Exact controllability for the fourth order Schrödinger equation *

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Abstract

In this paper the boundary controllability of the fourth order Schrödinger equation in bounded domains is studied. By means of an L^2-Neumann boundary control, we prove that the solution is exactly controllable in $H^{-2}(\Omega)$ for arbitrarily small time. The method of proof combines both the HUM (Hilbert Uniqueness Method) and multiplier techniques.

AMS Subject Classifications.
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1 Introduction

Let Ω be a nonempty open bounded domain in $\mathbb{R}^n$ $(n \in \mathbb{N})$ with C^3 boundary Γ, Γ_0 be a nonempty open subset of Γ, and $T > 0$ be a given time duration. Fix some $x_0 \in \mathbb{R}^n$, put

$$\Gamma_0 \stackrel{\triangle}{=} \left\{ x \in \Gamma \;\middle|\; (x - x_0) \cdot \nu(x) > 0 \right\}, \tag{1.1}$$

where $\nu(x)$ is the unit outward normal vector of Ω at $x \in \Gamma$. We consider the following controlled fourth order linear schrödinger equation with a controller acting on the subset of

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the boundary

$$\begin{cases} iy_t + \Delta^2 y = 0, & \text{in } \Omega \times (0,T) \\ y = 0, \frac{\partial y}{\partial \nu} = v\chi_{\Gamma_0}, & \text{on } \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{1.2}$$

Here and henceforth, χ_{Γ_0} is the characteristic function of the set Γ_0 and Δ is the Laplacian in the space variable $x \in \Omega$. In (1.2), $y(\cdot,t)$ can be considered as the *probability amplitude of the state* and $v(\cdot,t)$ is the *control*. Both are complex valued functions. The *control space* of system (1.2) is chosen to be $L^2((0,T)\times\Gamma_0)$.

As we will show later in Section 4, the well-posedness of the system is given as follows: For any initial data $y_0 \in H^{-2}(\Omega)$ and $v \in L^2(\Omega)$, there exists a unique solution $y \in C([0,T]; H^{-2}(\Omega))$ of (1.2), in the transposition sense ([13]).

In this paper, we are interested in the exact (boundary) controllability problem of (1.2), which is stated as follows: Let y_0 be a given function in $H^{-2}(\Omega)$ and let $T > 0$ be given, whether there exists a boundary function v on $\Gamma_0 \times (0,T)$ such that the solution of the equation (1.2), satisfies $y(0) = y_0$ and $y(T) = 0$ in Ω? If such a control v exists, we say that the system (1.2) is *exactly controllable* from y_0 to the rest at time T by the boundary control v.

The fourth order Schrödinger equation arises in many scientific fields such as quantum mechanics, nonlinear optics and plasma physics, and has been intensively studied with fruitful references. For instance, the well-posedness and existence of the solutions has been shown ([7, 8, 16, 17]) by means of the energy method and harmonic analysis. However, it is still unknown for the corresponding controllability properties.

As far as we know, there are plenty of references concerning the controllability properties of second order schrödinger equations([14]). For the higher order operators, these control problems are mostly studied for parabolic cases, such as the approximate controllability of the nonlinear equation ([4]), the null boundary controllability of $1-d$ and $N-d$ case ([3, 10]), etc. Recent results ([2, 15, 18]) considered the exact observability and some equivalent assertions for the skew-adjoint operators, which can be seen as an abstract model for higher-order schrödinger equations.

By establishing the control theory for the linear fourth-order model (1.2), we hope it would be helpful to understand the phenomena of the high dimensional higher-order nonlinear systems. In this paper, we attempt to establish the boundary controllability properties of system (1.2) by means of the Hilbert Uniqueness Method (HUM) and the multiplier techniques. More precisely, by classical duality arguments ([12]), the above controllability property is equivalent to a (boundary) observability estimate of the following uncontrolled Schrödinger equation:

$$\begin{cases} i\varphi_t + \Delta^2 \varphi = 0, & \text{in } \Omega \times (0,T) \\ \varphi = 0, \dfrac{\partial \varphi}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T) \\ \varphi(x,0) = \varphi^0, & \text{in } \Omega. \end{cases} \tag{1.3}$$

Our first result is the observability inequality of (1.3), which reads as follows:

Theorem 1.1 *For equation* (1.3)*, the solution of* (1.3) *satisfies*

$$\|\varphi^0\|^2_{H_0^2(\Omega)} \le C\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 d\sigma dt, \qquad \forall \varphi^0 \in H_0^2(\Omega). \tag{1.4}$$

Here and thereafter, we will use C to denote a generic positive constant (depending only on T, Ω and Γ_0) which may vary from line to line.

