31st Youth Academic Annual Conference of Chinese Association of Automation
Wuhan, China; November 11-13, 2016

Controllability of a Simplified Reparable System

Fuli Wei and Chuang Zheng
Department of Mathematics, Beijing Normal University
Beijing, 100875, China
Email: chuang.zheng@bnu.edu.cn

Weiwei Hu
Department of Mathematics, Oklahoma State University
Stillwater, OK, 74078, USA

***Abstract*—In this work we consider the controllability of a coupled PDE-ODE system, which is a simplified model of the multi-state reparable system. We establish an observability inequality of its adjoint system and show that any initial condition of the device can be steered into any quasi steady state by a distributed control, except for a small interval of elapsed repair time near zero.**

***Keywords*—controllability; coupled system; reparable system; observability inequality; quasi steady state; distributed control**

I. Introduction

Reparable systems occur naturally in problems of product design, inventory systems, computer networking and complex manufacturing processes. A reparable system receives maintenance actions that restore/renew system components when they fail. These actions revise the overall function of the system. It is often of considerable interest to improve or optimize system reliability. Reliability is defined as the probability that the system, subsystem or component will operate successfully by a given time t. The mathematical model of coupled transport and integro-differential equations is employed in order to address the reliability characteristics of reparable systems (see [1]–[5]). The application of Markov chain and supplementary variable techniques are used to derive the general mathematical models.

In order to operate and maintain the system efficiently, it is natural to ask if it is possible to manipulate the system through, say, the renewal/replacement policy or incoming quality control so that it achieves the desired reliability in a given time. This question can be formulated as the problems of distributed or boundary controllability. In this paper, we are interested in the controllability of the reparable systems in terms of achieving a given reliability distribution by applying the internal controls. In particular, we consider a simple case of the mathematical model of a reparable multi-state device introduced in [1] with only one failure mode, which is a hybrid system of coupled transport and integral equations (PDE-ODE system):

$$\begin{cases} \dfrac{d}{dt}p_0 = -\alpha p_0 + \displaystyle\int_0^L (\mu p + \chi_0 u)dx, \\ p_t + p_x + \mu p + \chi_0 u = 0, \\ p(0,t) = \alpha p_0(t), \\ p(x,0) = p_1(x),\ p_0(0) = p_0, \end{cases} \tag{1}$$

where $p_0 = p_0(t), p = p(x,t), u = u(x,t), (x,t) \in Q = (0,L) \times (0,T)$ with $0 < L < \infty$, $\chi_0(x)$ is the characteristic function of the interval $(0, a_0)$, $0 < a_0 \leq L$, and $u(x,t)$ is the control input. System (1) describes a simple device which transfer its state between good state 0 and failure mode 1. Here α represents the constant failure rate of the device for failure mode; $\mu(x)$ represents the time-dependent repair rate when the device is in state 1 and has an elapsed repair time of $x \in [0, L]$; $p_0(t)$ represents the probability that the device is in state 0, i.e., the good state, at time t; $p(x,t)$ represents the probability density (with respect to repair time) that the failed device is in state 1 and has an elapsed repair time of x at time t. The control input $\chi_0(x)u$ represents the maintenance action exerted on a given domain $(0, a_0)$ for the failed mode. Note that, while receiving maintenance, the failed mode can transit into the good state. Consequently, $p_0(t)$ satisfies the first equation of (1) and is also influenced by u.

Furthermore, we assume that the repair rate has the following properties

$$\int_0^{l'} \mu_j(x)dx < \infty, \quad \forall l' < L, \quad \int_0^L \mu(x)dx = \infty. \tag{2}$$

The well-posedness and asymptotic properties of system (1) have been thoroughly studied in [6]–[8]. In the current work, we are interested in the controllability of (1) steered by internal controls to a quasi steady state, i.e. a time-independent pair $\big(\bar{p}_0, \bar{p}(x)\big)$ satisfying

$$\begin{cases} \alpha\bar{p}_0 = \displaystyle\int_0^L \big(\bar{p}(x)\mu(x) + \chi_0(x)\omega(x)\big)dx, \\ \bar{p}_x(x) = -\mu(x)\bar{p}(x) - \chi_0(x)\omega(x), \\ \bar{p}(0) = \alpha\bar{p}_0, \end{cases} \tag{3}$$

for some control input function ω. It is known that there are many rich results on the controllability of coupled PDE systems (see, for instance, [9]) as well as on the stability and control design of cascaded PDE-ODE systems ([10]–[12] and the references therein). To our best knowledge, the controllability of (1) has not been addressed.

