Nonlinear differential equation methods in image processing

Chuang Zheng[*] and Juan Luis Vázquez[†]

Contents

[*]Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain. *e-mail:* `chuang.zheng@uam.es`.

[†]Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma de Madrid, 28049 Madrid, Spain. *e-mail:* `juanluis.vazquez@uam.es`.

1 Introduction

Many mathematicians have been attracted by image processing and computer vision in recent years. This has been triggered by mathematically well-founded methods using e.g. *wavelets* or *nonlinear partial differential equations*, the latter being either stationary (usually minimization problems) or evolution equations. We are interested in the latter one, namely methods that are based on nonlinear diffusion techniques.

The nonlinear diffusion technique has evolved in a very fruitful way. It is closely connected to a specific kind of multi-scale analysis called scale-space(with respect to the time variable of the Nonlinear PDE), and it has been used for image smoothing with simultaneous edge enhancement. Later on, close connections to regularization methods have been discovered, and related nonlinear methods have also entered computer vision fields such as motion analysis in image sequences or interactive segmentation.

In image treatment, it is generally desirable to smooth the homogenous regions of the picture with two scopes: noise elimination and image interpretation (pattern recognition). On the other side, we wish to keep the accurate location of the boundaries of these regions. Those boundaries are called "step edges" [1]. In the classical theory, these objects are defined as the curves where the gradient of the smoothed picture has a maximum. ("Edge point" is therefore related to the property that the Laplacian of the smooth signal at that point changes sign.)

This theory of smoothing comes from Marr and Hildreth [2] and has been improved by Vitkin [3], Koenderink [4] and Canny [7]. The low-pass filtering is generally made by convolution with Gaussian kernels of increasing variance. It is easy to understand the previous low-pass filtering: if the signal is noisy, the gradient will have a lot of irrelevant maxima which must be eliminated. Of course, strong oscillations can be due to different causes, for instance, the presence of "textures." Konederink [4] noticed that the convolution of the signal with Gaussians at each scale was equivalent to the solution of the heat equation with the signal as initial datum. Denote this datum by u_0; the "scale space" analysis associated with u_0 consists in solving the problem

$$u_t = \Delta u, \qquad u(0, x, y) = u_0(x, y). \tag{1.1}$$

for a function $u(t, x, y)$ where t is the smoothing parameter and not time! The solution of this equation for an initial datum with bounded quadratic norm is $u(t, x, y) = (G_t * u_0)(x, y)$, where

$$G_\sigma(x, y) = C\sigma^{-1} \exp(-(x^2 + y^2)/4\sigma)$$

is the Gauss function.

This is not the only option based on evolution PDE's. There are many articles considering some other similar types of diffusion equation. The preceding idea is, as we shall see, quite close to an important improvement of the edge detection theory proposed in the paper of P. Perona and J. Malik ([11]). Their main idea is to introduce a part of the edge detection step in the filtering itself, allowing an interaction between scales from the beginning of the algorithm. More precisely, they propose to replace the heat equation by a nonlinear equation

$$\frac{\partial u}{\partial t} = \mathrm{div}(c(x,y,t)Du), \qquad u(0,x,y) = u_0(x,y), \tag{1.2}$$

where $u_0(x,y)$ $u(t,x,y)$ have the same meaning as in (2.1), with $c(t,x,y)$ is defined as

$$c(t,x,y) = g(|D(t,x,y)|). \tag{1.3}$$

Here $g(s)$ is a smooth nonincreasing function with $g(0) = 1, g(s) \geq 0$ and $g(s) \to 0$ at infinity. The idea is that the smoothing process obtained by the equation is "conditional":

(a) If $D(x,y)$ is big, then the diffusion will be low and therefore the exact localization of the "edges" will be kept.

(b) If $Du(x,y)$ is small , then the diffusion will tend to smooth still more around (x,y).

Thus the choice of g corresponds to a sort of thresholding which has to be compared to the thresholding of $|Du|$ used in the final step of the classical theory explained above.The experimental results obtained by this equation are perceptually impressive and show that an "edge detector" based on this theory gives edges which remain much more stable across the scales, therefore making the backward following of edges across scales unnecessary.

The model of Malik and Perona had several serious, practical and theoretical difficulties.

(a) Assume that the signal is noisy, with the white noise for instance. Then **large gradients** $|Du|$ are introduced by the noise. Moreover, Du is in theory unbounded. Thus, the conditional smoothing introduced by the model will not give good results, since all these noise edges will be kept.

(b) The second difficulty arose from the equation itself. The function g in (1.2) needs to be considered carefully to obtain the available theory. Indeed, in order to obtain both existence and uniqueness of the solutions, g must verify that $sg(s)$ is nondecreasing. In practice we will find out that if for some functions g with $sg(s)$ nonincreasing, very close pictures could produce divergent solutions and therefore different edges.

The model which has been proposed by Catté, Coll, Lions and Morel in ([9]) is a synthesis of Malik and Perona's ideas which avoids the above-mentioned difficulties; it is robust in the presence of noise and consistent from the formal viewpoint mentioned above.

We shall define a new "selective smoothing of u_0 at scale $t^{1/2}$ based on estimate at the scale σ" as the function $u(t,x,y)$, verifying

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{div}(g(|DG_\sigma * u|)Du), & \mathrm{in}(0,T)\times\Omega, \\ u(0) = u_0, \end{cases} \tag{1.4}$$

where

$$G_\sigma(x,y) = C\sigma^{-1}\exp(-(x^2+y^2)/4\sigma). \tag{1.5}$$

Remark 1.1 *It is easily seen that $G(t,x,y) = G_t(x,y)$ is the fundamental solution of the heat equation. Therefore the term $(|DG_\sigma * u|)(t,x,y)$ which appears inside the divergence term of (1.4) is simply the gradient of the solution at time σ of the heat equation with $u(0,x,y)$ as initial datum. Thus the modification of the model of Malik and Perona is only to replace the gradient $|Du|$ by its estimate $|DG_\sigma * u|$. But this slight change of the model is enough to* **avoid both inconsistencies of the Malik and Perona model**.

This model is still not quite optimal. Indeed, it is not necessary to diffuse anisotropically at points where the gradient is low. we do not want to enhance , or even to preserve, the edges without contrast. Therefore , rather than (1.4), we shall prefer the following formulation, which separates the behavior for large gradients from the behavior small gradients:

$$\frac{\partial u}{\partial t} - (g(|DG_\sigma * u|)\Big((1 - h(|Du|))\Delta u + h(|Du|)|Du|\text{div}\frac{Du}{|Du|}\Big) = 0, \tag{1.6}$$

where $h(s)$ is a function such that

$$\begin{cases} h(s) = 0, & \text{if } s \le e, \\ h(s) = 1, & \text{if } s \ge 2e, \\ h(s) \text{ is smooth nondecreasing,} & \text{elsewhere.} \end{cases} \tag{1.7}$$

The parameter e is not a additional parameter. It is only a refinement of our contrast model. We know that if $|Du|$ is large, $g(|Du|)$ is small. Thus e must depend on the same contrast parameter as g: it is the upper bound of the interval where u is allowed to diffuse freely.

