,1) = 0, y_x(t,0) = y_x(t,1) = 0, & t \in (0,T) \\ y(0,x) = 0, & x \in \Omega. \end{cases} \tag{30}$$

with $f := q - p$ (real valued) and $R := u^p$.

Since $f \in L^2(\Omega;\mathbb{R})$ and $R \in H^1(0,T;L^\infty(\Omega))$ and $R(0,x) \in \mathbb{R}$ *a.e.* in Ω, we can take the even-conjugate extensions of y and R to the interval $(-T,T)$; i.e., we set $y(t,x) = -\bar{y}(-t,x)$ and $R(t,x) = \bar{R}(-t,x)$ for every $(t,x) \in (-T,0) \times \Omega$. Moreover, we have that $R \in H^1(-T,T;L^\infty(\Omega))$, and y satisfies the system (30) in $(-T,T) \times \Omega$. In the case when $R(0,x) \in i\mathbb{R}$, the proof is still valid by take odd-conjugate extensions.

Changing t into $t + T$, we may assume that y and R are defined on $(0,2T) \times \Omega$, instead of $(-T,T) \times \Omega$.

Let $z(t,x) = y_t(2T - t, x)$. Then z satisfies the following system:

$$\begin{cases} z_t + iz_{xxxx} + iq(x)z = if(x)R_t(t,x), & (t,x) \in (0,2T) \times \Omega \\ z(t,0) = z(t,1) = 0, z_x(t,0) = z_x(t,1) = 0, & t \in (0,2T) \\ z(T,x) = -if(x)R(T,x), & x \in \Omega. \end{cases} \tag{31}$$

Step 2. (Estimation on z) We will adapt the revised Carleman estimate (29) on z, the solution of (31).

We consider the weight function in the time interval $(0,2T)$, i.e.,

$$\theta = e^l, \qquad \varphi(t,x) = \frac{e^{3\mu\psi(x)}}{t(2T-t)} \qquad \text{and} \qquad l(t,x) = \lambda\frac{e^{3\mu\psi(x)} - e^{5\mu\|\psi\|_\infty}}{t(2T-t)}.$$

Let $v = \theta z$ and I_1 is taken as in (9). Denote by

$$J = \int_0^T \int_\Omega I_1 \theta \bar{z} dxdt.$$

Then we have

$$\begin{aligned}|J| \leq & \left(\int_0^T \int_\Omega |I_1|^2 dxdt\right)^{1/2} \left(\int_0^T \int_\Omega \theta^2 |z|^2 dxdt\right)^{1/2} \\ \leq & \lambda^{-7/2}\mu^{-4} \int_0^T \int_\Omega |I_1|^2 dxdt + \lambda^{7/2}\mu^4 \int_0^T \int_\Omega \theta^2 |z|^2 dxdt \\ \leq & C\lambda^{-7/2}\mu^{-4} \left(\int_0^T \int_\Omega |I_1|^2 dxdt + \lambda^7 \mu^8 \int_0^T \int_\Omega \varphi^7 \theta^2 |z|^2 dxdt\right).\end{aligned}$$

The last inequality comes from the fact that φ is bounded from below.

Applying the Carleman inequality (29) (with $2T$ instead of T) on z, we obtain

$$\begin{aligned}|J| \leq & C\lambda^{-\frac{7}{2}}\mu^{-4} \int_0^{2T} \left(\int_\Omega \theta^2 |fR_t|^2 dx + \left(\lambda^3\mu^3\varphi^3\theta^2|z_{xx}|^2 + \lambda\mu\varphi\theta^2|z_{xxx}|^2\right)(t,1)\right) dt \\ \leq & C\lambda^{-\frac{7}{2}}\mu^{-4} \int_\Omega e^{2l(T,x)}|f(x)|^2 dx \\ & + C\lambda^{-\frac{1}{2}}\mu^{-1} \int_0^{2T} |z_{xx}(t,1)|^2 dt + C\lambda^{-\frac{5}{2}}\mu^{-3} \int_0^{2T} |z_{xxx}(t,1)|^2 dt.\end{aligned} \tag{32}$$

The last inequality holds true due to the fact that $l(T,x) \geq l(t,x)$ for all $(t,x) \in (0,2T)\times\Omega$, that $\varphi^3\theta^2$ and $\varphi\theta^2$ are bounded from above in $(0,2T)\times\Omega$ and that $R_t \in L^2(0,2T; L^\infty(\Omega))$.

On the other hand, since $v = \theta z$, we have

$$\begin{aligned}J = & \int_0^T \int_\Omega I_1 \bar{v} dxdt \\ = & \int_0^T \int_\Omega i v_t \bar{v} dxdt + \int_0^T \int_\Omega \left((\Psi + \frac{1}{2}a_{2,xx})|v|^2 - a_2|v_x|^2 + |v_{xx}|^2\right) dxdt,\end{aligned}$$

hence,

$$\mathrm{Im}(J) = \frac{1}{2}\int_\Omega |v(T,x)|^2 dx = \frac{1}{2}\int_\Omega e^{2l(T,x)}|f(x)|^2 |R(T,x)|^2 dx.$$

Using the hypothesis on $R(T,x)$, it follows that

$$\mathrm{Im}(J) \geq \frac{r^2}{2}\int_\Omega e^{2l(T,x)}|f(x)|^2 dx. \tag{33}$$

Combining (32) and (33), we have that

$$\int_\Omega e^{2l(T,x)}|f(x)|^2 dx \leq C\left(\int_0^{2T} |z_{xx}(t,1)|^2 + |z_{xxx}(t,1)|^2\right) dt \tag{34}$$

for λ and μ large enough. Then (6) follows from (34) since

$$e^{2l(T,x)} \geq e^{2M} > 0, \qquad \text{with} \qquad M = \frac{\lambda}{T^2}(1 - e^{5\mu\|\psi\|_\infty}).$$