In this paper, we consider the controllability of the system (1) in Hilbert space X defined by $X = \mathbb{R}^+ \times L^2(0,L)$, with $|\cdot|_X = |\cdot| + |\cdot|_{L^2[0,L]}$. Our main Theorem is stated as follows:

Theorem 1: Let $0 < \delta \leq a_0$ and $L \leq T < L + \delta$. Then for any $P^0 = (p_0, p_1(x)), P^T = (\bar{p}_0, \bar{p}(x))$ given in X, one can find $u^\delta \in L^2(Q)$ such that the solution of (1) satisfies

$$p^{u^\delta}(x,T) = \bar{p}(x), \forall x \in [\delta, L]. \tag{4}$$

Moreover, we have

$$\begin{aligned}&\int_0^T\int_0^{a_0}\big(u^\delta(x,t)-\omega(x)\big)^2dxdt\\&\le\frac{C}{\delta^2}\Big(\int_0^L\Big|p_1(x)-\bar p(x)\Big|^2dx+\big|p_0-\bar p_0\big|\Big).\end{aligned}\tag{5}$$

where C is a constant, independent of δ.

The rest of the pager is organized as follows. In Section II, we present the adjoint system of (1) and give an observability estimate of the corresponding solutions, which plays the most important role in the proof of Theorem 1. In Section III, a variational approach is given and we obtain the controllability result of (1). Finally, we conclude the paper by Section IV.

II. Adjoint System and the Observability

In this section we will focus on the observability inequality of the adjoint system associated with the original controlled system (1), which is of the form

$$\begin{cases}-\dfrac{d}{dt}q_0=-\alpha q_0+\alpha q(0,t),\\-q_t-q_x+\mu q-\mu q_0=0,\\q(L,t)=0,\\Q(x,T):=(q_0(T),q(x,T))=(q_T,q_1(x)),\end{cases}\tag{6}$$

where $q_0=q_0(t),q=q(x,t),\mu=\mu(x),(x,t)\in Q$.

Our first result is based on the observability inequality of (6) for the case $t=L$:

Lemma 2.1: Assume that $T=L$. Let $0<\delta\le a_0$ be sufficiently small. Then there is a constant $C>0$, independent of δ, such that

$$\begin{aligned}&\int_0^L q^2(x,0)dx+q_0^2(0)\\&\le\frac{C}{\delta^2}\Big(\int_0^T\int_0^\delta(q_0(t)-q(x,t))^2dxdt+q_0^2(T)\Big)\\&\quad+C\int_0^\delta q^2(x,T)dx\end{aligned}\tag{7}$$

holds for any solution $(q_0(t),q(x,t))$ of the system (6).

Remark 2.1: Compared to the observability estimate of the first order hyperbolic equation in [13], an extra term $q_0^2(\cdot)$ appears in the both side of the inequality (7). It is reasonable since the solution of the coupled system lives in the functional space X_c with an appropriate norm, instead of the natural $L^2(0,L)$ space in [13].

Proof: The proof will be divided in several steps.

Step 1: Expression of the solutions.

We prefer to solve the problem forwards rather than backwards. Hence, we use the change of variable as follows: Set $q(x,t)=\varphi(L-x,T-t),q_0(t)=\varphi_0(T-t)$. Then (6) reduces to

$$\begin{cases}\dfrac{d\varphi_0}{dt}=-\alpha\varphi_0+\alpha\varphi(L,t),\\\varphi_t+\varphi_x+\mu_0\varphi-\mu_0\varphi_0=0,\\\varphi(0,t)=0,\\\varphi_0(0)=\varphi^0,\varphi(x,0)=\varphi_1(x),\end{cases}\tag{8}$$

where $\varphi_0=\varphi_0(t),\varphi=\varphi(x,t),\mu_0=\mu(L-x),(\varphi^0,\varphi_1(x))=\big(q_0(T),q(L-x,T)\big)$. Moreover, the domain of observer $Q_\delta=(0,\delta)\times(0,T)$ now has the form $Q_1=(L-\delta,L)\times(0,T)$.

Set $\varphi_1(x-t)=0$ for $x<t$, $\pi(x,y)=\exp(-\int_x^y\mu_0(\tau)d\tau)$, for $x\le y,y\in(0,L)$. The solution of (8) can be expressed as

$$\begin{aligned}\varphi(x,t)=&\pi(x-t,x)\varphi_1(x-t)\\&+\int_0^{x\wedge t}\mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds,\end{aligned}\tag{9}$$

and

$$\varphi_0(t)=\varphi_0(0)\exp(-\alpha t)+\int_0^t\alpha\varphi(L,s)\exp(-\alpha(t-s))ds.\tag{10}$$

where $x\wedge t$ denote the smaller number between x and t.