In the next 3 sections we will study that model. In Section 5 we will study the model of contour enhancement proposed by Sethian and Malladi /// and studied by G.I. Barenblatt and J.L. Vázquez [8] by free boundary techniques.

2 The ALM Mathematical Model

We propose and study a class of nonlinear parabolic differential equations for image processing of the following kind(see [10]):

$$\frac{\partial u}{\partial t} = g(|G * Du|)|Du|\text{div}(\frac{Du}{|Du|}), \qquad u(0,x,y) = u_0(x,y), \tag{2.1}$$

where $u_0(x,y)$ is the grey level of the image to be processed, $u(t,x,y)$ is its smoothed version depending on the "scale parameter" t, G is a smoothing kernel (for instance, a Gaussian), $G * Du$ is therefore a local estimate of Du for noise elimination, and $g(s)$ is a nonincreasing real function which tends to zero as $s \to \infty$. Roughly speaking, the interpretation of the terms of the equation are as follows.

(a) The term

$$|Du|\text{div}(\frac{Du}{|Du|}) = \Delta u - D^2u\frac{(Du, Du)}{|Du|^2} \tag{2.2}$$

represents a degenerate diffusion term, which diffuse the direction orthogonal to its gradient Du and does not diffuse at all in the direction of Du, since $\text{div}(\frac{Du}{|Du|})$ is zero at this direction. (Here and everywhere below, D^2u denotes the Hessian of u and we prove it in Lemma (6.1).) The aim of the degenerate diffusion term is to make u smooth on both side of an "edge" with a minimal smoothing of the edge itself.

(b) The term $g(G * Du)$ is used for the "enhancement" of the edges. Indeed, it controls the speed of the diffusion: if Du has a small mean in a neighborhood of a point x, the point x is considered the interior point of a smooth region of the image and the diffusion is therefore strong; If Du has a large mean value on the neighborhood of x, x is considered an edge point and the diffusion spread is lowered, since $g(s)$ is small for large s.

Thus, the proposed model performs a selective smoothing of the image, where the "edges" are relatively enhanced and preserved as much as possible.

3 Mathematical properties

For making the model in the above section useful, we have to give out some properties for the solution u.

(a) The existence of the solution.

(b) The uniqueness of the solution.

(c) The stability of the equation.

Since we need to apply the model to computer, we need the numerical schemes and make sure for the scale h, the numercial solutions are converge to the continuous ones.

For both of the aims, we claim that for such a class of nonlinear diffusion models the following properties can be established.

(a) (Well-posedness and smoothness results)

There exists a unique solution $u(t, x, y)$ in the distributional sense which belongs to the space $C^\infty(\overline{\Omega} \times (0, \infty))$ and depends continuously on u_0 with respect to the $L^2(\Omega)$ norm.

(b) (Maximum Principle) Let

$$a := \inf_{\Omega} u_0, \qquad b := \sup_{\Omega} u_0. \tag{3.1}$$

Then,

$$a \le u(t, x, y) \le b \qquad \text{on} \quad \Omega \times [0, \infty). \tag{3.2}$$

(c) (Average grey-level invariance) The average grey level

$$\mu := \frac{1}{|\Omega|} \int_{\Omega} u_0(x, y) dx dy \tag{3.3}$$

is not affected by nonlinear diffusion filtering:

$$\mu := \frac{1}{|\Omega|} \int_{\Omega} u(t, x, y) dx dy \qquad \text{for all} \quad t > 0. \tag{3.4}$$

(d) (Convergence to a constant steady state) For large scale parameter we have

$$\lim_{t \to \infty} u(t, x, y) = \mu \in L^p(\Omega), \qquad 1 \le p < \infty. \tag{3.5}$$

4 Basic Mathematical Theory of the Model

In this section, we wish to show that (2.1) is well posed. In fact, we shall know uniqueness and existence of Lipschitz solutions (for Lipschitz initial data) for a general class of equations which contain our model. Indeed, we first observe that (2.1) take the form

$$\frac{\partial u}{\partial t} - g(u * DG)a_{ij}(Du)\partial_{ij}u = 0 \text{in } [0, +\infty] \times \mathbb{R}^n, \tag{4.1}$$

where we denote by $\partial_i u = \partial u/\partial x_i$ and use the convention on repeated indices. Mathematically we can take any $n \geq 2$, although in the physical models n is no more than 3. Next we assume that

$$g \in C^{1,1}(\mathbb{R}^n, \mathbb{R}^n), \qquad g(p) > 0 \text{ for all } p \text{ in } \mathbb{R}^n, \tag{4.2}$$

$$D^\alpha G \in L^1(\mathbb{R}^n) \text{ for all } |\alpha| \leq 2, \tag{4.3}$$

$$a_{ij}(p)\xi_i\xi_j \geq 0 \text{ for all } p \in \mathbb{R}^n \setminus \{0\}, \ \ \xi \in \mathbb{R}^n, \tag{4.4}$$

$$a_{ij} \text{ is continuous and bounded on } \mathbb{R}^n \setminus \{0\}. \tag{4.5}$$

It is trivial fact to check that (2.1) is indeed of the above form. Of course the application of such model to image treatment requires solving (4.1) only in a subdomain Ω of $\mathbb{R}^n$(in fact, a rectangle in $\mathbb{R}^n$), that we can assume to be convex and piecewise smooth to simplify the presentation. To fix ideas we should think of $\Omega = [0, 1]^n$. In that case, we have to prescribe boundary conditions for u and $\partial\Omega$, and the most natural choice of for image processing is Neumann boundary conditions, i.e.,

$$\frac{\partial u}{\partial \nu} = 0, \qquad \text{on } \partial\Omega, \tag{4.6}$$

where ν denote the unit exterior normal. Indeed, the Neumann condition corresponds to the reflection of the picture across the boundary and has the advantage of not imposing any value one the boundary and not creating "edges" on it.

To simplify the presentation, we shall work with periodic boundary conditions or, in other words, solve (4.1)with solutions satisfying $u(x + h) = u(x)$ for all x in $\mathbb{R}^n$, h in $\mathbf{Z}^n$. Moreover we set $u(-x, y) = u(x, y)$ if $-1 \leq x \leq 0$, and $0 \geq y \geq 1$, etc. It is easy seen that with this extension, u can be assumed to be periodic on $(2\mathbf{Z})^n$.

Of course, we complement (4.1)with an initial condition

$$u(0, x) = u_0(x) \qquad \text{in } \mathbb{R}^n, \tag{4.7}$$

where u_0 is continuous on $\mathbb{R}^n$ and periodic, as above.

