Controllability of Hopfield Impulsive Neural Network Systems with Infinite Delay in Banach Spaces

Jinxiang TANG and Chuang ZHENG

School of Mathematics & Laboratory of Mathematics and Complex Systems, Beijing Normal University, 100875 Beijing, China.
E-mail: jinxiangtang@yeah.net(Jinxiang Tang)
E-mail: chuang.zheng@bnu.edu.cn(Chuang Zheng)

Abstract: The paper considers the controllability of Hopfield impulsive neural network systems with infinite delay in Banach spaces. Taking into account the influence of the impulsive controls, we obtain a sufficient condition of the controllability which is weaker than the existing result in [14](without the $Kalman\ rank\ condition$). An example is given to illustrate our results.

Key Words: Controllability; Impulsive controls; Infinite delay

1 Introduction

Nowadays one of the most popular models of artificial neural networks is the so-called Hopfield-type neural network which is described by time continuous functional differential equations with time delays ([1–3]). Meanwhile, impulsive effect exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications, etc. The stability of such systems has been intensively studied in the literature (see [4–6] and the references therein).

On the other hand, controllability plays a central role throughout the history of modern control theory and engineering, together with the properties such as observability, stability, quadratic optimal control, pole assignment, structural decomposition and observer design, etc. There are amount of results related to the controllability properties of evolution processes characterizing by impulses and delay effects (see, for instance, [7–14]).

By applying Schauder's fixed point theorem, Li et al. ([11]) studied the sufficient condition of the controllability property of impulsive functional differential systems with finite delay. Moreover, Chang ([14]) extended the result with infinite delay. However, none of them considered the specific impact of the impulsive process and the consequent controls.

Our goal in this paper is to study in detail the influence of the impulsive controls. We are interested in the following Hopfield impulsive neural network systems with infinite delay:

$$\begin{cases} \dfrac{dx(t)}{dt} = -Ax(t) + B\displaystyle\int_{-\infty}^{0} dv(\theta) f(x(t+\theta)) \\ \qquad\qquad + Cu(t), t \neq t_k, t \in [t_0, b] = J, \\ \Delta x(t_k) = D^k u(t_k) x(t_k^-), k = 1, 2, \cdots, \rho, \\ x_{t_0}(\theta) = \phi(\theta), \theta \in (-\infty, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the state variable, $u(t) \in \mathbf{R}^m$ is the control input, A, B, C are time-invariant matrices, $v(\theta) : (-\infty, 0] \to \mathbf{R}^n$ is a bounded variation kernel function and without loss of generality, $v(\theta)$ is normalized such that $\int_{-\infty}^{0} |dv(\theta)| = \int_{-\infty}^{0} \sqrt{\sum_{i,j=1}^{n} (dv_{ij}(\theta))^2} = 1$, $f : \mathbf{B} \to \mathbf{R}^n$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump of x at time t_k. $D^k u(t_k) = (d_{ij}^k u(t_k))_{n\times n}$ with $d_{ij}^k : \mathbf{R}^m \to \mathbf{R}$ for $i, j = 1, \cdots, n$, $k = 1, 2, \cdots, \rho$, and remark that if $k = 1, 2, \cdots, \rho - 1, d_{ij}^k = 0, i \neq j$. The histories $x_t(\theta) : (-\infty, 0] \to \mathbf{R}^n$ with $x_t(\theta) = x(t + \theta)$ for any $\theta \leq 0$, is a component of an abstract phase space $\mathbf{B}$, which will be introduced in Section 2.

As we will show in Section 3, by considering the influence of the impulsive controls, we present a weaker sufficient condition by which one can provide the controllability of neural networks with infinite delay, even the $Kalman\ rank\ condition$ ([14]) does not hold.

The rest of the paper is organized as follows. By using the technique in [15], we introduce an abstract space $\mathbf{B}$ in Section 2. In Section 3, we establish a controllability result for mild solution of the system (1) based on Schauder's fixed point theorem. Finally, an example is presented to illustrate our result in Section 4.

2 Preliminaries

In this section, we shall introduce some basic definitions, notations and lemmas which are used throughout this paper.

we introduce a phase space $\mathbf{B}$, which has been used in [15] and is modified in [16]. Note that $\mathbf{B}$ is a linear function space mapping $(-\infty, 0]$ into $\mathbf{R}^n$ and endowed with a semi-norm $|\cdot|_{\mathbf{B}}$. We assume that $\mathbf{B}$ satisfies the following two axioms:

(A) If $x : (-\infty, b] \to \mathbf{R}^n, b > 0$, is such that $x_{t_0} \in \mathbf{B}$ and $x(t)$ is piecewise continuous in $[t_0, b]$, then for every $t \in [t_0, b]$, the following conditions hold:

(i) x_t is in $\mathbf{B}$.

(ii) $|x_t|_{\mathbf{B}} \leq k_1(t - t_0) \sup_{t_0 \leq s \leq t} |x(s)| + M(t - t_0)|\phi|_{\mathbf{B}}$,

where $k_1, M : [0, +\infty) \to (0, +\infty), k_1$ is continuous, M is locally bounded, that is, for any $\beta \in [0, \infty)$ there exists a neighbourhood U of β such that

$$\sup_{t\in U \bigcap [0,\infty)} M(t) < \infty.$$

and k_1, M are independent of x.

(B) The space $\mathbf{B}$ is complete.

Remark 2.1 Note that two examples of the phase space $\mathbf{B}$ has been provided in [16] to provide the existence of such space. Moreover, an example is also given in Section 4.

This work is supported by the CSC and the Fundamental Research Funds for the Central Universities under Grant 2013YB56.