Step 2: We know that

$$\begin{aligned}&\Big|\int_0^t\mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds\Big|\le|\varphi_0(t)|\\&\quad+|\varphi_0(0)|+C\int_0^t(|\varphi_0(s)|+|\varphi(L,s)|)ds.\end{aligned}\tag{11}$$

Indeed, (11) can be obtained from the definition of $\pi(x,y)$ as follows.

$$\begin{aligned}&\int_0^t\mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds\\&=\int_0^t\mu_0(x-s)e^{-\int_0^s\mu_0(x-\tau)d\tau}\varphi_0(t-s)ds\\&=-\exp(-\int_0^s\mu_0(x-\tau)d\tau)\varphi_0(t-s)|_0^t\\&\quad+\int_0^t\frac{d\varphi_0(t-s)}{ds}e^{-\int_0^s\mu_0(x-\tau)d\tau}ds.\end{aligned}\tag{12}$$

Since

$$\frac{d\varphi_0(t-s)}{ds}=\alpha\varphi_0(t-s)-\alpha\varphi(L,t-s),$$

using (12) and the above equation, we obtain

$$\begin{aligned}&\int_0^t\mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds\\=&\varphi_0(t)-\exp(-\int_0^s\mu_0(x-\tau)d\tau)\varphi_0(0)\\&+\int_0^t\alpha\varphi_0(t-s)\exp(-\int_0^s\mu_0(x-\tau)d\tau)ds\\&-\int_0^t\alpha\varphi(L,t-s)\exp(-\int_0^s\mu_0(x-\tau)d\tau)ds.\end{aligned}\tag{13}$$

Taking the absolute value on both sides of (13) gives (11).

Step 3: We now simplify the right hand side of (11).

By the integration of (9) and (10) in Q, using Gronwall's inequality if it is necessary, one can derive that

$$|\varphi_0(t)|\le C\big(|\varphi^0|+\int_0^t|\pi_0(s)\varphi_1(L-s)|ds\big),\tag{14}$$

and

$$\int_0^t |\varphi(L,s)|ds \le C\big(|\varphi^0| + \int_0^t |\pi_0(s)\varphi_1(L-s)|ds\big), \quad (15)$$

where$\pi_0(s) = \pi^2(L-s, L) = \exp(-\int_0^s \mu(\tau)d\tau)$.

Integrating (14) on $s \in (0,t)$ yields

$$\begin{aligned}&\int_0^t |\varphi_0(s)|ds\\ &\le C\Big(t|\varphi_0(0)| + \int_0^t\int_0^s |\pi_0(\tau)\varphi_1(L-\tau)|d\tau ds\Big)\\ &\le C\Big(t|\varphi_0(0)| + \int_0^t\int_0^t |\pi_0(\tau)\varphi_1(L-\tau)|d\tau ds\Big)\\ &\le C\Big(|\varphi_0(0)| + \int_0^t |\pi_0(s)\varphi_1(L-s)|ds\Big).\end{aligned} \quad (16)$$

Taking (14)$-$(16) into (11), we derive

$$\begin{aligned}&\big|\int_0^t \mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds\big|\\ \le& C\big(|\varphi_0(0)| + \int_0^t |\pi_0(s)\varphi_1(L-s)|ds\big).\end{aligned} \quad (17)$$

Step 4: Estimate on the term $\pi(x-t,x)\varphi_1(x-t)$.

In this step we will prove the following estimate:

$$\begin{aligned}\pi^2(x-t,x)\varphi_1^2(x-t) \le C\Big((\varphi(x,t) - \varphi_0(t))^2\\ + |\varphi^0|^2 + \int_0^t \pi_0^2(s)\varphi_1^2(L-s)ds\Big),\end{aligned} \quad (18)$$

where C is a constant independent of δ.

