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Explicit representation of invariant measures for 2-death processes

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ABSTRACT

Construction of invariant measures is a challenging problem for Markov chains with multi-exit. In this paper, we obtain explicit representation of invariant measures for 2-death processes. And furthermore, the criterion for ergodicity is also derived based on this formula. In addition, several examples are demonstrated to verify the validity of our result.

1. Introduction and main result

Consider a continuous-time and homogeneous Markov chain $\{X(t) : t \geqslant 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with the transition probability matrix $P(t) = (p_{ij}(t))$ on the countable state space $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. In this paper we assume that the considered transition rate Q-matrix $Q = (q_{ij})$ is totally stable and conservative, which means $q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ for all $i \in \mathbb{Z}_+$.

Given a Q-matrix, if there exists such a positive integer m that for all $i \geqslant 0$,

$$q_{i,i+m} > 0, \quad q_{ij} = 0, \quad j > i + m,$$

we call it a Q-matrix with uniformly finite range upward; symmetrically, if for all $i \geqslant m$,

$$q_{i,i-m} > 0, \quad q_{ij} = 0, \quad 0 \leqslant j < i - m,$$

it is called a Q-matrix with uniformly finite range downward. In particular, in the case of $m = 1$, the Q-matrix is also called single birth Q-matrix and single death one, or skip-free upwardly one and skip-free downwardly one, respectively. For simplicity, we call them m-birth Q-matrix and m-death one respectively. For the systematic results of single birth processes and single death ones, refer to Chen (2004, 1999), Chen and Zhang (2014), Mao and Zhang (2004), Zhang (2001, 2018), Zhang and Zhou (2019), Mao et al. (2022). In this paper, we focus on the $m = 2$ case, namely 2-death processes. Until now, there is still a big gap except the criteria for recurrence and uniqueness of 2-death processes. One can see Zhang (2023) for reference. In Li and Li (2021) they consider the down/up crossing property of weighted Markov collision processes which are of 2-death in fact.

In the past decade years, due to the advantage of single exit, single birth processes are always regarded as the largest class of Markov chains which are expected to obtain explicit criteria in various aspects. Besides, although m-death processes maybe of infinite exit and the existence of invariant measures $(\mu_n)_{n\geqslant 0}$ and stationary distributions are difficult to deal with, but the answer for single death processes has been obtained. In Wang and Zhang (2014) and Zhang (2016), the recursive formula and direct representation for $(\mu_n)_{n\geqslant 0}$ are obtained respectively. However, for 2-death processes, this question has been unsolved for a long time since the dimension

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of the solutions to the equation $\mu Q = 0$ is larger than one, which is different for single death processes which could determine $(\mu_n)_{n\geqslant 0}$ explicitly together with the positive property for $(\mu_n)_{n\geqslant 0}$. Fortunately, we obtain the answer recently by investigating the coefficients of formal representation for $(\mu_n)_{n\geqslant 0}$.

Now consider a 2-death Q-matrix $Q = (q_{ij})$ which means that for all $i \geqslant 2$,

$$q_{i,i-2} > 0, \quad q_{ij} = 0, \quad 0 \leqslant j < i-2.$$

Let

$$q_i^{(\ell)} = \sum_{k\geqslant \ell} q_{ik}, \quad \ell \geqslant i \geqslant 0.$$

By the conservativeness of the Q-matrix, $q_0^{(0)} = 0$.

Let

$$G_n^{(n)} = 1, \quad n \geqslant 1; \quad G_i^{(n)} = \frac{1}{q_{i,i-2}} \sum_{k=i+1}^{n} q_i^{(k-1)} G_k^{(n)}, \quad 2 \leqslant i < n.$$

Define

$$G_0^{(n)} = \sum_{\ell=2}^{n} q_0^{(\ell-1)} G_\ell^{(n)}, \quad G_1^{(n)} = \sum_{\ell=2}^{n} q_1^{(\ell-1)} G_\ell^{(n)}, \quad n \geqslant 2. \tag{1.1}$$