Now, we consider the space

$$\mathbf{B}' = \{x : (-\infty, b] \to \mathbf{R}^n \mid x_k \in C(J_k, \mathbf{R}^n) \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k^-) = x(t_k), x_{t_0} = \phi \in \mathbf{B}, |\phi|_{\mathbf{B}} < \infty, k = 0, 1, \cdots, \rho\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}], k = 0, 1, \cdots, \rho$. Set $|x|_{\mathbf{B}'}$ be a semi-norm in $\mathbf{B}'$ defined by

$$|x|_{\mathbf{B}'} = |\phi|_{\mathbf{B}} + \sup\{|x(s)| : s \in [t_0, b]\}, \ x \in \mathbf{B}'.$$

To set the framework for our main controllability result, we will first use the following definitions.

Definition 2.1 A function $x : (-\infty, b] \to \mathbf{R}^n$ is called a mild solution of the problem (1) if $x_{t_0}(\theta) = \phi(\theta) \in \mathbf{B}$ on $(-\infty, 0]$; $x(t_k^+) - x(t_k^-) = D^k u(t_k) x(t_k^-), k = 1, 2, \cdots, \rho$; the restriction of $x(t)$ to the interval $J_k = (t_k, t_{k+1}] (k = 0, 1, \cdots, \rho)$ is continuous and the following integral equation:

$$x(t) = \prod_{j=1}^{k}(I + D^j u(t_j))e^{-A(t-t_0)}\phi(0) + \sum_{m=1}^{k}\prod_{j=m}^{k}(I + D^j u(t_j))\int_{t_{m-1}}^{t_m} e^{-A(t-s)}[B\int_{-\infty}^{0} dv(\theta)f(x(s+\theta)) + Cu(s)]ds + \int_{t_k}^{t} e^{-A(t-s)}[B\int_{-\infty}^{0} dv(\theta)f(x(s+\theta)) + Cu(s)]ds.$$

is satisfied.

Remark 2.2 Compared to the knowing definition of the mild solution in the existing results (see, for instance, Def 2.1 in [5]), the impulsive controls $D^j u$ appear on the first two items of the right hand side and will play a constructive role in the proof of the controllability.

Definition 2.2 The system (1) is said to be exactly controllable on the interval J if for every initial function $\phi \in \mathbf{B}$ and $x_1 \in \mathbf{R}^n$, there exists a control $u \in L^2(J, \mathbf{R}^m)$ such that the mild solution $x(t)$ of (1) satisfies $x(b) = x_1$.

Let us list the following hypotheses:

(H_1) : There exists positive constants M_1, M_2, M_3, M_4 such that $|e^{-At}| \leq M_1$, $|B| \leq M_2$, $|C| \leq M_3$, $\sup_{1\leq k\leq m} |D^k u(t_k)| \leq M_4$ when $t > t_0$.

(H_2) : $f : \mathbf{B} \to \mathbf{R}^n$ is continuous, there exists $a, d \in \mathbf{R}_+, \gamma \in [0, 1)$ such that $|f(\phi)| \leq a + d|\phi|_{\mathbf{B}}^{\gamma}, \phi \in \mathbf{B}$.

3 Controllability result

In this section, we will present and prove our main results.

Theorem 3.1 Suppose that $(H_1) - (H_2)$ are satisfied. Then the system (1) is exactly controllable on J provided that $x(t_\rho) \neq \mathbf{0}$.

Remark 3.1 Note that (1) is exactly controllable even that A, C does not satisfy the *Kalman rank condition*. This is due to the fact that the last impulsive controls (at time $t = t_\rho$) can change the state of the system provided that $x(t_\rho) \neq \mathbf{0}$. Indeed, let $x(t_\rho) = \mathbf{0}$ and $B = 0$, the impulsive control on time $t = t_\rho$ does not affect the system and the last impulsive processing of system (1) is exactly controllable in time b if and only if A, C satisfy the *Kalman rank condition*.

Let us first consider the one-step impulsive functional differential systems with infinite delay:

$$\begin{cases} \dfrac{dx(t)}{dt} = -Ax(t) + B\displaystyle\int_{-\infty}^{0} dv(\theta)f(x(t+\theta)) + Cu(t), t \in J, \\ \Delta x(t_0) = Du(t_0)x(t_0), \\ x_{t_0}(\theta) = \phi(\theta), \theta \in (-\infty, 0]. \end{cases} \tag{2}$$

From Definition 2.1, we obtain the mild solution of (2) is given by

$$\begin{cases} x(t) = e^{-A(t-t_0)}(I + Du(t_0))\phi(0) + \displaystyle\int_{t_0}^{t} e^{-A(t-s)} \times [B\int_{-\infty}^{0} dv(\theta)f(x(s+\theta)) + Cu(s)]ds, t \in J, \\ x_{t_0}(\theta) = \phi(\theta), \theta \in (-\infty, 0]. \end{cases}$$

To examine conditions of exact controllability of (2), we suppose that the final state $x_f \in \mathbf{R}^n$, and the final time $b > t_0$. Then exact controllability implies that for any given $\phi(\theta) \in \mathbf{B}$, there exists $u(t_0)$ and $u(t), t \in (t_0, b]$ such that

$$x_f = x(b) = e^{-A(b-t_0)}(I + Du(t_0))\phi(0) + \int_{t_0}^{b} e^{-A(b-s)} \times [B\int_{-\infty}^{0} dv(\theta)f(x(s+\theta)) + Cu(s)]ds.$$

Lemma 3.1 Suppose that $(H_1) - (H_2)$ are satisfied. Then the system (2) is exactly controllable on J provided that $x(t_0) = \phi(0) \neq 0$.