In fact, by (9) and (10), since $\varphi_1(x-t) = 0$ for $x < t$, it holds

$$\begin{aligned}&\pi(x-t,x)\varphi_1(x-t)\\ &= \varphi(x,t) + e^{-\alpha t}\varphi^0 + \int_0^t \alpha\varphi(L,s)e^{-\alpha(t-s)}ds\\ &- \varphi_0(t) - \int_0^t \mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds\end{aligned} \quad (19)$$

for $x > t$. Using (17) and Hölder inequality, we get

$$\begin{aligned}&\big|\int_0^t \mu_0(x-s)\pi(x-s,x)\varphi_0(t-s)ds\big|^2\\ \le& C\big(\varphi_0^2(0) + \int_0^t \pi_0^2(s)\varphi_1^2(L-s)ds\big).\end{aligned} \quad (20)$$

According to (15) and Hölder inequality, we have

$$\big(\int_0^t |\varphi(L,s)|ds\big)^2 \le C\big(\varphi_0^2(0) + \int_0^t \pi_0^2(s)\varphi_1^2(L-s)ds\big). \quad (21)$$

Using (19)$-$ (21), by Hölder inequality yields (18).

Step 5: Now we show the estimate on $\varphi(x,T)$ and $\varphi_0(T)$.

We first prove that for $\varphi(x,T)$ it holds

$$\begin{aligned}&\int_0^L \varphi^2(x,T)dx\\ \le& \frac{C}{\delta^2}\int_0^T\int_{L-\delta}^L \big((\varphi(x,t) - \varphi_0(t))^2 + \varphi_0^2(0)\big)dxdt\\ &+ C\int_{L-\epsilon}^L \pi^2(s,L)\varphi_1^2(s)ds.\end{aligned} \quad (22)$$

Indeed, following the same procedure as in [13], we get

$$\begin{aligned}&\int_0^L \pi^2(s,L)\varphi_1^2(s)ds\\ \le& \frac{C}{\epsilon^2}\Big(\int_0^T\int_{L-\delta}^L \big((\varphi(x,t) - \varphi_0(t))^2 + \varphi_0^2(0)\big)dxdt\\ &+ \epsilon^2\int_{L-\epsilon}^L \pi^2(s,L)\varphi_1^2(s)ds\Big),\end{aligned} \quad (23)$$

where C here is a new constant independent of δ, $\frac{\delta}{2} < \epsilon \le \delta$. From (23), we derive

$$\begin{aligned}&\int_0^L \pi_0^2(s)\varphi_1^2(L-s)ds\\ =& \int_0^L \pi^2(s,L)\varphi_1^2(s)ds\\ \le& \frac{C}{\epsilon^2}\Big(\int_0^T\int_{L-\delta}^L \big((\varphi(x,t) - \varphi_0(t))^2 + \varphi_0^2(0)\big)dxdt\\ &+ \epsilon^2\int_{L-\epsilon}^L \pi^2(s,L)\varphi_1^2(s)ds\Big).\end{aligned} \quad (24)$$

On the other hand, the solution of (8) satisfies

$$\int_0^L \varphi^2(x,T)dx \le C\big(\varphi_0^2(0) + \int_0^T \pi_0^2(s)\varphi_1^2(L-s)ds\big) \quad (25)$$

From (24) and (25), we have

$$\begin{aligned}&\int_0^L \varphi^2(x,T)dx\\ \le& C\Big(\varphi_0^2(0) + \frac{C}{\epsilon^2}\big(\int_0^T\int_{L-\delta}^L \big((\varphi(x,t) - \varphi_0(t))^2\\ &+ \varphi_0^2(0)\big)dxdt + \epsilon^2\int_{L-\epsilon}^L \pi^2(s,L)\varphi_1^2(s)ds\big)\Big)\\ \le& \frac{C}{\epsilon^2}\int_0^T\int_{L-\delta}^L \big((\varphi(x,t) - \varphi_0(t))^2 + \varphi_0^2(0)\big)dxdt\\ &+ C\int_{L-\delta}^L \pi^2(s,L)\varphi_1^2(s)ds.\end{aligned} \quad (26)$$

Moreover, since $\frac{\delta}{2} < \epsilon \le \delta$, we have

$$\begin{aligned}&\int_0^L \varphi^2(x,T)dx\\ \le& \frac{C}{\delta^2}\int_0^T\int_{L-\delta}^L \big((\varphi(x,t) - \varphi_0(t))^2 + \varphi_0^2(0)\big)dxdt\\ &+ C\int_{L-\delta}^L \pi^2(s,L)\varphi_1^2(s)ds,\end{aligned} \quad (27)$$

which obviously equals to (22).