Remark 4.1 *(4.1) is a second-order parabolic equation with possible high degeneracy and two types of nonlinear terms, namely, a quasilinear term $(a_{ij}(Du)\partial_{ij}u)$ and a nonlocal term $g * DG$. This is why it is important to work here with viscosity solutions.*

Now we give out the definition of the viscosity solution to the equation (4.1):

Definition 4.1 *Let u in $C([0,T]\times\mathbb{R}^n)$ for some T in $(0,\infty)$. Then u is a viscosity solution of (4.1) if u is both a subsolution and a supersolution, with the definition:*

(a) u is a viscosity subsolution of(4.1), if for all ϕ in $C^2(\mathbb{R}\times\mathbb{R}^n)$, the following condition holds at any point (t_0,x_0) in $(0,T]\times\mathbb{R}^n$ which is a local maximum of $(u-\phi)$

$$\begin{aligned}&\tfrac{\partial\phi}{\partial t}(t_0,x_0)-g(u*DG(t_0,x_0))a_{ij}(D\phi(t_0,x_0))\partial_{ij}\phi(t_0,x_0)\le 0 && \text{if } D\phi(t_0,x_0)\neq 0,\\ &\tfrac{\partial\phi}{\partial t}(t_0,x_0)-g(u*DG(t_0,x_0))\limsup_{p\to 0}a_{ij}(p)\partial_{ij}\phi(t_0,x_0)\le 0 && \text{if } D\phi(t_0,x_0)=0.\end{aligned}\tag{4.8}$$

(b) u is a viscosity supersolution of (4.1), if for all ϕ in $C^2(\mathbb{R}\times\mathbb{R}^n)$, the following condition holds at any point (t_0,x_0) in $(0,T]\times\mathbb{R}^n$ which is a local minimum of $(u-\phi)$

$$\begin{aligned}&\tfrac{\partial\phi}{\partial t}(t_0,x_0)-g(u*DG(t_0,x_0))a_{ij}(D\phi(t_0,x_0))\partial_{ij}\phi(t_0,x_0)\ge 0 && \text{if } D\phi(t_0,x_0)\neq 0,\\ &\tfrac{\partial\phi}{\partial t}(t_0,x_0)-g(u*DG(t_0,x_0))\liminf_{p\to 0}a_{ij}(p)\partial_{ij}\phi(t_0,x_0)\ge 0 && \text{if } D\phi(t_0,x_0)=0.\end{aligned}\tag{4.9}$$

Theorem 4.1 *Let u_0, v_0 be, respectively, Lipschitz continuous and continuous on Ω. Let $W^{1,\infty}(\mathbb{R}^n))$ denote the space of bounded Lipschitz continuous functions in $\mathbb{R}^n$.*

(1) **(Existence of the solution)** *The system (4.1)-(4.7) has a unique viscosity solution u in $C([0,+\infty)\times\mathbb{R}^n)\cap L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n))$ for any $T<\infty$. Moreover,*

$$\inf_{\mathbb{R}^n}u_0\le u(x,t)\le\sup_{\mathbb{R}^n}u_0.$$

(2) **(Stability of the solution)** *Let v in $C([0,+\infty)\times\mathbb{R}^n)$ be a viscosity solution of (4.1) satisfying (4.7) with u_0 replaced by v_0. Then for all T in $[0,+\infty)$, there exists a constant K which depends only on $\|u_0\|_{W^{1,\infty}}$ and $\|v_0\|_{L^\infty}$ such that*

$$\sup_{0\le t\le T}\|u(t,\cdot)-v(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}\le K\|u_0-v_0\|_{L^\infty(\mathbb{R}^n)}.\tag{4.10}$$

Proof : We first prove the uniqueness and stability estimate of the solution. Following with the arguments given in Crandall, Ishii, and Lions [6], and we consider a maximum point (t_0,x_0,y_0) of

$$u(t,x)-v(t,y)-(4\varepsilon)^{-1}|x-y|^4-\lambda t,\qquad t\in[0,T],\qquad x,y\in\mathbb{R}^n,\tag{4.11}$$

where $T,\varepsilon,\lambda\in(0,\infty)$ will be determined later.

We first assume that $t_0 > 0$. Then we find a and b in $\mathbb{R}$, X, Y, $(n\times n)$ symmetric matrices such that

$$a - b = \lambda, \qquad \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \begin{pmatrix} A+\mu A^2 & -A-\mu A^2 \\ -A-\mu A^2 & A+\mu A^2 \end{pmatrix} \tag{4.12}$$

for each $\mu > 0$,

$$\begin{array}{l} a - g((u*DG)(t_0,x_0))a_{ij}(\varepsilon^{-1}|x_0-y_0|^2(x_0-y_0))X_{ij} \leq 0, \\ b - g((v*DG)(t_0,x_0))a_{ij}(\varepsilon^{-1}|x_0-y_0|^2(x_0-y_0))Y_{ij} \geq, 0 \end{array} \tag{4.13}$$

where

$$A = \varepsilon^{-1}|x_0-y_0|^2 I_n + 2\varepsilon^{-1}(x_0-y_0)\otimes(x_0-y_0),$$

so that

$$A^2 = \varepsilon^{-2}|x_0-y_0|^4 I_n + 8\varepsilon^{-2}|x_0-y_0|^2(x_0-y_0)\otimes(x_0-y_0).$$

It is easy to show that $x_0 \neq y_0$. Suppose it be true, we have $A = 0$ so that by (4.12), $X \leq 0$ and $Y \geq 0$. We then rewrite (4.13) as

$$\begin{array}{l} a - g((u*DG)(t_0,x_0)) \limsup_{p\to 0} a_{ij}(p)X_{ij} \leq 0, \\ b - g((v*DG)(t_0,x_0)) \liminf_{p\to 0} a_{ij}(p)Y_{ij} \geq, 0 \end{array} \tag{4.14}$$

Hence, in particular, $a \leq 0, b \geq 0$, which comes out the contradiction of $a - b = \lambda > 0$.

Therefore, $x_0 \neq y_0$ and we may write and use (4.13). We next choose $\mu = \varepsilon|x_0-y_0|^{-2}$ and we deduce

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{2}{\varepsilon}\begin{pmatrix} B & -B \\ -B & B \end{pmatrix}, \tag{4.15}$$

where $B = |x_0-y_0|^2 I_n + 5(x_0-y_0)\otimes(x_0-y_0)$.