Proof: First the $n \times n$ matrix $\tilde{W}$ is defined by

$$\tilde{W} = \int_{t_0}^{b} e^{-A(b-s)}CC^T e^{-A^T(b-s)}ds.$$

According to the rank of matrix $\tilde{W}$, the proof is split into three cases:

*Case*1. Assume that $\tilde{W}$ is invertible, i.e. rank$[\tilde{W}] = n$, and there exists a positive M_5 such that $|\tilde{W}^{-1}| \leq M_5$ when $t \geq t_0$.

Then given any $n \times 1$ vectors $x_f, \phi(0)$, choose

$$u(t) = C^T e^{-A^T(b-t)}\tilde{W}^{-1}[x_f - e^{-A(b-t_0)}(I + Du(t_0)) \times \phi(0) - \int_{t_0}^{b} e^{-A(b-s)}B\int_{-\infty}^{0} dv(\theta)f(x(s+\theta))ds], \quad t \in J.$$

It shall be shown that when using this control the operator Γ defined by

$$(\Gamma x)(t) = \begin{cases} e^{-A(t-t_0)}(I + Du(t_0))\phi(0) + \displaystyle\int_{t_0}^{t} e^{-A(t-s)}[B\int_{-\infty}^{0} dv(\theta) \times f(x(s+\theta)) + Cu(s)]ds, t \in J, \\ \phi(t), \quad t \in (-\infty, t_0]. \end{cases}$$

has a fixed point. This fixed point is then a mild solution of the system (2). Clearly, $x(b) = (\Gamma x)(b) = x_f$, which implies that the system is exactly controllable.

For $\phi \in \mathbf{B}$, we define $\check{\phi}$ by

$$\check{\phi}(t) = \begin{cases} e^{-A(t-t_0)}(I + Du(t_0))\phi(0), t \in J, \\ \phi(t), t \in (-\infty, t_0], \end{cases}$$

then $\check{\phi} \in \mathbf{B}'$. Let

$$x(t) = y(t) + \check{\phi}(t), \ -\infty < t \le b.$$

It is easy to see that y satisfies $y_{t_0}(\theta) = 0, \theta \in (-\infty, 0]$ and

$$\begin{aligned} y(t) = &\int_{t_0}^{t} e^{-A(t-s)} B \int_{-\infty}^{0} dv(\theta) f(y(s+\theta) \\ &+ \check{\phi}(s+\theta))ds + \int_{t_0}^{t} e^{-A(t-\eta)} CC^T e^{-A^T(b-\eta)} \\ &\times \tilde{W}^{-1}[x_f - e^{-A(b-t_0)}(I + Du(t_0))\phi(0) \\ &- \int_{t_0}^{b} e^{-A(b-s)} B \int_{-\infty}^{0} dv(\theta) f(y(s+\theta) \\ &+ \check{\phi}(s+\theta))ds]d\eta, \end{aligned}$$

if and only if x satisfies

$$\begin{aligned} x(t) = &e^{-A(t-t_0)}(I + Du(t_0))\phi(0) + \int_{t_0}^{t} e^{-A(t-s)} B \\ &\times \int_{-\infty}^{0} dv(\theta) f(x(s+\theta))ds + \int_{t_0}^{t} e^{-A(t-\eta)} \\ &\times CC^T e^{-A^T(b-\eta)} \tilde{W}^{-1}[x_f - e^{-A(b-t_0)} \\ &\times (I + Du(t_0))\phi(0) - \int_{t_0}^{b} e^{-A(b-s)} B \\ &\times \int_{-\infty}^{0} dv(\theta) f(x(s+\theta))ds]d\eta, \end{aligned}$$

and $x_t(\theta) = \phi(\theta), \theta \in (-\infty, 0]$.

Define $\mathbf{B}'' = \{y \in \mathbf{B}' \mid y_{t_0} = 0 \in \mathbf{B}\}$. For any $y \in \mathbf{B}''$,

$$\begin{aligned} |y|_{\mathbf{B}'} &= |y_{t_0}|_{\mathbf{B}} + \sup\{|y(s)|, s \in [t_0, b]\} \\ &= \sup\{|y(s)|, s \in [t_0, b]\}, \end{aligned}$$

thus $(\mathbf{B}'', |\cdot|_{\mathbf{B}'})$ is a Banach space.

Set $A(\eta) = \{y \in \mathbf{B}'' \mid |y|_{\mathbf{B}'} \le \eta\}$ for some $\eta > 0$. Then for any $y \in A(\eta)$, from the axioms of $\mathbf{B}$, we have

$$\begin{aligned} |y_t + \check{\phi}_t|_{\mathbf{B}} &\le k_1(t-t_0) \sup_{t_0 \le s \le b} |y(s) + \check{\phi}(s)| \\ &+ M(t-t_0)|y_{t_0} + \check{\phi}_{t_0}|_{\mathbf{B}} \\ &\le k_1(t-t_0)[\sup_{t_0 \le s \le b} |y(s)| + \sup_{t_0 \le s \le b} |\check{\phi}(s)|] \\ &+ M(t-t_0)[|y_{t_0}|_{\mathbf{B}} + |\check{\phi}_{t_0}|_{\mathbf{B}}] \\ &\le k_1(t-t_0)[\eta + M_1(1+M_4)|\phi(0)|] \\ &+ M(t-t_0)|\phi(t)|_{\mathbf{B}} = q'', \ t \in J. \end{aligned} \tag{3}$$

Let $\varphi : \mathbf{B}'' \to \mathbf{B}''$ be an operator defined by $(\varphi y)(t) = 0, t \in (-\infty, t_0]$ and

$$\begin{aligned} (\varphi y)(t) = &\int_{t_0}^{t} e^{-A(t-s)} B \int_{-\infty}^{0} dv(\theta) f(y(s+\theta) + \check{\phi}(s \\ &+ \theta))ds + \int_{t_0}^{t} e^{-A(t-\eta)} CC^T e^{-A^T(b-\eta)} \tilde{W}^{-1}[x_f \\ &- e^{-A(b-t_0)}(I + Du(t_0))\phi(0) - \int_{t_0}^{b} e^{-A(b-s)} B \\ &\times \int_{-\infty}^{0} dv(\theta) f(y(s+\theta) + \check{\phi}(s+\theta))ds]d\eta. \end{aligned}$$

Obviously the operator Γ has a fixed point is equivalent to φ has one. So it turns out to prove that φ has a fixed point. The proof will be given in several steps.