Estimate of $\varphi_0^2(T)$: From the estimate of $\varphi_0(t)$ in (14), we derive

$$\varphi_0^2(t) \le C\big(\varphi_0^2(0) + \int_0^t \pi_0^2(s)\varphi_1^2(L-s)ds\big),$$

Hence,

$$\varphi_0^2(T) \le C\big(\varphi_0^2(0) + \int_0^T \pi_0^2(s)\varphi_1^2(L-s)ds\big). \tag{28}$$

Since (28), and $T = L$, we get

$$\varphi_0^2(T) \le C\big(\varphi_0^2(0) + \int_0^L \pi_0^2(s)\varphi_1^2(L-s)ds\big). \tag{29}$$

Combining (24) and (29), we obtain

$$\begin{aligned}\varphi_0^2(T) \le & \frac{C}{\epsilon^2}\int_0^T\int_{L-\delta}^L \big((\varphi(x,t)-\varphi_0(t))^2+\varphi_0^2(0)\big)dxdt \\ & + C\int_{L-\delta}^L \pi^2(s,L)\varphi_1^2(s)ds + C\varphi_0^2(0) \\ \le & \frac{C}{\epsilon^2}\int_0^T\int_{L-\delta}^L \big((\varphi(x,t)-\varphi_0(t))^2+\varphi_0^2(0)\big)dxdt \\ & + C\int_{L-\delta}^L \pi^2(s,L)\varphi_1^2(s)ds.\end{aligned} \tag{30}$$

Since $\frac{\delta}{2} < \epsilon \le \delta$, then

$$\begin{aligned}\varphi_0^2(T) \le & \frac{C}{\delta^2}\int_0^T\int_{L-\delta}^L \big((\varphi(x,t)-\varphi_0(t))^2+\varphi_0^2(0)\big)dxdt \\ & + C\int_{L-\delta}^L \pi^2(s,L)\varphi_1^2(s)ds.\end{aligned} \tag{31}$$

Step 6: Finally, by the change of variables, we obtain

$$\begin{aligned}\int_0^L q^2(x,0)dx \le & \frac{C}{\delta^2}\Big(\int_0^T\int_0^\delta \big(q(x,t)-q_0(t)\big)^2dxdt \\ & + q_0^2(T)\Big) + C\int_0^\delta q^2(x,T)dx,\end{aligned} \tag{32}$$

and

$$\begin{aligned}q_0^2(0) \le & \frac{C}{\delta^2}\Big(\int_0^T\int_0^\delta \big(q(x,t)-q_0(t)\big)^2dxdt + q_0^2(T)\Big) \\ & + C\int_0^\delta q^2(x,T)dx,\end{aligned} \tag{33}$$

where C is a constant independent of δ.

Summarizing (32) and (33), we get the desired inequality (7) and completes the proof of Lemma 2.1. ∎

III. Internal Control

This section is devoted to the proof of Theorem 1. Thus, let $0 < \delta \le a_0$ be arbitrary but fixed and let $P^T(x) = \big(\bar{p}_0, \bar{p}(x)\big)$ be any pair satisfying (3). We first prove the result for $T = L$. To this purpose we consider the optimal control problem: Minimize

$$\begin{aligned}J_\lambda(u) = & \int_0^T\int_0^{a_0}(u-\omega)^2dxdt + \frac{1}{\lambda}\int_\delta^L (p(x,T) \\ & -\bar{p}(x))^2dx + \int_0^\delta (p(x,T)-\bar{p}(x))^2dx \\ & + (p_0(T)-\bar{p}_0)^2\end{aligned} \tag{34}$$

subjected to the state system (1). For each $\lambda > 0$, problem (34) has a unique solution $u_\lambda \in L^2(Q)$ satisfying

$$u_\lambda(x,t) - \omega(x) = -(q_{0\lambda}(t) - q_\lambda(x,t)), \tag{35}$$

where $q_{0\lambda}(t), q_\lambda(x,t)$ are the solution to the dual system (6) with the Cauchy final condition

$$\begin{aligned}& q_\lambda(x,T) = \big(\frac{1}{\lambda}\chi(x) + 1 - \chi(x)\big)\big(p_\lambda(x,T)-\bar{p}(x)\big), \\ & q_{0\lambda}(T) = p_{0\lambda}(T) - \bar{p}_0,\end{aligned} \tag{36}$$

where $\chi(x)$ is the characteristic function of the interval (δ, L) and $\big(p_{0\lambda}(t), p_\lambda(x,t)\big)$ denotes the solutions of (1) corresponding to the control $u_\lambda(x,t)$.