We then set

$$\begin{array}{l} g_1 = g((u*DG)(t_0,x_0)), \qquad g((v*DG)(t_0,x_0)), \\ a = (a_{ij}(\varepsilon^{-1}|x_0-y_0|^2(x_0-y_0)))_{1\leq i,j\leq n}, \end{array}$$

and we consider the matrix

$$\Gamma = \begin{pmatrix} g_1 a & (g_1g_2)^{1/2}a \\ (g_1g_2)^{1/2}a & g_2 a \end{pmatrix}.$$

Obviously, Γ is a nonnegative symmetric matrix so that multiplying (4.15) to the left by Γ and taking the trace we find

$$\begin{array}{rl} g_1 a_{ij}X_{ij} - g_2 a_{ij}X_{ij} \leq & 2\varepsilon^{-1}(g_1^{1/2}-g_2^{1/2})^2 \mathrm{trace}(aB) \\ & C_0\varepsilon^{-1}(g_1^{1/2}-g_2^{1/2})^2|x_0-y_0|^2 \end{array} \tag{4.16}$$

for some C_0 which depends only on $(a_{ij}(p))_{1\leq i,j\leq n}$. Next if we combine (4.12), (4.13) and (4.16) we obtain

$$\lambda \leq C_0\varepsilon^{-1}(g_1^{1/2}-g_2^{1/2})^2|x_0-y_0|^2. \tag{4.17}$$

We now estimate $(g_1^{1/2} - g_2^{1/2})$. First of all, we observe that from the property of g, $g^{1/2}$ is Lipschitz on bounded sets, therefore

$$(g_1^{1/2} - g_2^{1/2}) \leq C_1|(u * DG)(t_0, x_0) - (v * DG)(t_0, y_0)|$$

for some C_1, depending only on g and on $\sup|u|, \sup|v|$.

But this last quantity is estimated by $C_2(\sup_{[0,T]\times\mathbb{R}^n}|u-v|+|x_0-y_0|)$, where C_2 depends only on G and on $\sup|u|, \sup|v|$. This allows us to deduce from (4.17) that

$$\lambda \leq C\left\{\left(\sup_{[0,T]\times\mathbb{R}^n}|u-v|\right)^2 \frac{|x_0-y_0|^2}{\varepsilon} + \frac{|x_0-y_0|^4}{\varepsilon}\right\}, \tag{4.18}$$

where $C = 2C_0C_1^2C_2^2$.

Next , we estimate $|x_0 - y_0|$. To this aim, we observe that

$$u(t_0, x_0) - v(t_0, y_0) - \frac{|x_0-y_0|^4}{4\varepsilon} - \lambda t_0 \geq u(t_0, y_0) - v(t_0, y_0) - \lambda t_0$$

and thus

$$\frac{|x_0-y_0|^4}{4\varepsilon} \leq L|x_0 - y_0|,$$

where L is a Lipschitz constant (in x) for u on $[0, T] \times \mathbb{R}^n$. Therefore, $|x_0 - y_0| \leq (4\varepsilon L)^{1/3}$. This bound and (4.18) finally yield

$$\lambda \leq M\left\{\varepsilon^{1/2} + \varepsilon^{1/3}\left(\sup_{[0,T]\times\mathbb{R}^n}|u-v|\right)^2\right\}, \tag{4.19}$$

where $M = \max((4L)^{2/3}, (4L)^{4/3})C$. Without loss of generality, we may assume $\sup_{[0,T]\times\mathbb{R}^n}|u-v| > 0$(otherwise, we conclude the results.) and we choose

$$\varepsilon^{1/3} = \delta \sup_{[0,T]\times\mathbb{R}^n}|u-v|, \qquad \lambda = (1+\delta+1/\delta)M\sup_{[0,T]\times\mathbb{R}^n}|u-v|,$$

where $\delta > 0$ will be determined later. These choices contradict (4.19). This contradiction proves, in fact, that $t_0 = 0$. Therefore,

$$u(t,x) - v(t,y) - \frac{|x-y|^4}{4\varepsilon} - \lambda t \leq \sup_{x,y\in\mathbb{R}^n}\left\{u_0(x) - v_0(y) - \frac{|x-y|^4}{4\varepsilon}\right\}. \tag{4.20}$$

In particular, we may choose $x = y$ in (4.20) while the right-hand side can be estimated by $\sup_{\mathbb{R}^n}(u_0 - v_0) + \sup_{r\geq 0}(Lr - r^4/4\varepsilon$.

We finally obtain

$$\begin{aligned}\sup_{[0,T]\times\mathbb{R}^n}(u-v) \leq\ & \sup_{\mathbb{R}^n}|u_0 - v_0| + \frac{3}{4}L^{4/3}\delta\sup_{[0,T]\times\mathbb{R}^n}|u-v| \\ & + M(1+\delta+1/\delta)T\sup_{[0,T]\times\mathbb{R}^n}|u-v|.\end{aligned} \tag{4.21}$$

Exchanging the role of u and v, and choosing $\delta = L^{-4/3}$, we deduce

$$\sup_{[0,T]\times\mathbb{R}^n} |u - v| \le 4 \sup_{\mathbb{R}^n} |u_0 - v_0| + KT \sup_{[0,T]\times\mathbb{R}^n} |u - v|, \tag{4.22}$$

where $K = 4M(1 + \delta + 1/\delta)$. In order to conclude, we choose $T = t_1 = 1/2K$, and we find

$$\sup_{[0,t_1]\times\mathbb{R}^n} |u - v| \le 8 \sup_{\mathbb{R}^n} |u_0 - v_0|. \tag{4.23}$$

Therefore, if T is an arbitrary time in $[0, \infty)$ and $N \ge 1$ is such that $Nt_1 \le T$, we deduce easily by reiterating this argument that

$$\sup_{[0,t_1]\times\mathbb{R}^n} |u - v| \le 8^N \sup_{\mathbb{R}^n} |u_0 - v_0|. \tag{4.24}$$

This finish the proof of the stability and uniqueness of the solution.

We next prove the existence of claim in the first part. We begin by remarking that definition of viscosity solutions immediately implies that if u is a solution, then

$$\inf_{\mathbb{R}^n} u_0 - \delta t \le u \le \sup_{\mathbb{R}^n} u_0 + \delta t \qquad \text{on} \ \ [0, +\infty] \times \mathbb{R}^n \qquad \text{for all} \ \ \delta > 0.$$

Therefore, we have

$$\inf_{\mathbb{R}^n} \le u \le \sup_{\mathbb{R}^n} u_0 \qquad \text{on} \ \ [0, +\infty] \times \mathbb{R}^n. \tag{4.25}$$

Indeed, set $\phi(x,t) = \sup_{\mathbb{R}^n} u_0 + \delta t$ and assume that $u - \phi$ has local maximum at a point (t_0, x_0) with $t_0 > 0$. Then by the definition of subsolution, we get by the second relation of (4.9) that $\partial\phi/\partial t(t_0, x_0) \le 0$. Thus $\delta \le 0$, which yields a contradiction and therefore $u - \phi$ attains its maximum, for $t_0 = 0$.