Step 1. $\varphi(A(\eta)) \subset A(\eta)$ for some $\eta > 0$.

If this is not true, then for each positive number $\eta > 0$, there exists a function $y^\eta(t) \in A(\eta)$, but $\varphi(y^\eta(t)) \notin A(\eta)$, i.e. $|\varphi(y^\eta(t))| > \eta$ for some $t \in J$. However, on the other hand, we have from (H_1), (H_2) and (3),

$$\begin{aligned} \eta < |\varphi(y^\eta(t))| &\le M_1 M_2 (b-t_0)(a + d(q'')^\gamma) + (b-t_0) \\ &\times (M_1 M_3)^2 M_5[|x_f| + M_1(1+M_4) \\ &\times |\phi(0)|] + (b-t_0)(M_1 M_3)^2 M_5[M_1 \\ &\times M_2(b-t_0)(a + d(q'')^\gamma)] \\ &= (1 + (b-t_0)(M_1 M_3)^2 M_5)[M_1 M_2(b \\ &- t_0)(a + d(q'')^\gamma)] + (b-t_0)(M_1 M_3)^2 \\ &\times M_5[|x_f| + M_1(1+M_4)|\phi(0)|]. \end{aligned}$$

Dividing both sides by η and noting that

$$\begin{aligned} q'' = &k_1(t-t_0)[\eta + M_1(1+M_4)|\phi(0)|] \\ &+ M(t-t_0)|\phi(t)|_{\mathbf{B}} \to \infty(\eta \to \infty), \end{aligned}$$

we obtain

$$\begin{aligned} &\lim_{\eta \to \infty} \inf \frac{a + d(q'')^\gamma}{\eta} \\ &= \lim_{\eta \to \infty} \inf(\frac{a + d(q'')^\gamma}{q''} \times \frac{q''}{\eta}) \\ &= d \inf_{t_0 \le t \le b} k_1(t-t_0) \lim_{\eta \to \infty} \inf(\frac{1}{(q'')^{1-\gamma}}) = 0. \end{aligned}$$

Thus, we have $1 < 0$. This is a contradiction. Hence for some positive number $\eta > 0, \varphi(A(\eta)) \subset A(\eta)$.

Step 2. $\varphi : \mathbf{B}'' \to \mathbf{B}''$ is continuous.

Let $\{y^{(n)}\}_0^\infty \subset \mathbf{B}''$, with $y^{(n)} \to y \in \mathbf{B}''$. In view of (H_2), we have

$$\begin{aligned} &|\varphi y^{(n)} - \varphi y|_{\mathbf{B}'} \\ &= \sup_{t \in J} |\int_{t_0}^{t} e^{-A(t-s)} B \int_{-\infty}^{0} dv(\theta)[f(y_s^{(n)} \\ &+ \check{\phi}_s) - f(y_s + \check{\phi}_s)]ds + \int_{t_0}^{t} e^{-A(t-\eta)} C \\ &\times C^T e^{-A^T(b-\eta)} \tilde{W}^{-1}\{- \int_{t_0}^{b} e^{-A(b-s)} B \end{aligned}$$

$$\times \int_{-\infty}^{0} dv(\theta)[f(y_s^{(n)} + \check{\phi}_s) - f(y_s + \check{\phi}_s)]ds\}d\eta|$$
$$\leq M_1M_2(b-t_0)|\int_{-\infty}^{0} dv(\theta)(f(y_s^{(n)}$$
$$+ \check{\phi}_s) - f(y_s + \check{\phi}_s))ds| + (M_1M_3)^2M_5$$
$$\times (b-t_0)^2M_1M_2|\int_{-\infty}^{0} dv(\theta)f(y_s^{(n)}$$
$$+ \check{\phi}_s) - f(y_s + \check{\phi}_s)ds| \to 0.$$

Thus, φ is continuous.

Step 3. φ maps $A(\eta)$ into an equicontinuous family.

Let $y \in A(\eta), \tau_1, \tau_2 \in J$. Then $t_0 < \tau_1 < \tau_2 \leq b$, in view of $(H_1), (H_2)$ and (3), we have