As a direct consequence of Theorem 1.1, the controllability property of (1.2) is stated as follows:

Theorem 1.2 *Let $T > 0$, Γ_0 be defined by* (1.1) *and $\Sigma_0 = \Gamma_0 \times (0, T)$. Then, for any $y_0 \in H^{-2}(\Omega)$, there exists $v \in L^2(\Gamma_0 \times (0, T))$ such that the unique solution $y \in C([0, T]; H^{-2}(\Omega))$ of* (1.2) *satisfies $y(T) = 0$.*

Remark 1.1 *Without loss of generality, the final state $y(T)$ is driven to the rest. This is due to the fact that system* (1.2) *is linear and time reversible. This phenomenon happens in finite dimensional linear controlled systems, and the situation is completely different in the case of the time irreversible case, such as the heat equation.*

The rest of the paper is organized as follows. An identity of the fourth order Schrödinger operator is given in Section 2 by choosing suitable multiplier and playing carefully with the boundary terms. In Section 3, we show the observability estimate (1.4). The well-posedness and the exact controllability of the system (1.2) are both given in Section 4. Finally we state some open problems and further comments in the last section.

2 Identity via multipliers

This section is addressed to establish two fundamental identities by multipliers. Let $f \in L^2(0, T; H_0^2(\Omega))$, we consider the system

$$\begin{cases} i\theta_t + \Delta^2\theta = f, & \text{in } \ \Omega \times (0, T) \\ \theta = 0, \dfrac{\partial\theta}{\partial\nu} = 0, & \text{on } \ \partial\Omega \times (0, T) \\ \theta(x, 0) = \theta^0, & \text{in } \ \Omega. \end{cases} \tag{2.1}$$

First, we show the following one:

Lemma 2.1 *Let $q = q(x, t) \in C^3(\overline{Q}, \mathbb{R}^n)$, with $\overline{Q}$ is the closed set of Q. For every solution* *of* (2.1) *with* $f \in \mathcal{D}(Q)$ *and* $\varphi^0 \in \mathcal{D}(\Omega)$, *the following identity holds:*

$$\begin{aligned} 0 \;=\; & \frac{i}{2}\int_\Omega \theta\nabla\bar\theta\cdot q\Big|_0^T - \frac{i}{2}\int_Q (\theta\nabla\bar\theta_t + \bar\theta\nabla\theta_t)\cdot q - \frac{i}{2}\int_Q \theta\nabla\bar\theta\cdot q_t \\ & -\frac{1}{2}\int_\Sigma \Big|\Delta\theta\Big|^2 q\cdot\nu - \frac{1}{2}\int_Q \Big(\nabla\theta H(\Delta\bar\theta) - \nabla\bar\theta H(\Delta\theta)\Big)\cdot q \\ & +\frac{1}{2}\int_Q \sum_{i,j}\Big(\theta_{x_ix_j}\Delta\bar\theta q^j_{x_i} + \theta_{x_i}\Delta\bar\theta q^j_{x_ix_j} + 3\Delta\theta\bar\theta_{x_ix_j}q^j_{x_i} + 2\Delta\theta\bar\theta_{x_j}q^j_{x_ix_i}\Big) \\ & +\frac{1}{2}\int_Q \Big(\Delta\theta\nabla\bar\theta\cdot\nabla\mathrm{div}_x\, q + \Delta\theta\bar\theta\Delta\mathrm{div}_x\, q\Big) - \int_Q f(\nabla\bar\theta\cdot q + \frac{1}{2}\bar\theta\mathrm{div}_x\, q) \end{aligned} \tag{2.2}$$

where $H(f)$ *is the Hessian Matrix of* f.