Next, we prove an a priori estimate on Du. This estimate will be formal at that level and will be justified later. In fact, we consider a smooth solution u of

$$\frac{\partial u}{\partial t} - g(\omega * DG) a_{ij}(Du) \partial_{ij} u = 0 \qquad \text{in}]0, +\infty[\times \mathbb{R}^n, \tag{4.26}$$

where a_{ij} is now supposed to be smooth on $\mathbb{R}^n$, and $\omega \in L^\infty(]0, +\infty] \times \mathbb{R}^n)$. We are going to show that

$$\|DU(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le e^{Ct} \|Du_0\|_{L^\infty(\mathbb{R}^n)}, \tag{4.27}$$

where C depends only on $\sup_{|p|\le R} |D^2 g(p)|$ and $\sup_p |a_{ij}(p)|$ with $\mathbb{R} = \|w\|_{L^\infty(\mathbb{R}^n)} \|DG\|_{L^1(\mathbb{R}^n)}$. Everywhere below, C will denote a positive constants depending only on these quantities. To prove the a priori estimate (4.27), we use the " classical" Bernstein method and derive a parabolic inequality for $|Du|^2$. To this end, we differentiate (4.26) with respect to x_k, and we find

$$\begin{aligned} &\frac{\partial u_k}{\partial t} - g(\omega * DG) a_{ij}(Du) \partial_{ij} u_k - \frac{\partial g}{\partial l}(\omega * DG) \cdot (\omega * \partial_{lk} G) a_{ij}(Du) \partial_{ij} u \\ &\qquad -g(\omega * DG) \frac{\partial a_{ij}}{\partial l}(Du) \partial_{lk} u = 0 \qquad \text{in} \ \ ]0, +\infty[\times \mathbb{R}^n, \end{aligned} \tag{4.28}$$

where we denote by $u_k = \partial_k u$. Hence, we obtain by multiplying by u_k

$$\begin{aligned}\frac{\partial|Du|^2}{\partial t} - g(\omega * DG)a_{ij}(Du)\partial_{ij}(|Du|^2) - \frac{\partial g}{\partial l}(\omega * DG)\frac{\partial a_{ij}}{\partial l}(Du)\partial_l(|Du|^2)\\ = -2g(\omega * DG)\partial a_{ij}(Du)u_{ki}u_{kj} + 2\frac{\partial g}{\partial l}(\omega * DG)\\ \cdot(\omega * \partial_{lk}G)a_{ij}(Du)u_k\partial_{ij}u \qquad \text{in } \;]0,+\infty[\times\mathbb{R}^n,\end{aligned} \tag{4.29}$$

Next, we observe that in terms of constants C only depending on $\sup|\omega|$ and g, we have

$$|\omega * \partial_{lk}G| \leq C, \qquad \left|\frac{\partial g}{\partial l}\omega * DG\right| \leq C(g(\omega * DG))^{1/2},$$

and

$$|a_{ij}(Du)u_{ij}| \leq (a_{ij}(Du)u_{ki}u_{kj})^{1/2}.$$

(This last inequality is purely algebraic and only uses that $a_{ij}x_ix_j$ is nonnegative.)

Inserting these bounds in (4.29) and using the Cauchy-Schwarz inequality we get

$$\begin{aligned}\frac{\partial|Du|^2}{\partial t} - g(\omega * DG)a_{ij}(Du)\partial_{ij}(|Du|^2) - \frac{\partial g}{\partial l}(\omega * DG)\frac{\partial a_{ij}}{\partial l}(Du)\partial_l(|Du|^2)\\ \leq C|Du|^2 \qquad \text{in } \;]0,+\infty[\times\mathbb{R}^n.\end{aligned} \tag{4.30}$$

We then deduce easily (4.27) by applying the maximum principle. In order to conclude, we only have to approximate (4.1) by a (slightly) simpler one of a similar form for which we will be able to produce smooth solutions. Then, we will conclude using the above a priori estimate (which will be valid on the approximated solutions). To this end, we consider u_0^ε in $C^\infty(\mathbb{R}^n)$ (periodic) such that $u_0^\varepsilon \to u_0$ uniformly, $\|Du_0^\varepsilon\|_{L^\infty} \leq \|Du_0\|_{L^\infty}, \|u_0^\varepsilon\|_{L^\infty} \leq \|u_0\|_{L^\infty}$. We also introduce $g_\infty = g + \varepsilon, a_{ij}^\varepsilon = \varepsilon\delta_{ij} + a_{ij}^\varepsilon$, where the a_{ij}^ε tends monotonically to a_{ij}, satisfy

$$a_{ij}(p)\xi_i\xi_j \geq 0 \;\text{ for all } p \in \mathbb{R}^n \setminus \{0\}, \;\; \xi \in \mathbb{R}^n,$$

and have compact support in $\mathbb{R}^n\backslash\{0\}$.

Using the general theory of quasilinear uniformly parabolic equations (Ladyzhenskaya, Solonnikov and Uralt'seva [5]), it can easily be checked that there exists u^ε smooth on $[0,+\infty[\times\mathbb{R}^n$ solution of

$$\frac{\partial u^\varepsilon}{\partial t} - g_\varepsilon(\omega * DG)a_{ij}^\varepsilon(Du^\varepsilon)\partial_{ij}u^\varepsilon = 0 \qquad \text{in}]0,+\infty[\times\mathbb{R}^n. \tag{4.31}$$

In view of the general consistency-stability properties of viscosity solutions, there just remains to show that u^ε (or a subsequence) converges uniformly on $[0,T] \times \mathbb{R}$ to some function u in $C([0,T] \times \mathbb{R}^n) \cap L^\infty(0,T;W^{1,\infty}(\mathbb{R}^n))$ for any $T < \infty$. This will follow from the Ascoli-Arzela theorem. Indeed, we may now apply the proof of estimate (4.27), and we find for all t in $[0,T]$

$$\|Du^\varepsilon(t,\cdot)\|_{L\infty(\mathbb{R}^n)} \leq e^{Ct}\,\|Du_0^\varepsilon\|_{L\infty(\mathbb{R}^n)} \leq \|Du_0\|_{L\infty(\mathbb{R}^n)} \leq C_T. \tag{4.32}$$

In other words,

$$|u^\varepsilon(t,x) - u^\varepsilon(t,y)| \le C_T|x-y| \qquad \text{for any } x,y \text{ in } \mathbb{R}^n \text{ and } t \text{ in } [0,T], \tag{4.33}$$

where C_T denotes various constants independent of ε, t, x, y. In additions, this estimate combined with (4.31 yields, by a (somewhat) standard argument, that

$$|u^\varepsilon(t,x) - u^\varepsilon(s,x)| \le C_T|s-t|^{1/2} \qquad \text{for any } x \text{ in } \mathbb{R}^n \text{ and } s,t \text{ in } [0,T]. \tag{4.34}$$

We conclude then by combining (4.33) and (4.34).