$$|(\varphi y)(\tau_1) - (\varphi y)(\tau_2)|_{\mathbf{B}'}$$
$$\leq \sup_{t\in J}|\int_{t_0}^{\tau_1} e^{-A(\tau_1-s)}B\int_{-\infty}^{0} dv(\theta)f(y(s+\theta)+\check{\phi}(s$$
$$+\theta))ds + \int_{t_0}^{\tau_1} e^{-A(\tau_1-\eta)}CC^Te^{-A^T(b-\eta)}\tilde{W}^{-1}[x_f$$
$$- e^{-A(b-t_0)}(I+Du(t_0))\phi(0) - \int_{t_0}^{b} e^{-A(b-s)}B\int_{-\infty}^{0}$$
$$\times dv(\theta)f(y(s+\theta)+\check{\phi}(s+\theta))ds]d\eta - \int_{t_0}^{\tau_2} e^{-A(\tau_2-s)}$$
$$\times B\int_{-\infty}^{0} dv(\theta)f(y(s+\theta)+\check{\phi}(s+\theta))ds$$
$$- \int_{t_0}^{\tau_2} e^{-A(\tau_2-\eta)}CC^Te^{-A^T(b-\eta)}\tilde{W}^{-1}[x_f - e^{-A(b-t_0)}$$
$$\times (I+Du(t_0))\phi(0) - \int_{t_0}^{b} e^{-A(b-s)}B\int_{-\infty}^{0} dv(\theta)$$
$$\times f(y(s+\theta)+\check{\phi}(s+\theta))ds]d\eta|$$
$$\leq \int_{t_0}^{\tau_1} |e^{-A(\tau_1-s)} - e^{-A(\tau_2-s)}||B|\int_{-\infty}^{0} |dv(\theta)|$$
$$\times (a+d(q'')^\gamma)ds + \int_{t_0}^{\tau_1} |e^{-A(\tau_1-\eta)} - e^{-A(\tau_2-\eta)}||C|^2$$
$$\times |e^{-A^T(b-\eta)}||\tilde{W}^{-1}|[|x_f| + M_1(1+M_4)|\phi(0)| + M_1$$
$$\times M_2\int_{t_0}^{b}\int_{-\infty}^{0} |dv(\theta)|(a+d(q'')^\gamma)ds]d\eta$$
$$+ \int_{\tau_1}^{\tau_2} |e^{-A(\tau_2-s)}||B|\int_{-\infty}^{0} |dv(\theta)|(a+d(q'')^\gamma)ds$$
$$+ \int_{\tau_1}^{\tau_2} |e^{-A(\tau_2-\eta)}||C|^2|e^{-A^T(b-\eta)}||\tilde{W}^{-1}|[|x_f|$$
$$+ M_1(1+M_4)|\phi(0)| + M_1M_2\int_{t_0}^{b}|B|$$
$$\times \int_{-\infty}^{0} |dv(\theta)|(a+d(q'')^\gamma)ds]d\eta.$$

The right-hand side is independent of $y \in A(\eta)$ and tends to zero as $\tau_1 - \tau_2 \to 0$. Thus, φ maps $A(\eta)$ into an equicontinuous family. The equicontinuities for the cases $\tau_1 < \tau_2 \leq t_0$ and $\tau_1 \leq t_0 < \tau_2$ are obvious.

So, by Steps $1-3$ together with the Arzela-Ascoli theorem, we can conclude that $\varphi : A(\eta) \to A(\eta)$ is completely continuous. As a consequence of Schauder's fixed theorem, we deduce that φ has a fixed point $\bar{y} \in A(\eta)$. Let $\bar{x}(t) = \bar{y}(t) + \check{\phi}(t), t \in (-\infty, b]$. Then $\bar{x}(t)$ is a fixed point of the operator Γ which is a mild solution of the problem (1).

*Case*2. Suppose that $\tilde{W}$ is not invertible and the rank is d $(0 \leq d < n)$, it is easy to see that $rank[C, AC, \cdots, A^{n-1}C] = d$. Then there exists an $n \times n$ invertible matrix S such that the matrices $\tilde{A} = S^{-1}AS,\ \tilde{B} = S^{-1}B,\ \tilde{C} = S^{-1}C$ have the block structure given by

$$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ \mathbf{0} & A_3 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tilde{C} = \begin{bmatrix} C_1 \\ \mathbf{0} \end{bmatrix},$$

where A_1, A_2, A_3, B_1, B_2 and C_1 are, respectively, $d\times d$, $d\times(n-d)$, $(n-d)\times(n-d)$, $d\times n$, $(n-d)\times n$ and $d\times m$ matrices. See [13]. Thus with the change of variable $x = Sy$, the system (2) is transformed into

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -A_1y_1(t) - A_2y_2(t) + B_1\displaystyle\int_{-\infty}^{0} dv(\theta) \\ \qquad \times f(Sy(s+\theta)) + C_1u(t), t \in J, \\ \dfrac{dy_2(t)}{dt} = -A_3y_2(t) + B_2\displaystyle\int_{-\infty}^{0} dv(\theta)f(Sy(s \\ \qquad +\theta)), t \in J, \\ y_1(t_0^+) = N^{11}y_1(t_0^-) + N^{12}y_2(t_0^-), \\ y_2(t_0^+) = N^{21}y_1(t_0^-) + N^{22}y_2(t_0^-), \\ y_{t_0}(\theta) = S^{-1}\phi(\theta), \theta \in (-\infty, 0]. \end{cases} \quad (4)$$

where $N^{11}, N^{12}, N^{21}, N^{22}$ are, respectively, $d\times d$, $d\times(n-d)$, $(n-d)\times d$ and$(n-d)\times(n-d)$ matrices such that

$$S^{-1}[I+Du(t_0)]S = \begin{bmatrix} N^{11} & N^{12} \\ N^{21} & N^{22} \end{bmatrix}.$$