Remark 2.1 *For convenience, we drop all* $dx, d\sigma, dt$ *terms in all integrals here and thereafter. More precisely, we write* $\int_\Omega(\cdot)$, $\int_Q(\cdot)$, $\int_\Sigma(\cdot)$ *and* $\int_{\Sigma_0}(\cdot)$ *instead of* $\int_\Omega(\cdot)dx$, $\int_{\Omega\times(0,T)}(\cdot)dxdt$, $\int_{\Gamma\times(0,T)}(\cdot)d\sigma dt$ *and* $\int_{\Gamma_0\times(0,T)}(\cdot)d\sigma dt$, *respectively.*

Proof: *Step 1:* Multiplying (1.3) by $\nabla\bar\theta\cdot q + \frac{1}{2}\bar\theta\mathrm{div}_x\, q$, integrating on Q of the left hand side of (1.3) (abbreviated by ILHS), we get

$$\begin{aligned} \mathrm{ILHS} \;=\; & \frac{i}{2}\int_\Omega \theta\nabla\bar\theta\cdot q\Big|_0^T - \frac{i}{2}\int_Q (\theta\nabla\bar\theta_t + \bar\theta\nabla\theta_t)\cdot q - \frac{1}{2}\int_Q \nabla(\Delta\theta)\bar\theta\cdot\nabla\mathrm{div}_x\, q \\ & -\frac{i}{2}\int_Q \theta\nabla\bar\theta\cdot q_t - \frac{1}{2}\int_\Sigma \Delta\theta H(\bar\theta)q\cdot\nu - \frac{1}{2}\int_Q \Big(\nabla\theta H(\Delta\bar\theta) - \nabla\bar\theta H(\Delta\theta)\Big)\cdot q \\ & +\int_Q \sum_{i,j}\Big(\frac{1}{2}\Delta\theta\bar\theta_{x_ix_j}q^j_{x_i} - \frac{1}{2}\theta_{x_i}\Delta\bar\theta_{x_j}q^j_{x_i} - \Delta\theta_{x_i}\bar\theta_{x_j}q^j_{x_i}\Big). \end{aligned} \tag{2.3}$$

In fact, ILHS $= \int_Q (i\theta_t + \Delta^2\theta)(\nabla\bar\theta\cdot q + \frac{1}{2}\bar\theta\mathrm{div}_x\, q)$ equals to $(A+B)+C+D$ with the notation

$$A+B \;=\; \frac{i}{2}\int_\Omega \theta\nabla\bar\theta\cdot q\Big|_0^T - \frac{i}{2}\int_Q (\theta\nabla\bar\theta_t\cdot q + \bar\theta\nabla\theta_t\cdot q) - \frac{i}{2}\int_Q \theta\nabla\bar\theta\cdot q_t, \tag{2.4}$$

$$C \;=\; \int_Q \Delta^2\theta\nabla\bar\theta\cdot q, \qquad D = \frac{1}{2}\int_Q \Delta^2\theta\bar\theta\mathrm{div}_x\, q. \tag{2.5}$$

(2.6)

Taking into account the boundary conditions, we arrive at

$$C \;=\; -\int_Q \sum_{i,j}\Delta\theta_{x_i}\bar\theta_{x_j}q^j_{x_i} - I, \tag{2.7}$$

$$D \;=\; \frac{1}{2}\int_Q \nabla\bar\theta H(\Delta\theta)q + \frac{1}{2}I - \frac{1}{2}\int_Q \nabla(\Delta\theta)\cdot\bar\theta\nabla\mathrm{div}_x\, q, \tag{2.8}$$

with

$$I=\int_Q \nabla(\Delta\theta)H(\bar\theta)q.$$

Moreover,

$$\begin{aligned} I &= \int_\Sigma \Delta\theta H(\bar\theta)q\cdot\nu-\int_Q \Delta\theta(\nabla(\Delta\bar\theta)\cdot q+\sum_{i,j}\bar\theta_{x_ix_j}q^j_{x_i}) \\ &= \int_\Sigma \Delta\theta H(\bar\theta)q\cdot\nu+\int_Q \nabla\theta H(\Delta\bar\theta)q+\int_Q\sum_{i,j}(\theta_{x_i}\Delta\bar\theta_{x_j}q^j_{x_i}-\Delta\theta\bar\theta_{x_ix_j}q^j_{x_i}). \end{aligned} \tag{2.9}$$