This completes the proof of Theorem 1.2. □

5. **Further comments and open problems.**

1. In this paper we derive a boundary Carleman estimate for the fourth order Schrödinger operator. It is well known that based on (4), we can derive the observability inequality and, consequently, prove the controllability property of the controlled system with two boundary controls. As a direct consequence of this methodology, it is very likely to expect that the controllability property holds for the fourth order Schrödinger equation with nontrivial potential q. Such result is much more general than the existing one in [24], which is for trivial potential q, even though only one boundary control is needed. It would be interesting to know whether two controls on the boundary are necessary with the nontrivial potential q.
2. It is well known that the Carleman estimate is a useful tool to analyze inverse problems. In fact, it has been studied for second order Schrödinger operator not only in bounded domain, but also in an unbounded strip ([4]) or on a tree ([11]). One could expect similar results in different domains. Meanwhile, it is still a challenging problem whether one can construct Carleman inequalities for fourth order equations on higher dimensions.
3. Note that there are fruitful literatures considering the numerical approximation results for the second order Schrödinger equations. Similar to the discrete Carleman estimate constructed by parabolic equation (see [3]), it would be interesting to find out the discrete analogue of (4) for space semi-discretized Schrödinger equation as the first step to solve discrete problems.

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REFERENCES

[1] L. Baudouin and J. P. Puel, Uniqueness and stability in an inverse problem for the Schrödinger equation, *Inverse Problems*, **18** (2002), 1537–1554.

[2] M. Bellassoued and M. Choulli, Stability estimate for an inverse problem for the magnetic Schrödinger equation from the Dirichlet-to-Neumann map, *J. Funct. Anal.*, **258** (2010), 161–195.

[3] F. Boyer, F. Hubert and J. Le Rousseau, Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications, *SIAM J. Control Optim.*, **48** (2010), 5357–5397.

[4] L. Cardoulis, M. Cristofol and P. Gaitan, Inverse problem for the Schrödinger operator in an unbounded strip, *J. Inverse Ill-Posed Probl.*, **16** (2008), 127–146.

[5] K.-J. Engel and R. Nagel, *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, 194, Springer-Verlag, New York, 2000.

[6] S. Ervedoza, C. Zheng and E. Zuazua, On the observability of time-discrete conservative linear systems, *J. Funct. Anal.*, **254** (2008), 3037–3078.

[7] G. Eskin, Inverse problems for the Schr̈dinger operators with electromagnetic potentials in domains with obstacles, *Inverse Problems*, **19** (2003), 985–996.

[8] E. Fernández-Cara and E. Zuazua, The cost of approximate controllability for heat equations: The linear case, *Adv. Differential Equations*, **5** (2000), 465–514.

[9] X. Fu, Sharp observability inequalities for the 1-D plate equation with a potential, *Chin. Ann. Math. Ser. B.*, **33** (2012), 91–106.

[10] C. Hao, L. Hsiao and B. Wang, Wellposedness for the fourth order nonlinear Schrödinger equations, *J. Math. Anal. Appl.*, **320** (2006), 246–265.

[11] L. Ignat, A. F. Pazoto and L. Rosier, Inverse problem for the heat equation and the Schrödinger equation on a tree, *Inverse Problems*, **28** (2012), 015011, 30 pp.

[12] V. I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: Fourth-order nonlinear Schrödinger-type equations, *Phys. Rev. E*, **53** (1996), R1336–R1339.

[13] V. I. Karpman and A. G. Shagalov, Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion, *Phys. Rev. D*, **144** (2000), 194–210.

[14] I. Lasiecka, R. Triggiani and X. Zhang, Global uniqueness, observability and stabilization of nonconservative Schrödinger equations via pointwise Carleman estimates. I. $H^1(\Omega)$-estimates, *J. Inverse Ill-Posed Probl.*, **12** (2004), 43–123.

[15] E. Machtyngier, Exact controllability for the Schrödinger equation, *SIAM J. Control Optim.*, **32** (1994), 24–34.

[16] N. Mandache, Exponential instability in an inverse problem for the Schrödinger equation, *Inverse Problems*, **17** (2001), 1435–1444.

[17] A. Mercado, A. Osses and L. Rosier, Inverse problems for the Schrödinger equation via Carleman inequalities with degenerate weights, *Inverse Problems*, **24** (2008), 015017, 18 pp.

[18] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case, *Dyn. Partial Differ. Equ.*, **4** (2007), 197–225.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations,* Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[20] B. Pausader, The cubic fourth-order Schrödinger equation, *J. Funct. Anal.*, **256** (2009), 2473–2517.

[21] M. Yamamoto, Uniqueness and stability in multidimensional hyperbolic inverse problems, *J. Math. Pures Appl.*, **78** (1999), 65–98.

[22] G. Yuan and M. Yamamoto, Carleman estimates for the Schrödinger equation and applications to an inverse problem and an observability inequality, *Chin. Ann. Math. Ser. B.*, **31** (2010), 555–578.

[23] X. Zhang, Exact controllability of semilinear plate equations, *Asympt. Anal.*, **27** (2001), 95–125.

[24] C. Zheng and Z. Zhou, Exact controllability for the fourth order Schrödinger equation, *Chin. Ann. Math. Ser. B.*, **33** (2012), 395–404.

[25] Z. Zhou, Observability estimate and null controllability for one-dimensional fourth order parabolic equation, *Taiwanese J. Math.*, **16** (2012), 1991–2017.

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