In convention that $q_{1,-1} = q_{0,-2} = 1$. Note that $q_0^{(0)} = 0$. Then we can extend the definitions of $\{G_i^{(n)}\}$ for all $0 \leqslant i \leqslant n$ as

$$G_n^{(n)} = 1, \quad n \geqslant 0; \quad G_k^{(n)} = \frac{1}{q_{k,k-2}} \sum_{k+1\leqslant \ell\leqslant n} q_k^{(\ell-1)} G_\ell^{(n)}, \quad 0 \leqslant k < n.$$

Equivalently, it is not difficult to check by Lemma 2.1 that

$$G_k^{(n)} = \sum_{i=k}^{n-1} \frac{G_k^{(i)} q_i^{(n-1)}}{q_{i,i-2}}, \quad 0 \leqslant k < n.$$

Now we first state our main result.

Theorem 1.1. *For an irreducible and regular 2-death Q-matrix, the invariant measures of the corresponding process can be represented as $\mu_0 = 1$ and*

$$\mu_n = \frac{1}{q_{n,n-2}} \left(G_0^{(n)} + \mu_1 G_1^{(n)} \right), \quad n \geqslant 2,$$

where

$$\mu_1 \in \left[-\inf_{m\geqslant 1} \left(\frac{G_0^{(2m+1)}}{G_1^{(2m+1)}} \right), -\sup_{m\geqslant 1} \left(\frac{G_0^{(2\ m)}}{G_1^{(2\ m)}} \mathbb{1}_{\{G_1^{(2\ m)} \neq 0\}} \right) \right]. \tag{1.2}$$

Here the constant multiple factor is not counted. Moreover, assume that the process is recurrent. Then it is ergodic if and only if

$$\inf_{m\geqslant 1} \left(\frac{G_0^{(2m+1)}}{G_1^{(2m+1)}} \right) = \sup_{m\geqslant 1} \left(\frac{G_0^{(2\ m)}}{G_1^{(2\ m)}} \mathbb{1}_{\{G_1^{(2\ m)} \neq 0\}} \right) \quad \text{and} \quad \mu := \sum_{n=0}^{\infty} \mu_n < +\infty.$$

Meanwhile the stationary distribution of the process is $\pi_n = \frac{\mu_n}{\mu}$ for all $n \geqslant 0$.

The paper is organized as follows. Section 2 contains some preliminaries which are key technical tools used throughout the rest of the article. Sections 3 is devoted to the proof of our main result. In addition, several examples are given to illustrate the validity of our result.

2. Preliminaries

In this section we prove two lemmas. The first lemma is presented as follows, in which some notations are partly modified from Zhang (2023).

Lemma 2.1. *Define*

$$g_n^{(n)} = 1, \quad n \geqslant 1; \quad g_{i-1}^{(n)} = \frac{1}{q_{i,i-2}} \sum_{k=i}^{n} q_i^{(k)} g_k^{(n)}, \quad 2 \leqslant i \leqslant n.$$

The following relation holds:

$$g_\ell^{(n)} = \sum_{i=\ell+1}^{n} \frac{g_\ell^{(i-1)} q_i^{(n)}}{q_{i,i-2}}, \quad 1 \leqslant \ell < n, \tag{2.1}$$

and

$$g_{i-1}^{(n)} = G_i^{(n+1)}, \quad 2 \leqslant i \leqslant n+1, \tag{2.2}$$

i.e.,

$$G_\ell^{(n)} = \sum_{i=\ell}^{n-1} \frac{G_\ell^{(i)} q_i^{(n-1)}}{q_{i,i-2}}, \quad 2 \leqslant \ell < n.$$

Proof. In fact,

$$g_{n-1}^{(n)} = \frac{1}{q_{n,n-2}} \sum_{k=n}^{n} q_n^{(k)} g_k^{(n)} = \frac{q_n^{(n)}}{q_{n,n-2}} = \sum_{i=n}^{n} \frac{g_{n-1}^{(i-1)} q_i^{(n)}}{q_{i,i-2}}.$$