Combining (2.7),(2.8) and (2.9) we get

$$\begin{aligned} C+D &= -\frac{1}{2}\int_\Sigma \Delta\theta H(\bar\theta)q\cdot\nu-\int_Q\sum_{i,j}\Big(\Delta\theta_{x_i}\bar\theta_{x_j}q^j_{x_i}+\frac{1}{2}\theta_{x_i}\Delta\bar\theta_{x_j}q^j_{x_i}-\frac{1}{2}\Delta\theta\bar\theta_{x_ix_j}q^j_{x_i}\Big) \\ &\quad -\frac{1}{2}\int_Q(\nabla\theta H(\Delta\bar\theta)q-\nabla\bar\theta H(\Delta\theta)q)-\frac{1}{2}\int_Q\nabla(\Delta\theta)\bar\theta\cdot\nabla\mathrm{div}_x\, q. \end{aligned} \tag{2.10}$$

Finally, from (2.4) and (2.10) we obtain the desired identity (2.3).

Step 2: Integrating by parts with respect to x, we get

$$\int_Q\sum_{i,j}\theta_{x_i}\Delta\bar\theta_{x_j}q^j_{x_i} = -\int_Q\sum_{i,j}(\theta_{x_ix_j}\Delta\bar\theta q^j_{x_i}+\theta_{x_i}\Delta\bar\theta q^j_{x_ix_j}), \tag{2.11}$$

$$\int_Q\sum_{i,j}\Delta\theta_{x_i}\bar\theta_{x_j}q^j_{x_i} = -\int_Q\sum_{i,j}\Delta\theta(\bar\theta_{x_ix_j}q^j_{x_i}+\bar\theta_{x_j}q^j_{x_ix_i}), \tag{2.12}$$

$$\int_Q\nabla(\Delta\theta)\bar\theta\cdot\nabla\mathrm{div}_x\, q = -\int_Q\Delta\theta\Big(\nabla\bar\theta\cdot\nabla\mathrm{div}_x\, q+\bar\theta\Delta(\mathrm{div}_x\, q)\Big). \tag{2.13}$$

On the other side, since $\theta=0$ and $\frac{\partial\theta}{\partial\nu}=0$ on the boundary Σ, we have $\theta_{x_i}=0, i=1,\cdots,n$ and $\theta_{x_i,x_j}=\frac{\partial\theta_{x_i}}{\partial\nu}\nu_j, i,j=1,\cdots,n$ for any $x\in\Gamma$. Consequently, for any $x\in\Gamma$, it holds

$$\begin{aligned} \sum_{i,j}q_i\theta_{x_ix_j}\nu_j &= \sum_{i,j,k}q_i\theta_{x_j,x_k}\nu_k\nu_i\nu_j=(\sum_i q_i\nu_i)\sum_{j,k}(\theta_{x_j,x_k}\nu_k\nu_j) \\ &= (\sum_i q_i\nu_i)\sum_k\frac{\partial\theta_{x_k}}{\partial\nu}\nu_k=(\sum_i q_i\nu_i)\sum_k\theta_{x_k,x_k}=\Delta\theta(q\cdot\nu). \end{aligned}$$

Hence,

$$\int_\Sigma\Delta\theta H(\bar\theta)q\cdot\nu=\int_\Sigma\Delta\theta\sum_{i,j}q_i\bar\theta_{x_ix_j}\nu_j=\int_\Sigma\Big|\Delta\theta\Big|^2q\cdot\nu. \tag{2.14}$$

Taking (2.11)-(2.14) into (2.3) and putting the right hand side of (2.1) into account, we finish the proof of (2.2). □

The conservation laws hold for the solutions of (1.3)

Lemma 2.2 *For any positive time t, the solution φ of* (1.3) *satisfies*

$$\|\varphi(t)\|_{L^2(\Omega)} = \|\varphi(0)\|_{L^2(\Omega)}\,; \tag{2.15}$$
$$\|\nabla\varphi(t)\|_{L^2(\Omega)} = \|\nabla\varphi(0)\|_{L^2(\Omega)}\,; \tag{2.16}$$
$$\|\Delta\varphi(t)\|_{L^2(\Omega)} = \|\Delta\varphi(0)\|_{L^2(\Omega)}\,. \tag{2.17}$$

Remark 2.2 *Note that in quantum mechanics, the conservation of the norms validates the Born's statistical interpretation of the probability amplitude function $\varphi(x,t)$. More precisely, $\int_\Omega |\varphi(x,t)|^2dx$ represents the probability of finding the particle in domain Ω at time t and the conservation law provides the particle will not disappear in Ω.*

Proof: We use multipliers $\bar\varphi, \Delta\bar\varphi$ and $\bar\varphi_t$ on (1.3) and we achieve the above identities (2.15), (2.16) and (2.17), respectively.