By subtracting (3) into (1) with $p_0(t) = p_{0\lambda}(t), p(x,t) = p_\lambda(x,t)$ and letting $l_\lambda(x,t) = p_\lambda(x,t) - \bar{p}(x), l_{0\lambda}(t) = p_{0\lambda}(t) - \bar{p}_0$, we have that

$$\begin{cases}\frac{d}{dt}l_{0\lambda} = -\alpha l_{0\lambda} + \int_0^L \big(\mu l_\lambda + \chi_0(u_\lambda-\omega)\big)dx, \\ l_{\lambda,t} + l_{\lambda,x} = -\mu l_\lambda - \chi_0\big(u_\lambda - \omega\big), \\ l_\lambda(0,t) = \alpha l_{0\lambda}, \\ l_{0\lambda}(0) = p^0 - \bar{p}_0, l_\lambda(x,0) = p_1(x) - \bar{p}(x).\end{cases} \tag{37}$$

Multiplying $(q_0(t), q(x,t))$ the solutions of the adjoint system (6) on (37), integrating on $(0,T)$ and $(0,L)\times(0,T)$ respectively, taking into accounts those boundary conditions, we finally get

$$\begin{aligned}& \int_0^L l_\lambda(x,0)q(x,0)dx + l_{0\lambda}(0)q_0(0) \\ = & -\int_0^T\int_0^L \chi_0(u_\lambda-\omega)\big(q_0-q\big)dxdt \\ & + \int_0^L l_\lambda(x,T)q(x,T)dx + l_{0\lambda}(T)q_0(T).\end{aligned} \tag{38}$$

We choose the initial data of (6) as the form of (36) and denote the corresponding solution by $(q_{0\lambda}, q_\lambda)$. Then we have

$$\begin{aligned}& \int_0^L l_\lambda(x,0)q_\lambda(x,0)dx + l_{0\lambda}(0)q_{0\lambda}(0) \\ = & \int_0^T\int_0^L \chi_0(x)\big(q_{0\lambda}(t)-q_\lambda(x,t)\big)^2dxdt \\ & + \int_0^L l_\lambda(x,T)q_\lambda(x,T)dx + l_{0\lambda}(T)q_{0\lambda}(T).\end{aligned} \tag{39}$$

The above inequality implies that

$$\begin{aligned} &\int_0^L (p_1(x)-\bar{p}(x))q_\lambda(x,0)dx + (p_0-\bar{p}_0)q_{0\lambda}(0) \\ &= \int_0^T\int_0^L \chi_0(x)\big(q_{0\lambda}(t)-q_\lambda(x,t)\big)^2 dxdt \\ &+ \frac{1}{\lambda}\int_\delta^L \Big(p_\lambda(x,T)-\bar{p}(x)\Big)^2 dx \\ &+ \int_0^\delta \Big(p_\lambda(x,T)-\bar{p}(x)\Big)^2 dx + \Big(p_{0\lambda}(T)-\bar{p}_0\Big)^2. \end{aligned} \tag{40}$$

By the observability inequality of *Lemma 2.1* and Schwartz inequality, we have

$$\begin{aligned} &\text{LHS of (40)} \\ \le&\Big(\int_0^L (p_1(x)-\bar{p}(x))^2dx\Big)^{\frac{1}{2}}\Big(\int_0^L (q_\lambda(x,0))^2dx\Big)^{\frac{1}{2}} \\ &+|p_0-\bar{p}_0|\big(q_{0\lambda}^2(0)\big)^{\frac{1}{2}} \\ \le&\frac{C}{\delta}\Big(\Big|p_1-\bar{p}\Big|_{L^2(0,L)}+|p_0-\bar{p}_0|\Big)\Big(\big(\int_0^T\int_0^\delta (q_{0\lambda}(t) \\ &-q_\lambda(x,t))^2dxdt+q_{0\lambda}^2(T)\big)+\int_0^\delta q_\lambda^2(x,T)dx\Big)^{\frac{1}{2}}. \end{aligned}$$

Consequently,

$$\begin{aligned} &\text{LHS of (40)} \\ \le&\frac{C^2}{2\delta^2}\Big(\Big|p_1-\bar{p}\Big|_{L^2(0,L)}+|p_0-\bar{p}_0|\Big)^2 \\ &+\frac{1}{2}\int_0^T\int_0^\delta (q_{0\lambda}(t)-q_\lambda(x,t))^2dxdt \\ &+\frac{1}{2}\Big(p_{0\lambda}(T)-\bar{p}_0\Big)^2+\frac{1}{2}\int_0^\delta\Big(p_\lambda(x,T)-\bar{p}(x)\Big)^2dx. \end{aligned}$$