From Definition 2.1 and system (4), we obtain

$$\begin{cases} y_2(t) = e^{-A_3(t-t_0)}(N^{21}y_1(t_0^-) + N^{22}y_2(t_0^-)) \\ \qquad + \displaystyle\int_{t_0}^{t} e^{-A_3(t-s)}B_2\int_{-\infty}^{0} dv(\theta)f(Sy(s+\theta))ds, \\ y_1(t) = e^{-A_1(t-t_0)}(N^{11}y_1(t_0^-) + N^{12}y_2(t_0^-)) \\ \qquad + \displaystyle\int_{t_0}^{t} e^{-A_1(t-s)}[B_1\int_{-\infty}^{0} dv(\theta)f(Sy(s+\theta)) \\ \qquad - A_2y_2(s) + C_1u(s)]ds, \end{cases} \quad (5)$$

for $t_0 \leq t \leq b$. Exact controllability implies for any given initial state $\phi(\theta) \in \mathbf{B}$, there exists $u(t_0)$ and $u(t)(t \in (t_0, b])$ such that

$$y_2(b) = e^{-A_3(b-t_0)}(N^{21}y_1(t_0^-) + N^{22}y_2(t_0^-)) + \int_{t_0}^{b} e^{-A_3(b-s)}B_2\int_{-\infty}^{0} dv(\theta)f(Sy(s+\theta))ds, \quad (6)$$

$$y_1(b) = e^{-A_1(b-t_0)}(N^{11}y_1(t_0^-) + N^{12}y_2(t_0^-)) + \int_{t_0}^{b} e^{-A_1(b-s)}[B_1\int_{-\infty}^{0} dv(\theta)f(Sy(s+\theta)) - A_2y_2(s) + C_1u(s)]ds. \quad (7)$$

According to (6) we obtain

$$y_2(b) - \int_{t_0}^{b} e^{-A_3(b-s)}B_2\int_{-\infty}^{0} dv(\theta)f(Sy(s+\theta))ds$$
$$= e^{-A_3(b-t_0)}(N^{21}y_1(t_0^-) + N^{22}y_2(t_0^-))$$

$$\Leftrightarrow e^{A_3(b-t_0)}[y_2(b) - \int_{t_0}^{b} e^{-A_3(b-s)} B_2 \int_{-\infty}^{0} dv(\theta) \times f(Sy(s+\theta))ds] = N^{21}y_1(t_0^-) + N^{22}y_2(t_0^-).$$

Therefore, we can choose appropriate impulsive control matrix $Du(t_0)$ to hold the above equation.

In order to discuss the exact controllability of (7), we define the $n \times n$ constant matrix as

$$V = \int_{t_0}^{b} e^{-A_1(b-s)} C_1 C_1^T e^{-A_1^T(b-s)} ds.$$

It's easy to know the matrix V is invertible, and now we assume that there exists a positive constant M_5' such that $|V^{-1}| \leq M_5'$.

For any given $d \times 1$ vector $y_1(b)$, in view of (7), let

$$\begin{aligned} u(t) = {} & C_1^T e^{-A_1^T(b-t)} V^{-1}\{y_1(b) - e^{-A_1(b-t_0)}(N^{11}y_1(t_0^-) \\ & + N^{12}y_2(t_0^-)) - \int_{t_0}^{b} e^{-A_1(b-s)}\{B_1 \int_{-\infty}^{0} dv(\theta) f(S \\ & \times y(s+\theta)) - A_2[e^{A_3(b-s)}(y_2(b) - \int_{t_0}^{b} e^{-A_3(b-\xi)} \\ & \times B_2 \int_{-\infty}^{0} dv(\theta) f(Sy(\eta+\theta)))d\xi + \int_{t_0}^{s} e^{-A_3(s-\eta)} \\ & \times B_2 \int_{-\infty}^{0} dv(\theta) f(Sy(\eta+\theta))d\eta)]\}ds\}. \end{aligned}$$

In the subsequent, in view of $(H_1), (H_2)$ and Schauder's fixed point theorem, we can use the similar method of Case 1 to obtain the controllability of system (2). This finishes the proof.

To examine conditions of exact controllability of system (1), we transform the system (1) into

$$\begin{cases} \dfrac{dx(t)}{dt} = -Ax(t) + B\displaystyle\int_{-\infty}^{0} dv(\theta) f(x(t+\theta)) \\ \qquad + Cu(t), t \in [t_\rho, b], \\ x(t_\rho^+) - x(t_\rho^-) = D^\rho u(t_\rho) x(t_\rho^-), \\ x_{t_\rho}(\theta) = \tilde{\phi}(\theta), \theta \in (-\infty, 0], \end{cases} \tag{8}$$

where

$$\tilde{\phi}(\theta) = \begin{cases} \displaystyle\prod_{j=1}^{k}(I + D^j u(t_j)) e^{-A(t-t_0)}\phi(0) + \sum_{i=1}^{k}\prod_{j=i}^{k}(I \\ \displaystyle + D^j u(t_j)) \int_{t_{i-1}}^{t_i} e^{-A(t-s)}[B\int_{-\infty}^{0} dv(\theta) \\ \displaystyle \times f(x(s+\theta)) + Cu(s)]ds + \int_{t_k}^{t} e^{-A(t-s)} \\ \displaystyle \times [B\int_{-\infty}^{0} dv(\theta) f(x(s+\theta)) + Cu(s)]ds, \\ k = 1, 2, \cdots, \rho-1, t \in (t_0, t_\rho], \\ \phi(t), t \in (-\infty, t_0], \end{cases}$$

where $u(t) \in L^2([t_0, t_\rho], \mathbf{R}^m)$ is chosen arbitrarily. It is easy to see that discussing the controllability of system (1) is equivalent to considering the exact controllability of system (2). As an immediate result of Lemma 3.1, we can obtain Theorem 3.1.