3 Observability

Proposition 3.1 *For every $T>0$, there exist $c_i = c_i(T,\Omega) > 0 (i = 1,2)$ such that*

$$\int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \leq c_1 \left\|\varphi^0\right\|^2_{H_0^2(\Omega)} \tag{3.1}$$

and

$$\left\|\varphi^0\right\|^2_{H_0^2(\Omega)} \leq c_2 \int_0^T\int_{\Gamma_0} |\Delta\varphi|^2 \tag{3.2}$$

for every solution $\varphi = \varphi(x,t)$ of the problem (1.3) *with $\varphi^0 \in H_0^2(\Omega)$.*

Proof: For the admissibility inequality (3.1) we choose $q = q(x) \in C^3(\bar Q, \mathbb{R}^n)$ such that $q = \nu$ on Γ (See Lions [13] for the construction of this vector field), take the real part of the identity (2.2) with $f = 0$, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Sigma |\Delta\varphi|^2 q\cdot\nu \;=\;& -\frac{1}{2}\mathrm{Im}\int_\Omega \varphi\nabla\bar\varphi\cdot q\Big|_0^T + \frac{1}{2}\mathrm{Re}\int_Q \Big(\Delta\varphi\nabla\bar\varphi\cdot\nabla\mathrm{div}_x\, q + \Delta\varphi\bar\varphi\Delta\mathrm{div}_x\, q\Big)\\ &+\frac{1}{2}\int_Q\sum_{i,j}\Big(\varphi_{x_ix_j}\Delta\bar\varphi q^j_{x_i} + \varphi_{x_i}\Delta\bar\varphi q^j_{x_ix_j} + 3\Delta\varphi\bar\varphi_{x_ix_j}q^j_{x_i} + 2\Delta\varphi\bar\varphi_{x_j}q^j_{x_ix_i}\Big)\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{1}{2}\int_\Sigma |\Delta\varphi|^2 \;\leq\;& k_1\|q\|_{L^\infty(\Omega)}(\|\varphi(T)\|^2_{L^2(\Omega)} + \|\nabla\varphi(T)\|^2_{L^2(\Omega)} + \|\varphi(0)\|^2_{L^2(\Omega)} + \|\nabla\varphi(0)\|^2_{L^2(\Omega)})\\ &+k_2\|q\|_{W^{2,\infty}(\Omega)}\int_0^T\Big((\|H(\varphi)\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)})\|\Delta\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)}\Big)\\ &+k_3\|q\|_{W^{3,\infty}(\Omega)}\int_0^T(\|\Delta\varphi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} + \|\Delta\varphi\|_{L^2(\Omega)}\|\varphi\|_{L^2(\Omega)}).\end{aligned}$$

Combining with the conservation law in Lemma 2.2, we obtain

$$\int_0^T \int_{\Gamma_0} |\Delta\varphi|^2 \le c_1 \left\|\varphi^0\right\|^2_{H_0^2(\Omega)}, \qquad \forall\, \varphi^0 \in \mathcal{D}(\Omega).$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^2(\Omega)$, the estimates (3.1) holds for every solution of the problem (1.3) with the initial data $\varphi^0 \in H_0^2(\Omega)$.

Now we prove (1.4). We choose $q(x,t) = m(x) = x - x_0$, using (2.2) we obtain

$$\int_\Sigma m\cdot\nu|\Delta\varphi|^2 = -\mathrm{Im}\int_\Omega \varphi\nabla\bar{\varphi}\cdot m\Big|_0^T + 4T\int_\Omega |\Delta\varphi|^2.$$

Furthermore, there exists a $\varepsilon > 0$ such that

$$\left|\mathrm{Im}\int_\Omega \varphi\nabla\bar{\varphi}\cdot m\Big|_0^T\right| \le c_\varepsilon \left\|\varphi^0\right\|^2_{L^2(\Omega)} + \varepsilon\left\|\varphi^0\right\|^2_{H_0^1(\Omega)}.$$