Since $\delta \le a_0$, above inequality and (40) means that

$$\begin{aligned} &\int_0^T\int_0^{a_0}\big(q_{0\lambda}(t)-q_\lambda(x,t)\big)^2dxdt \\ &+\frac{1}{\lambda}\int_\delta^L\Big(p_\lambda(x,T)-\bar{p}(x)\Big)^2dx \\ &+\int_0^\delta\Big(p_\lambda(x,T)-\bar{p}(x)\Big)^2dx+\Big(p_{0\lambda}(T)-\bar{p}_0\Big)^2 \\ \le&\frac{C}{\delta^2}\Big(\Big|p_1-\bar{p}\Big|_{L^2(0,L)}+|p_0-\bar{p}_0|\Big)^2, \end{aligned} \tag{41}$$

where C is a constant independent of δ and λ.

The First consequence of (41) is that the sequence $\{\chi_0 u_\lambda\}$ is bounded in $L^2(Q)$. Moreover,

$$\lim_{\lambda\mapsto 0}\int_\delta^L\Big(p_\lambda(x,T)-\bar{p}(x)\Big)^2dx=0. \tag{42}$$

On the other hand, we can get the following estimate:

$$\| p_{0\lambda}\|_{L^2(0,T)}\le C_0, \| p_\lambda\|_{L^2(Q)}\le C_0, \tag{43}$$

where C_0 is a constant. By (42) and (43) we may conclude that there exists a subsequence such that

$$\begin{aligned} &p_{0\lambda}\mapsto p_0^\delta \quad \text{weakly} \quad \text{in} \quad L^2(0,T) \\ &p_\lambda\mapsto p^\delta, \quad \text{weakly} \quad \text{in} \quad L^2(Q) \\ &\chi_0 u_\lambda\mapsto \chi_0 u^\delta \quad \text{weakly} \quad \text{in} \quad L^2(Q) \\ &p_\lambda(x,T)\mapsto p^\delta(x,T), \text{with } p^\delta(x,T)=\bar{p}(x),\ \forall x\ge\delta. \end{aligned}$$

By the weak formulation of the problem (1), it is easy to show that $p_0^\delta=p_0^{u^\delta}, p^\delta=p^{u^\delta}$. This completes the proof for the case $T=L$.

Next we extend the result for any time $L<T<L+\delta$. In fact, if u^* is the control such that

$$p^{u^*}(x,L)=\bar{p}(x), \quad \forall x\in(\bar{\delta},L),$$

for some $\bar{\delta}\in(0,\delta)$. Define

$$u(x,t)=\begin{cases} u^*(x,t), & x\in(0,L), t\in(0,L), \\ \omega(x), & x\in(0,L), t\in(L,T), \end{cases} \tag{44}$$

where $L<T<L+\delta$. It is easy to see that for $x\in(0,L)$,

$$\begin{aligned} &p_0^u(t)=p_0^{u^*}(t), \quad p^u(x,t)=p^{u^*}(x,t), \quad t\in(0,L) \\ &p^u(x,t)-\bar{p}(x)=v(x,t-L), \quad t\in(L,T), \end{aligned} \tag{45}$$

where $v(x,t)$ is the solution of the problem

$$\begin{cases} v_t+v_x=-\mu v, & (x,t)\in(0,L)\times(0,\delta). \\ v(x,0)=p^{u^*}(x,T)-\bar{p}(x), & x\in(0,L). \end{cases} \tag{46}$$

Integrating along the characteristic lines, we obtain the solution of (46):

$$v(x,t)=\Big(p^{u^*}(x-t,L)-\bar{p}(x-t)\Big)\exp\big(-\int_0^t\mu(x-\tau)d\tau\big),$$

with $(x,t)\in(0,L)\times(0,\delta)$, and

$$\begin{aligned} v(x,t-L)=&\Big(p^{u^*}(x-t+L,L)-\bar{p}(x-t+L)\Big) \\ &\times\exp\big(-\int_0^{t-L}\mu(x-\tau)d\tau\big), \end{aligned} \tag{47}$$

with $(x,t)\in(0,L)\times(L,L+\delta)$.

If $x-T+L>\bar{\delta}$, then

$$p^{u^*}(x-T+L,L)-\bar{p}(x-T+L)=0. \tag{48}$$

From (45), (47), and (48), we obtain

$$p^u(x,T)-\bar{p}(x)=0, \quad x-T+L>\bar{\delta},$$

or even more so for $x>\delta$. Therefore, with the control u defined in (44), we have $p^u(x,T)=\bar{p}(x)$ for $x\in[\delta,L]$ if $\bar{\delta}\le\delta-T+L$.