Let us sketch the proof of (4.34). It follows upon remarking that if $s \le t \le T$,

$$\|u^\varepsilon(t,\cdot) - u^\varepsilon_\delta\|_{L\infty(\mathbb{R}^n)} \le \frac{C_T(t-s)}{\delta} + \|u^\varepsilon(s,\cdot) - u^\varepsilon_\delta\|_{L\infty(\mathbb{R}^n)}\,, \tag{4.35}$$

where δ is arbitrary in $]0,\infty[$, $u^\varepsilon_\delta \in W^{2,\infty}(\mathbb{R}^n)$, $\|u^\varepsilon(s,\cdot) - u^\varepsilon_\delta\|_{L\infty(\mathbb{R}^n)} \le C_T(\delta)$ and $\|D^2 u^\varepsilon_\delta\|_{L\infty(\mathbb{R}^n)} \le C_T/\delta$.

Indeed, we have for $s \le t \le T$, $x \in \mathbb{R}^n$,

$$\Big|\frac{\partial u^\varepsilon_\delta}{\partial t} - g_\varepsilon(u^\varepsilon * DG)a^\varepsilon_{ij}(Du^\varepsilon)\partial_{ij}u^\varepsilon_\delta\Big| \le \frac{C_T}{\delta}, \tag{4.36}$$

and (4.35) is deduced from the maximum principle. We then derive (4.34) by choosing $\delta = (t-s)^{1/2}$.

5 Model of image contour enhancement

The theory of degenerate parabolic equations of the forms

$$u_t = (\Phi(u_x))_x \tag{5.1}$$

is used to analyze the process of contour enhancement in image processing, based on the evolution model of Sethian and Malladi [12]:

$$u_t = \frac{(1+u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1+u_x^2)u_{yy}}{1+u_x^2+u_y^2}. \tag{5.2}$$

This model is two-dimensional (compare with previous one). Due to the degenerate character of the diffusivity at high gradient values, a new one-dimensional free boundary problem with singular boundary data was introduced by G.I. Barenblatt and J.L. Vázquez [8], in order to analyze the process of front compression.

Definition 5.1 (Basic free boundary problem)
Given an increasing function $u_0(x)$ defined in a interval (a, b) with end values $u(a+) = 0, u(b-) = 1$, to find a continuous function $u(x, t)$ and continuous curves $x = l(t)$ and $x = r(t)$ such that

(i) $l(0) = a, r(0) = b$, and $l(t) < r(t)$ for some time interval $t \in (0, T)$.

(ii) u solves the following problem in $\Omega = \{(x, t) : 0 < t < T, l(t) < x < r(t)\}$:

$$(P_{II}) \qquad \begin{cases} u_t = \Phi(u_x)_x & \text{in } \Omega \\ u(x,0) = u_0(x) & \text{for } a \le x \le b \\ u(l(t),t) = 0, u_x(l(t),t) = +\infty & \text{for } 0 < t < T, \\ u(r(t),t) = 1, u_x(r(t),t) = +\infty & \text{for } 0 < t < T. \end{cases} \tag{5.3}$$

The regularity required from u as a solution of the problem will depend on the generality of the data. At least u will be continuous in the closure of Ω. Furthermore, to avoid unnecessary generality we will ask u to be smooth in the interior of Ω. Finally, the requirement of monotonicity is not intrinsic from the mathematical point of view, but it suits the application and allows for the use of the powerful conjugate formulations. The result that are proved by G.I. Barenblatt and J.L. Vázquez in [8] is as follows:

Theorem 5.1 *Let Φ be a flux function that is defined, smooth and $\Phi'(s) > 0$ for all $s > 0$. Assume moreover that $\Phi(\infty)$ is finite. Then for every increasing function $u_0(x)$ defined in a interval $[a, b]$ with $u(a) = 0$, $u(b) = 1$ and $u_0' \ge c > 0$ there exists a unique continuous function $u(x, t)$ which is defined in a set Ω as above, is smooth and strictly monotone in x for $0 < u < 1$, and there exist continuous curves $l(t)$ and $r(t)$, such that the triple (u, l, r) solves problem (P_{II}) in Ω_T. Besides, $u_x \ge c > 0$ whenever $0 < u < 1$.*

5.1 Conjugate formulation

When dealing with smooth monotone solutions $u_x > 0$ we can invert the variables x and u and write $x = X(u,t)$. Then $u_x \cdot x_u = 1$, and after some computations we get the partial differential equation satisfied by x as a function of u and t:

$$x_t = (\Psi(x_u))_u, \tag{5.4}$$

where Ψ is the conjugate flux function (conjugate to Φ), defined for $s > 0$ as

$$\Psi(s) = -\Phi(1/s). \tag{5.5}$$

This is due to the fact that

$$\frac{d}{dt}x = \frac{d}{dt}(x(u(x,t),t)) = x_u u_t + x_t = 0,$$

which means

$$x_t = -x_u u_t = -x_u(\Phi(u_x))_x = \frac{-1}{u_x}(\Phi(u_x))_x = \frac{-1}{u_x}(\Phi(u_x))_u u_x = (-\Phi(\frac{1}{x_u}))_u.$$

We now show how to use the conjugate formulations to solve the original problem. We assume that Φ a flux function defined for all $s > 0$ and such that $\Phi(\infty)$ is finite, say, $\Phi(\infty) = 0$. Then $\Psi(0) = 0$. This is the class of flux functions for which the conjugate problem looks simpler. Since we have assume that Φ is smooth, so is Ψ in its domain of definition.

(i) Assume that u_0 is continuous and strictly monotone in the interval $I = \{a < x < b\}$, with $u_0(a) = 0, u_0(b) = 1$, and C^1 smooth inside I with du_0/dx bounded below away from zero, we define the inverse function $x = h(u) = u_0^{-1} \mapsto [a,b]$, which satisfies the following equation

$$\begin{cases} X_t = (\Psi(X_u))_u & \text{for } 0 < u < 1, t > 0 \\ X(u,0) = h(u) & \text{for } 0 \le u \le 1 \\ X_u(u,t) = 0 & \text{for } u = 0, 1. \end{cases} \tag{5.6}$$

(ii) Set

$$\omega_0(u) = \frac{1}{u_{0,x}(h(u))}, \tag{5.7}$$

which is defined for $0 \le u \le 1$ and is positive, bounded and smooth inside, i.e., for $u \in (0,1)$. We then solve the conjugate problem

$$\begin{cases} \omega_t = \Psi(\omega)_{uu} & \text{for } 0 < u < 1, t > 0 \\ \omega(u,0) = \omega_0(u) & \text{for } 0 \le u \le 1 \\ \omega(u,t) = 0 & \text{for } u = 0, 1. \end{cases} \tag{5.8}$$

As initial data we choose a nonnegative, bounded function ω_0. Under these conditions Problem (5.8) has a unique solution by virtue of well-known nonlinear parabolic theories.