Thus

$$4T\left\|\varphi^0\right\|^2_{H_0^2(\Omega)} \le C\Big(\int_{\Sigma_0} m\cdot\nu|\Delta\varphi|^2 + c_\varepsilon\left\|\varphi^0\right\|^2_{L^2(\Omega)} + \varepsilon\left\|\varphi^0\right\|^2_{H_0^1(\Omega)}\Big). \tag{3.3}$$

To conclude the proof of (1.4) it is enough to prove the following estimates:

$$\left\|\varphi^0\right\|^2_{L^2(\Omega)} \le C\int_{\Sigma_0} m\cdot\nu|\Delta\varphi|^2, \tag{3.4}$$

$$\left\|\varphi^0\right\|^2_{H_0^1(\Omega)} \le C\int_{\Sigma_0} m\cdot\nu|\Delta\varphi|^2. \tag{3.5}$$

We argue by contradiction. We only state the proof of (3.5) and the one for (3.4) can be obtained directly with the Poincaré inequality. If (3.5) is not satisfied for any $C > 0$, there exists a sequence $\{\varphi_n\}$ of solutions of (1.3) such that

$$\|\varphi_n(0)\|_{H_0^1(\Omega)} = 1, \qquad \forall\, n \in \mathbb{N} \tag{3.6}$$

and

$$\int_{\Sigma_0} m\cdot\nu|\Delta\varphi_n|^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{3.7}$$

Obviously, $\{\varphi_n(0)\}$ is bounded in $H_0^1(\Omega)$ and from (3.3) it is also bounded in $H_0^2(\Omega)$. Then

$$\{\varphi_n\} \text{ is bounded in } L^\infty(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-2}(\Omega)).$$

Thus, by extracting a subsequence (that we will still note by $\{\varphi_n\}$) we will have

- $\varphi_n \to \varphi$ in $L^\infty(0,T;H_0^2(\Omega))$ weak*;
- $(\varphi_n)_t \to \varphi_t$ weakly in $L^\infty(0,T;L^2(\Omega))$ weak*.

The function $\varphi \in L^\infty(0,T;H_0^2(\Omega)\cap W^{1,\infty}(0,T;H^{-2}(\Omega))$ is clearly a solution of (1.3) and, from the compactness of the embedding (see Simon [19])

$$L^\infty(0,T;H_0^2(\Omega)) \cap W^{1,\infty}(0,T;H^{-2}(\Omega)) \to C([0,T];H_0^1(\Omega))$$

and (3.6), we deduce

$$\|\varphi(0)\|_{H_0^1(\Omega)} = 1. \tag{3.8}$$

On the other hand, (3.7) implies

$$\Delta\varphi = 0 \qquad \text{on } \Sigma_0,$$

which, combined with (1.3), implies $\varphi \equiv 0$, from Holmgren's Uniqueness Theorem (see Hörmander [9, Chap. V, Thm. 5.3.3]). This is in contradiction with (3.8). This ends the proof of (3.5).

Taking (3.4) and (3.5) into account, (1.4) is a direct consequence of (3.3). □

4 Well-posedness and exact controllability

We say that $y \in L^\infty(0,T;H^{-2}(\Omega))$ is a solution of (1.2) in the transposition sense if and only if

$$\int_0^T \langle y(t),\bar{f}(t)\rangle_{(H^{-2}(\Omega),H_0^2(\Omega))}dt + i\langle y(0),\bar{\theta}(0)\rangle_{(H^{-2}(\Omega),H_0^2(\Omega))} + \int_\Sigma v\Delta\bar{\theta}d\Sigma = 0 \tag{4.1}$$

for every $f \in L^2(0,T;H_0^2(\Omega))$, where $\theta = \theta(x,t)$ is the solution of the problem (2.1) with $\theta(T) = 0$.