4 An example

In this section, we shall give an example to illustrate our result. Consider the following second-order nonlinear functional differential equations of the form:

$$\begin{cases} \dfrac{dx(t)}{dt} = -Ax(t) + B\displaystyle\int_{-\infty}^{0} dv(\theta) f(x(t+\theta)) \\ \qquad + Cu(t), t \neq t_k, t \in [t_0, b] = J, \\ x(t_k^+) - x(t_k^-) = D^k u(t_k) x(t_k^-), k = 1, 2, \\ x_{t_0}(\theta) = \phi(\theta), \theta \in (-\infty, 0], \end{cases} \tag{9}$$

where $\phi(t) = (\phi^1(t), \phi^2(t))^T$,

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$D^1u(t_1) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, D^2u(t_2) = \begin{bmatrix} d_1^1u(t_2) & d_2^1u(t_2) \\ d_1^2u(t_2) & d_2^2u(t_2) \end{bmatrix}.$$

Set a space of piecewise continuous functions. For any $r \in \mathbf{R}$, let

$$\begin{aligned} \mathbf{B} = \{ & \phi : (-\infty, 0] \to \mathbf{R}^n |\ \phi \text{ is piecewise continuous,} \\ & \text{and for any } \theta, \phi(\theta) \text{ is locally bounded, } e^{r\theta}\phi(\theta) \\ & \to \text{ a limit as } \theta \to -\infty\}, \end{aligned}$$

and let

$$|\phi|_{\mathbf{B}} = \sup_{-\infty \leq \theta \leq 0} e^{r\theta}|\phi|,$$

it's easy to see that $(\mathbf{B}, |\cdot|_{\mathbf{B}})$ is a Banach space.

(a) It's easy to see that $x : (-\infty, b] \to \mathbf{R}^n, b > 0$, is such that $x_{t_0} := \phi \in \mathbf{B}$ and $x(t)$ is piecewise continuous in $[t_0, b], x_t$ is in $\mathbf{B}$.

(b) There exists continuous function $k_1(\beta) = \sup_{-\beta \leq \theta \leq 0} e^{r\theta}$ and locally bounded function $M(\beta) = e^{-r\beta}$ satisfy $|x_t|_{\mathbf{B}} \leq k_1(t-t_0) \sup_{t_0 \leq s \leq t} |x(s)| + M(t-t_0)|\phi|_{\mathbf{B}}$. Indeed,

$$\begin{aligned} |x_t|_{\mathbf{B}} & \leq \max\{\sup_{-(t-t_0) \leq \theta \leq 0} e^{r\theta}|x_t(\theta)|, \sup_{\theta \leq -(t-t_0)} e^{r\theta}|x_t(\theta)|\} \\ & \leq \sup_{-(t-t_0) \leq \theta \leq 0} e^{r\theta}|x_t(\theta)| + \sup_{\theta \leq -(t-t_0)} e^{r\theta}|x_t(\theta)| \\ & \leq \sup_{-(t-t_0) \leq \theta \leq 0} e^{r\theta} \sup_{t_0 \leq s \leq t} |x(s)| + e^{-r(t-t_0)} \sup_{\theta \leq 0} e^{r\theta} \\ & \quad \times |x_{t_0}(\theta)| \\ & = k_1(t-t_0) \sup_{t_0 \leq s \leq t} |x(s)| + M(t-t_0)|\phi|_{\mathbf{B}}. \end{aligned}$$

(c) In (H_2), let activation function $f(x(t+\theta)) = (e^{r\theta}x(t+\theta))^3, r \in \mathbf{R}, \gamma \in [0, 1)$, we know it's a continuous function such that

$$\begin{aligned} |f(x(t+\theta))| & = [(e^{3r\theta}x^3(t+\theta))^2]^{\frac{1}{2}} \\ & \leq [\sup_{-\infty \leq \theta \leq 0}((e^{r\gamma\theta}x^\gamma(t+\theta)))^2 \cdot (e^{r(3-\gamma)\theta} \\ & \quad \times x^{(3-\gamma)}(t+\theta)))^2]^{\frac{1}{2}} \\ & := d[(|x(t+\theta)|_{\mathbf{B}})^{2\gamma}]^{\frac{1}{2}} = d|x(t+\theta)|_{\mathbf{B}}^\gamma, \end{aligned}$$

where $a = 0, d = \sup_{-\infty \leq \theta \leq 0} |e^{r(3-\gamma)\theta}x^{(3-\gamma)}(t+\theta)|$.

(d) In (H_1), $A, B, C, D^k u(t_k)$ are constant matrixes.

(e) Choose bounded variation kernel matrix

$$v(\theta) = \begin{bmatrix} \frac{\sqrt{2}}{2}e^{\theta} & 0 \\ 0 & \frac{\sqrt{2}}{2}e^{\theta} \end{bmatrix}.$$

Using the method of variation of constants, we can receive part of the solution of system (9):

$$\begin{cases} x_2(t_2) = 4e^{t_2-t_0}\phi^2(0), \\ x_1(t_2) = 3e^{t_2-t_0}\phi^1(0) + \frac{3\sqrt{2}}{2}\int_{t_0}^{t_1} e^{t_2-s}\int_{-\infty}^{0}(x(t \\ \quad +\theta))^2 e^{(1+3r)\theta}[x_1(s+\theta)+x_2(s+\theta)]d\theta ds \\ \quad +3\int_{t_0}^{t_1} e^{t_2-s}u(s)ds + \frac{\sqrt{2}}{2}\int_{t_1}^{t_2} e^{t_2-s}\int_{-\infty}^{0} \\ \quad \times (x(t+\theta))^2 e^{(1+3r)\theta}[x_1(s+\theta)+x_2(s \\ \quad +\theta)]d\theta ds + \int_{t_1}^{t_2} e^{t_2-s}u(s)ds, \end{cases}$$

then for any given $\phi(\theta) \in \mathbf{B}$, we can let $u(t) = u_1(t), t \in [t_0, t_2]$ to get $x_1(t_2) \neq \mathbf{0}, x(t_2) \neq \mathbf{0}$. By the method of variation of constants, $x(t_2)$ and $D^2u(t_2)$, we receive