But note that now we are dealing with the homogeneous Dirichlet problem. The solution can be obtained as limit of the solutions $\omega_\varepsilon(u,t) \geq \varepsilon$ of the nondegenerate problem with initial data $\omega_{0,\varepsilon}(u) = \omega_0(u) + \varepsilon, \varepsilon > 0$. In the monotone limit we get

$$\lim_{t\to 0} \omega_\varepsilon(u.t) = \omega(x,t)$$

which is nonnegative, continuous and bounded. Under the additional assumption that ω_0 is locally bounded away from zero, it is easy proved that the solution $\omega(u,t)$ is positive, hence classical, in a strip

$$S_T = \{(u,t) : \ \ 0 < u < 1, 0 < t < T\}.$$

(iii) Next, we pass to the integrated version using the formula

$$z(u,t) = \int_\Gamma \omega du + \Psi(\omega)_u dt, \tag{5.9}$$

where Γ is any piece-wise smooth curve in (u,t) space starting from a fixed point, say $u = 1/2$, $t = 0$ and arriving at a generic point (u,t). In this way we obtain a solution of the integrated equation $z_t = (\Phi(z_u))_u$, much as we did in the case of the original pair of formulations.

Moreover, thanks to the fact that $\partial z/\partial u = \omega > 0$, we can invert the dependence between z and u to get a function $u = u(z,t)$ that is easily shown to satisfy the equation

$$u_t = (\Phi(u_z))_z.$$

Besides, u is a monotone function of z and takes the values $u = 0$ and $u = 1$ respectively at the left and right endpoints of the domain of definition

$$\Omega_z = \{l(t) < z < r(t)\}, \qquad l(t) = z(0,t), \qquad r(t) = z(1,t).$$

where $z(\cdot,t)$ is the function defined in (5.9). Therefore, $u(z,t)$ is a candidate to solve our original problem if we identify the independent variable z with $x - c$, where c is determined by the relation $u_0(c) = 1/2$.

In order to check that we have solved the original problem (P_{II}) we still have to check some particulars. It is clear that $v = u_x$ is related to the original ω by the formula

$$v(x,t) = \frac{1}{\omega(u,t)},$$

which simply states the derivative rule for the inverse function, and u in the second member is given by $u(z,t), z = x - c$, as explained before. Here comes an important point: since ω takes on zero boundary values, $v(x,t)$ diverges at the endpoints of its domain of definition, Ω. In other words, the solutions of the original problem $u = u(x,t)$ enjoy the property of infinite gradients at the endpoints of the strip where they are defined. Since $\Phi(\infty) = 0$ this also means zero flux at these points, a reasonable requirement, which explains why this condition has to be imposed on Φ.

As for the initial data, we have the mass formula

$$x = \int_{1/2}^{u} \omega_0(u)du + c, \qquad \text{with } u_0(c) = \frac{1}{2}, \tag{5.10}$$

so that x ranges over an interval $[a, b]$ when u goes from 0 to 1, i.e., $a = z(0,0) + c$, $b = z(1,0) + c$. This rule is accompanied by the rule $u = \int_{l(t)}^{x} v(x,t)dx$. Moreover, in the particular case that $\omega_0(u)$ is symmetric, we have $c(t) = c_0 = \frac{1}{2}$. Now $x(u,t)$ can be calculated by

$$x = \int_{1/2}^{u} \omega(u,t)du + c. \tag{5.11}$$

5.2 Numerical analysis of problem (P_{II})

In the previous section we have analyzed the solution of the problem (P_{II}). First we inverse the variable x and u to get system (5.6), then by setting $w = x_u$ we get system (5.8), which is a well defined parabolic partial differential equation with Dirichlet boundary condition. In this section we will compute the solution with programs. The two programs in this section are tested in Matlab 6.1.

The key role in the program is

$$X(u,t) = \int_{1/2}^{u} \omega(u,t)du + \int_{0}^{t} (\Psi(w))_u(1/2, s)ds + c,$$

with $u_0(c) = 1/2$. When the initial data $\omega_0(u)$ is symmetrical, the second term in the right side should be zero, consequently the formula (5.11) holds.

With the operator $\Phi(s) = -1/s$, the computation have been done with symmetrical data $\omega_0(u) = \sin(\pi u)$ and asymmetrical data $\omega_0(u) = 8x^3 - 8x^5$. Moreover, a extreme initial data

$$\omega_0(u) = f(u) = \begin{cases} 0, & 0 \le u \le 0.15 \\ 2 \times 10^3(x - 0.15), & 0.15 \le u \le 0.2 \\ 100, & 0.2 \le u \le 0.4 \\ 2 \times 10^3(0.45 - x), & 0.4 \le u \le 0.45 \\ 0, & 0.45 \le u \le 1. \end{cases}$$

Program. The program to produce the solution of the free boundary problem by this method is given as the follows:

```
clear

clc

h=0.05;       % space mesh size
x=[0:h:1];    % space interval
```

```
N=length(x); % number of space point   j

k=0.0005;    %time iteration size
t=[0:k:0.5]; %time interval
M=length(t); %number of time iteration s

%---initial data of the conjugate equation---
%---we can shift the initial data here-------
u0=zeros(1,N);

%u0=pi*sin(pi*x);

%u0=8*x.^3-8*x.^5;

%j=1; for j=4:8
%    u0(j)=100;
%end

%---------------------------------------

%----establish matrix A----------------
e=ones(N,1); A=spdiags([e -2*e e],-1:1,N,N); u(:,1)=u0';
%--------------------------------------

%---calculus of the solution u--------
s=1; for s=1:(M-1)
    u(:,s+1)=u(:,s)+(k/(h*h))*A*u(:,s);
    u(1,s+1)=0;u(N,s+1)=0;
end
%------------------------------------

%-------figure 1   u(j,s)-------------
figure(2) s=1; while s<=1000 hold on
        plot(x,u(:,s));
        s=s+20;
    end
    hold off
%------------------------------------

%--the derivative at the point x=1/2--
Du=(1/(2*h))*(u(12,:)-u(10,:));
```

```
% figure(2)
% plot(t,Du);
%-------------------------------------
%-----time integration of Du-----------
TDu=zeros(1,M);
g=1;
TDu(1)=0;
for g=1:(M-1)
    TDu(g+1)=TDu(g)+k*Du(g+1);
end
figure(3)
hold on
plot(t,Du,'green'); plot(t,TDu);
hold off
%-------------------------------------
%---space integration of u(1/2)-----------
Iu0=zeros(1,M);
Iu(1,:)=Iu0; C=zeros(1,M); j=1;
 for  j=1:((N-1)/2+1)
      C=C+h*u(j,:);
 end
figure(4)
plot(t,C);
%-------------------------------------
%---space integration of u(x)------------
j=1;
for j=1:(N-1)
    Iu(j+1,:)=Iu(j,:)+h*u(j,:);
end

j=1; for  j=1:N
    Iu(j,:)=Iu(j,:)-C+TDu;
end
figure(5)
plot(Iu(:,1),x)
%-------------------------------------
%-------figure we needed--------------
p=1;
figure(6)
 while p<=1000
    hold on
```

```
    plot(Iu(:,p),x);
    p=p+5;
end
hold off
%------------------------------------
```

The results are given in Figures $1, 2, 3$, with respect to three types of initial datum $\omega_0(u)$. Horizontal axis is x, vertical is u, and the curves are parameterized by time, evolving with increasing t towards the sharp front.