The following proposition claims the existence of a unique solution of system (1.2) in the sense of transposition:

Proposition 4.1 *Let $v \in L^2(\Sigma)$. Then there exists a unique solution $y \in C([0,T];H^{-2}(\Omega))$ in the transposition sense, of the problem* (1.2) *with initial data $y_0 \in H^{-2}(\Omega)$. Furthermore, the map $v \mapsto y$ is linear and continuous from $L^2(\Sigma)$ into $C([0,T];H^{-2}(\Omega))$.*

Proof: Without loss of the generality, we assume that $y_0 = 0$, which is due to the time reversibility of system (1.2). It is not hard to prove that

$$\|\theta(t)\|_{H_0^2(\Omega)} \le \|f\|_{L^1(0,T;H_0^2(\Omega))}, \qquad \forall\, t \in [0,T].$$

Applying the identity (2.2) with a vector field $q = \nu$ on Γ and using the above estimate we obtain

$$\|\Delta\theta\|_{L^2(\Sigma)} \le c\,\|f\|_{L^1(0,T;H_0^2(\Omega))}.$$

Hence, we have

$$\left|\mathrm{Re}\int_\Sigma v\Delta\bar{\theta}d\Sigma\right| \le \|v\|_{L^2(\Sigma)}\,\|\Delta\theta\|_{L^2(\Sigma)} \le c\,\|v\|_{L^2(\Sigma)}\,\|f\|_{L^1(0,T;H_0^2(\Omega))}. \tag{4.2}$$

It means that the map from f into Re $\int_\Sigma v\Delta\bar{\theta}d\Sigma$ is linear and continuous from $L^1(0,T;H_0^2(\Omega))$ into $\mathbb{R}$.

Hence, there exists a unique $y \in L^\infty(0,T;H^{-2}(\Omega))$ that satisfies (4.1) for every $f \in L^1(0,T;H_0^2(\Omega))$.

From (4.1) and (4.2) we have

$$\|y\|_{L^\infty(0,T;H^{-2}(\Omega))} \le c\,\|v\|_{L^2(\Sigma)}\,. \tag{4.3}$$

Thus, the map $v \mapsto y$ is continuous from $L^2(\Sigma)$ into $L^\infty(0,T;H^{-2}(\Omega))$.

Moreover, $y \in C([0,T];H^{-2}(\Omega))$. Indeed, we consider $\{v_n\}_{n\in N} \subset \mathcal{D}(0,T;C^2(\Gamma))$ such that

$$v_n \;\to v \text{ strongly in } \quad L^2(\Sigma). \tag{4.4}$$

Let y_n be the solution of (1.2) with boundary condition v_n. Since v_n is regular, in particular, we have $y_n \in C([0,T];H^{-2}(\Omega))$.

From (4.3) and (4.4), we have

$$y_n \;\to y \;\text{ in } \quad L^\infty(0,T;H^{-2}(\Omega)).$$

Since $C([0,T];H^{-2}(\Omega))$ is a closed subspace of $L^\infty(0,T;H^{-2}(\Omega))$, we have $y \in C([0,T];H^{-2}(\Omega))$. $\square$

Proof of Theorem 1.2: We consider the problem

$$\begin{cases} iy_t + \Delta^2 y = 0, & \text{in } \Omega\times(0,T) \\ y=0,\, \frac{\partial y}{\partial\nu} = v\chi_{\Gamma_0}, & \text{on } \partial\Omega\times(0,T) \\ y(T)=0, & \text{in } \Omega. \end{cases} \tag{4.5}$$

It is easy to see that, by multiplying (4.5) by $\bar{\varphi}$, taking the real part, and integrating it by parts, we have the following identity:

$$\langle -iy(0),\varphi^0\rangle = \int_{\Sigma_0} |\Delta\varphi|^2 d\Sigma \qquad \forall\, \varphi^0 \in \mathcal{D}(\Omega),$$

where φ is the corresponding solution of system (1.3) with initial data φ^0. Let Λ be a linear continuous operator from $H_0^2(\Omega)$ into $H^{-2}(\Omega)$ defined by $\Lambda\varphi^0 = -iy(0)$, where $y = y(x,t)$ is the solution of the problem (4.5).