We note that from estimate (41) and equality (35), we see that the estimate

$$\begin{aligned} &\int_0^T\int_0^{a_0}\big(u_\lambda(x,t)-\omega(x)\big)^2dxdt \\ \le&\frac{C}{\delta^2}\Big(\Big|p_1-\bar{p}\Big|_{L^2(0,L)}+|p_0-\bar{p}_0|\Big)^2 \end{aligned} \tag{49}$$

is valid, so that, letting $\lambda \to 0$, we get the estimate for the control u^δ,

$$\begin{aligned}&\int_0^T \int_0^{a_0} \left(u^\delta(x,t) - \omega(x)\right)^2 dxdt \\ \leq &\frac{C}{\delta^2}\left(\left|p_1 - \bar{p}\right|_{L^2(0,L)} + |p_0 - \bar{p}_0|\right)^2,\end{aligned}$$

where C is a constant, independent of $\delta, P^0(x) = \left(p_0, p_1(x)\right)$ and $P^T(x) = \left(\bar{p}_0, \bar{p}(x)\right)$.

Thus, the proof of Theorem 1 is complete.

IV. Conclusion

In this paper, we consider the controllability of a simplified multi-state reparable system. We prove that the whole energy of the solution of its adjoint system can be observed from the partial measurements of the solution. Following the standard procedure in [14], we address the observability of the adjoint system in order to obtain the controllability of its original system. Finally, we show that for any given initial condition, the device can be steered into any quasi steady state by a distributed control, except for a small interval of elapsed repair time near zero. A L^2-bound of the control is also given.

References

[1] W. K. Chung, "A reparable multistate device with arbitrarily distributed repairtimes," *Microelectron. Reliab.*, vol. 21, no. 2, pp. 255–256, 1981.

[2] W. K. Chung, "Reliability of imperfect switching of cold stanby systems with multiple non-critical and critical errors," *Microelectron. Reliab.*, vol. 35, no. 12, pp. 1479–1482, 1995.

[3] P. P. Gupta and S. C. Agarwal, "Cost function analysis of a 3-state reparable system," *Microelectron. Reliab.*, vol. 24, no. 1,pp. 51–53, 1984.

[4] A. Haji and A. Radl, "A semigroup approach to the Gnedenko system with single vacation of a repairman," *Semigroup Forum*, vol. 86, pp. 41–58, 2013.

[5] W. Li and J. H. Cao, "Some performance measures of transfer line consisting of two unreliable machines with reprocess rule," *J. Syst. Sci. Syst. Eng.*, vol. 7, no. 3, pp. 283–292, 1998.

[6] H. B. Xu, J. Y. Yu, and G. T. Zhu, "Asymptotic property of a reparable multi-state device," *Q. Appl. Math.*, vol. 63, no. 4, pp. 779–789, 2005.

[7] W. Hu, H. B. Xu, J. Y. Yu, and G. T. Zhu, "Exponential stability of a reparable multi-state device," *J. Syst. Sci. Complex.*, vol. 20, no. 3, pp. 437–443, 2007.

[8] W. Hu, "Differentiability and compactness of the c_0-semigroup generated by the reparable system with finite repair time," *J. Math. Anal. Appl.*, vol. 433, no. 2, pp. 1614–1625, 2016.

[9] X. Zhang and E. Zuazua, "Polynomial decay and control of a $1d$ hyperbolic-parabolic coupled system," *J. Differ. Equ.*, vol. 204, no. 2, pp. 380–438, 2004.

[10] Z. Zhou and C. Xu, "Stabilization of a second order ODE-heat system coupling at intermediate point," *Automatica J. IFAC.*, vol. 60, pp. 57–64, 2015.

[11] S. Tang and C. Xie, "State and output feedback boundary control for a coupled PDE-ODE system," *Syst. Control Lett.*, vol. 60, no. 8, pp. 540–545, 2011.

[12] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Control Lett.*, vol. 57, pp. 750–758, 2008.

[13] V. Barbu, M. Iannelli, and M Martcheva, "On the controllability of the Lotka-McKendrick model of population dynamics," *J. Math. Anal. Appl.*, vol. 253, no. 1, pp. 142–165, 2001.

[14] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*. New York: Springer-Verlag Berlin Heidelberg, 1971.