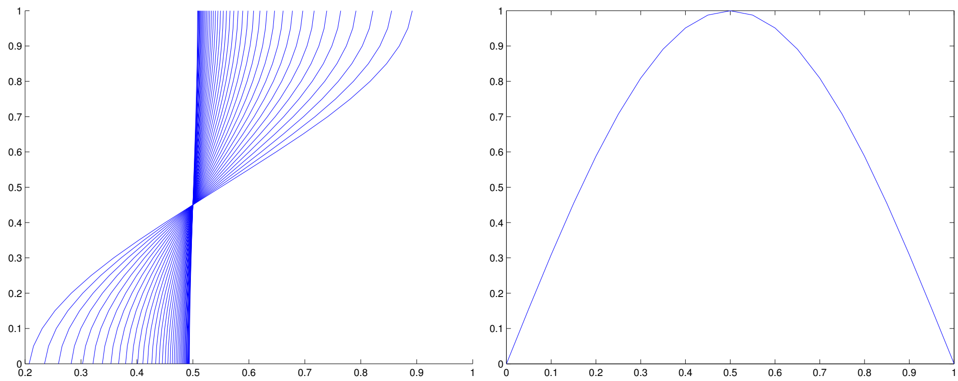

Figure 1: The case $\Phi(s) = -1/s$ with symmetrical data $\omega_0(u) = \sin(\pi u)$(right side).

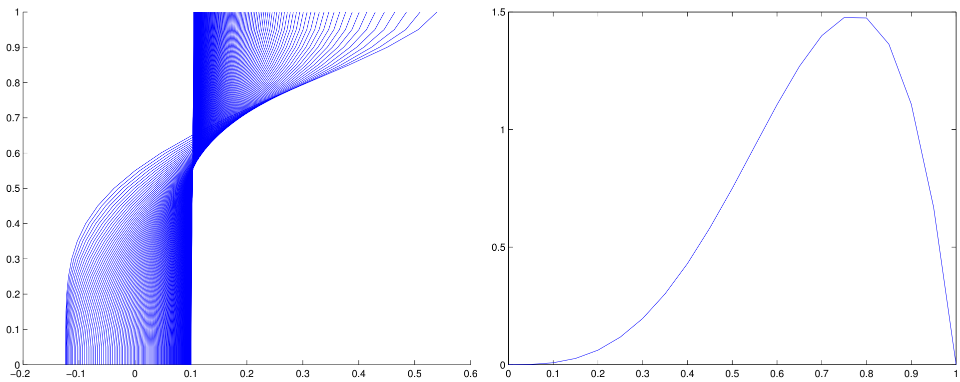

Figure 2: The case $\Phi(s) = -1/s$ with asymmetrical data $\omega_0(u) = 8x^3 - 8x^5$(right side).

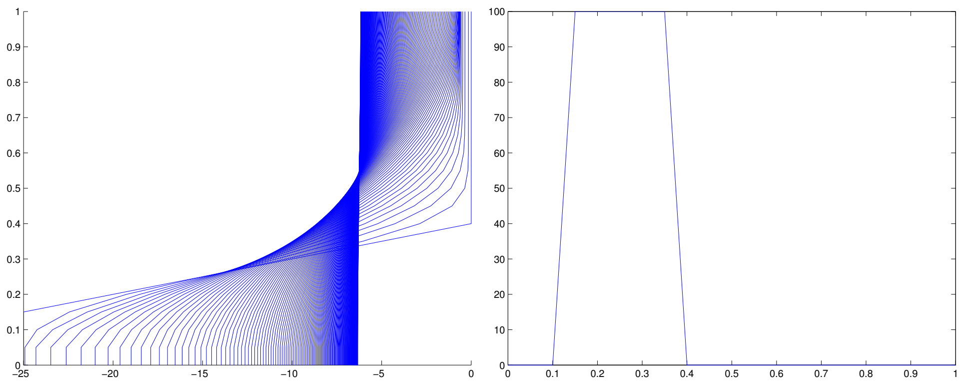

Figure 3: The case $\Phi(s) = -1/s$ with extra initial data $\omega_0(u) = f(u)$, which is in the right side.

6 Appendix

In this section we prove the following Calculus lemma:

Lemma 6.1 *Let* $\Omega \in \mathbb{R}^2$ *We have*

$$u_t = |Du| div \frac{Du}{|Du|} = \Delta u - \frac{1}{|Du|^2} D^2 u(Du, Du). \tag{6.1}$$

Proof: First we compute

$$\begin{aligned} u_t &= \sqrt{u_x^2 + u_y^2}\, \mathrm{div}\Big(\frac{u_x}{\sqrt{u_x^2+u_y^2}}, \frac{u_y}{\sqrt{u_x^2+u_y^2}}\Big) \\ &= \frac{u_x^2+u_y^2}{\sqrt{u_x^2+u_y^2}}\Big(\frac{u_{xx}\sqrt{u_x^2+u_y^2} - \frac{u_x u_{xx} + u_x u_y u_{xy}}{\sqrt{u_x^2+u_y^2}}}{\sqrt{u_x^2+u_y^2}} + \frac{u_{yy}\sqrt{u_x^2+u_y^2} - \frac{u_y u_{yy} + u_x u_y u_{xy}}{\sqrt{u_x^2+u_y^2}}}{\sqrt{u_x^2+u_y^2}}\Big) \\ &= \frac{\sqrt{u_x^2+u_y^2}(u_{xx}+u_{yy}) - \frac{u_x^2 u_{xx} + u_y^2 u_{yy} + 2u_x u_y u_{xy}}{\sqrt{u_x^2+u_y^2}}}{\sqrt{u_x^2+u_y^2}} \\ &= \frac{u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}}{u_x^2+u_y^2}. \end{aligned} \tag{6.2}$$

Next we have

$$\begin{aligned} u_t &= u_{xx} + u_{yy} - \frac{(u_x \;\; u_y)\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}}{u_x^2+u_y^2} \\ &= u_{xx} + u_{yy} - \frac{u_x^2 u_{xx} u_y^2 u_{yy} + 2u_x u_y u_{xy}}{u_x^2+u_y^2} \\ &= \frac{u_x^2 u_{yy} + u_y^2 u_{xx} - 2u_x u_y u_{xy}}{u_x^2+u_y^2}. \end{aligned} \tag{6.3}$$

Compare this two form we get out the conclusion. □

7 Conclusions

In this article we have examined some models in the mathematical theories of image processing. In particular we have analyzed two models:

- The ALM mathematical model in the paper of [10]. We introduced the model and checked the basic mathematical theories of the model in section 4.
- The model of image contour enhancement in the paper of [8]. We introduced the mathematical model and gave out the numerical implementation. Moreover, we can say, from the figures we have given, the black areas under both sides have the same surface.

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