From Proposition 3.1 we have $\langle\Lambda\varphi^0,\varphi^0\rangle \ge c\,\|\varphi^0\|^2_{H_0^2(\Omega)}$. Hence Λ is an isomorphism from $H_0^2(\Omega)$ to $H^{-2}(\Omega)$ and the theorem is proved. The control v is choosing by $v = \Delta\varphi$ on Σ_0 where φ is the solution of (1.2) with initial data $\varphi^0 = \Lambda^{-1}(-iy(0))$. $\square$

5 Further comments and open problems

1. Transmutation method. We derived Theorem 1.2 by means of multiplier techniques. One could expect a different proof by means of the transmutation method. Roughly speaking, the controllability of system (1.2) can be seen as a combination of the exact controllability of the Schrödinger equation on a segment([14]) and a plate equation ([11]), following the instruction in [15]. However, both methods cannot tell us whether the control domain is sharp. It is still an open problem.

2. Internal controllability. In this paper, we have only dealt with the L^2-Neumann boundary control. On the other hand, one can expect the same result with L^2 controls supported in a neighborhood of the boundary, by following the same methodology in [14]. Furthermore, for the controlled wave equation, the sharp control domain is the one satisfying GCC condition ([1]) instead of the one in (1.1). It is still an open problem whether same happens for system (1.2).

3. Carleman estimate. There are several different methods to derive observability inequalities. The Carleman estimate ([5, 6, 20]) is developed to derive the observability inequalities in a bounded domain with potentials. One may expect to solve the control problem for the fourth order schrödinger with potentials by means of the corresponding global Carleman estimate.

References

[1] Bardos, C., Lebeau, G. and Rauch, J., Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary. *SIAM J. Control Optim.*, **30**, 1992, 1024–1065.

[2] Burq, N. and Zworski, M., Geometric control in the presence of a black box. *J. Amer. Math. Soc.*, **17**, 2004, 443–471 (electronic).

[3] Chou, H. and Guo, Y., Null boundary controllability for a fourth order semilinear equation. *Taiwanese J. Math.*, **10**, 2006, 251–263.

[4] Díaz, J.I. and Ramos, A.M., On the approximate controllability for higher order parabolic nonlinear equations of Cahn-Hilliard type. *Internat. Ser. Numer. Math.*, **126**, 1998, 111–127.

[5] Fu, X., Yong, J. and Zhang, X., Controllability and observability of a heat equation with hyperbolic memory kernel. *J. Differential Equations*, **247**, 2009, 2395–2439.

[6] Fursikov, A. and Imanuvilov, O., Controllability of evolution equations. Lecture Notes Series, 34. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.

[7] Hao, C., Hsiao, L. and Wang, B., Wellposedness for the fourth order nonlinear Schrödinger equations. *J. Math. Anal. Appl.*, **320**, 2006, 246–265.

[8] Hao, C., Hsiao, L. and Wang, B., Well-posedness of Cauchy problem for the fourth order nonlinear Schrödinger equations in multi-dimensional spaces. *J. Math. Anal. Appl.*, **328**, 2007, 58–83.

[9] Hörmander, L., Linear partial differential operators. Springer Verlag, Berlin-New York, 1976.

[10] Lin, P. and Zhou, Z., Observability estimate for a one-dimensional fourth order parabolic equation. Proceedings of the 29th Chinese Control Conference, 2010, 830–832.

[11] Lions, J.L., Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1. *Recherches en Mathématiques Appliquées*, 8. Masson, Paris, 1988, 1–68.

[12] Lions, J.L., Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.*, **30**, 1988, 1–68.

[13] Lions, J.L. and Magenes, E., Problèmes aux Limites Non Homogènes et Applications, Dunod, Paris, 1968.

[14] Machtyngier, E., Exact controllability for the Schrödinger equation. *SIAM J. Control Optim.*, **32**, 1994, 24–34.

[15] Miller, L., Controllability cost of conservative systems: resolvent condition and transmutation. *J. Funct. Anal.*, **218**, 2005, 425–444.

[16] Pausader, B., Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case. *Dyn. Partial Differ. Equ.*, **4**, 2007, 197–225.

[17] Pausader, B., The cubic fourth-order Schrödinger equation. *J. Funct. Anal.*, **256**, 2009, 2473–2517.

[18] Ramdani, K., Takahashi, T., Tenenbaum, G. and Tucsnak, M., A spectral approach for the exact observability of infinite-dimensional systems with skew-adjoint generator.*J. Funct. Anal.*, **226**, 2005, 193–229.

[19] Simon, J. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, **146**, 1987, 65–96.

[20] Zhang, X., Exact controllability of semilinear plate equations. *Asympt. Anal.*, **27**, 2001, 95–125.