$$x_2(b) = e^{b-t_2}[d_1^2u(t_2)x_1(t_2) + d_2^2u(t_2)x_2(t_2)], \qquad (10)$$

and

$$\begin{aligned} x_1(b) &= e^{b-t_2}[d_1^1u(t_2)x_1(t_2) + d_2^1u(t_2)x_2(t_2)] \\ &\quad + \frac{\sqrt{2}}{2}\int_{t_2}^{b} e^{b-s}\int_{-\infty}^{0}(x(t+\theta))^2 e^{(1+3r)\theta} \\ &\quad \times [x_1(s+\theta)+x_2(s+\theta)]d\theta ds \\ &\quad + \int_{t_2}^{b} e^{b-s}u(s)ds. \end{aligned} \qquad (11)$$

In Eq. (10), since $x_1(t_2) \neq 0$, we can take $d_2^2u(t_2) = 0$ and obtain $d_1^2u(t_2) = \frac{e^{t_2-b}x_2(b)}{x_1(t_2)}$;

In Eq. (11), we can choose $d_1^1u(t_2) = 0, d_2^1u(t_2) = 0$, and receive

$$\begin{aligned} x_1(b) = \frac{\sqrt{2}}{2}\int_{t_2}^{b} e^{b-s}\int_{-\infty}^{0}(x(t+\theta))^2 e^{(1+3r)\theta}[x_1(s+\theta) \\ + x_2(s+\theta)]d\theta ds + \int_{t_2}^{b} e^{b-s}u(s)ds. \end{aligned}$$

Therefore, we can choose

$$u(t) = \begin{cases} u_1(t), t \in [t_0, t_2] \\ e^{b-t}[\int_{t_2}^{b}(e^{b-s})^2 ds]^{-1}\{x_1(b) - \frac{\sqrt{2}}{2}\int_{t_2}^{b} e^{b-s} \\ \times \int_{-\infty}^{0}(x(t+\theta))^2 e^{(1+3r)\theta}[x_1(s+\theta) \\ + x_2(s+\theta)]d\theta ds\}, t \in [t_2, b], \end{cases}$$

and get to know system (9) is exactly controllable.

5 Conclusion

In this paper, we establish a sufficient condition for the controllability of system (1) by introducing the impulsive controls and the abstract space **B**. The result is given by the fixed point method and by choosing an appropriate functional space **B** (as in Section 2). Compared to the existing results, Theorem 3.1 is weaker due to the fact that the $Kalman\ rank\ condition$ is removed from the assumptions. Finally, we present an example and shows that the impulsive controls can play the key role in the control process.

References

[1] Gopalsamy, K. and He, Xuezhong, Stability in asymmetric Hopfield nets with transmission delays, Phys. D 76 (1994), no. 4, 344-358.

[2] Hopfield, J. J., Neural networks and physical systems with emergent collective computational abilities, Proc. Nat. Acad. Sci. U.S.A. 79 (1982), no. 8, 2554-2558.

[3] Marcus. C. M. and Westervelt. R. M., Stability of analog neural networks with delay, Phys. Rev. A (3) 39 (1989), no. 1, 347-359.

[4] Ballinger, George Henri, Qualitative theory of impulsive delay differential equations, Thesis (Ph.D.)-University of Waterloo (Canada). 2000.

[5] Lakshmikantham, V., Bainov, D. D. and Simeonov, P. S., Theory of impulsive differential equations, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[6] Guan, Zhihong and Chen, Guanrong, On delayed impulsive Hopfield neural networks, Neural Networks 12 (1999), no. 2, 273-280.

[7] Liu, Bin and Marquez, Horacio J., Controllability and observability for a class of controlled switching impulsive systems, IEEE Trans. Automat. Control 53 (2008), no. 10, 2360-2366.

[8] Xie, Guangming, Yu, Junyan and Wang, Long, Necessary and sufficient conditions for controllability of switched impulsive control systems with time delay, Proe. IEEE Conf. Decision Control 45 (2006), 4093-4098.

[9] Xie, Guangming and Wang, Long, Controllability and observability of a class linear impulsive systems, J. Math. Anal. Appal. 304 (2005), 336-355.

[10] Xie, Guangming and Wang, Long, Necessary and sufficient conditons for controllability and observability of switched impulsive controls systems, IEEE Trans. Autom. Control 49 (2004), no. 6, 960-966.

[11] Li, Meili, Wang, Miansen and Zhang, Fengqin, Controllability of impulsive functional differential systems in Banach spaces, Chaos Solitons Fractals 29 (2006), no. 1, 175-181.

[12] George, R. K., Nandakumaran, A. K. and Arapostathis, A., A note on controllability of impulsive systems, J. Math. Anal. Appl. 241 (2000), no. 2, 276-283.

[13] Leela, S., McRae, Farzana A. and Sivasundaram, S., Controllability of impulsive differential equations, J. Math. Anal. Appl. 177 (1993), no. 1, 24-30.

[14] Chang, Yongkui, Controllability of impulsive functional differential systems with infinite delay in Banach spaces, Chaos Solitons Fractals 33 (2007), no. 5, 1601-1609.

[15] Hale, Jack K. and Kato, Junji, Phase space for retarded equations with infinite delay, Funkcial. Ekvac. 21 (1978), no. 1, 11-41.

[16] Chang, Yongkui, Arjunan, M. M. and Kavitha, V., Existence results for a second order impulsive functional differential eqution with state-dependent delay, Differ. Equ. Appl. 1 (2009), no. 3, 325-339.