Assume that Eq. (2.1) holds up to i. Now

$$\begin{aligned} g_{i-1}^{(n)} &= \frac{1}{q_{i,i-2}} \sum_{i\leqslant k\leqslant n} q_i^{(k)} g_k^{(n)} = \frac{1}{q_{i,i-2}} \left(q_i^{(n)} + \sum_{i\leqslant k\leqslant n-1} q_i^{(k)} g_k^{(n)} \right) \\ &= \frac{q_i^{(n)}}{q_{i,i-2}} + \frac{1}{q_{i,i-2}} \sum_{k=i}^{n-1} q_i^{(k)} \sum_{\ell=k+1}^{n} \frac{g_k^{(\ell-1)} q_\ell^{(n)}}{q_{\ell,\ell-2}} \\ &= \frac{q_i^{(n)}}{q_{i,i-2}} + \sum_{\ell=i+1}^{n} \frac{q_\ell^{(n)}}{q_{\ell,\ell-2}} \cdot \frac{1}{q_{i,i-2}} \sum_{k=i}^{\ell-1} q_i^{(k)} g_k^{(\ell-1)} \\ &= \frac{q_i^{(n)}}{q_{i,i-2}} + \sum_{\ell=i+1}^{n} \frac{g_{i-1}^{(\ell-1)} q_\ell^{(n)}}{q_{\ell,\ell-2}} \\ &= \sum_{\ell=i}^{n} \frac{g_{i-1}^{(\ell-1)} q_\ell^{(n)}}{q_{\ell,\ell-2}}, \end{aligned}$$

which means that Eq. (2.1) holds in the case of $\ell = i-1$. So the first assertion is checked. Then we turn to the second assertion to show that

$$g_{i-1}^{(n)} = G_i^{(n+1)}, \quad 2 \leqslant i \leqslant n+1.$$

The proof is also by induction on i with $i \leqslant n$. Indeed, $g_n^{(n)} = 1 = G_{n+1}^{(n+1)}$. Assume that the relation (2.2) holds until $i+1 \leqslant n+1$. Then it follows that

$$\begin{aligned} g_{i-1}^{(n)} &= \frac{1}{q_{i,i-2}} \sum_{k=i}^{n} q_i^{(k)} g_k^{(n)} = \frac{1}{q_{i,i-2}} \sum_{k=i}^{n} q_i^{(k)} G_{k+1}^{(n+1)} \\ &= \frac{1}{q_{i,i-2}} \sum_{k=i+1}^{n+1} q_i^{(k-1)} G_k^{(n+1)} = G_i^{(n+1)}. \end{aligned}$$

So, by induction, the relation holds for all $2 \leqslant i \leqslant n+1$.

Now the proof of Lemma 2.1 is finished. □

The second lemma is a key one.

Lemma 2.2. *Assume that the* 2*-death Q-matrix* $Q = (q_{ij})$ *is irreducible, then for all* $k \geqslant 0$ *and* $n \geqslant 0$*, it holds that*

$$G_k^{(k+2n)} > 0 \quad \text{and} \quad G_k^{(k+2n+1)} \leqslant 0.$$

Proof. It is obvious that for $k \geqslant 1$, $G_k^{(k)} = 1 > 0$ and

$$G_k^{(k+1)} = \frac{q_k^{(k)}}{q_{k,k-2}} = \begin{cases} 0, & \text{if } k = 0, \\ q_1^{(1)} = -q_{10}, & \text{if } k = 1, \\ -1 - \frac{q_{k,k-1}}{q_{k,k-2}}, & \text{if } k \geqslant 2. \end{cases}$$

So $G_k^{(k+1)} \leqslant 0$, $\forall k \geqslant 0$. Note that $G_0^{(1)} = 0$. Then, for $m \geqslant 2$ and $k \geqslant 0$,

$$\begin{aligned} G_k^{(k+m)} &= \frac{G_k^{(k+m-1)} q_{k+m-1}^{(k+m-1)}}{q_{k+m-1,k+m-3}} + \sum_{i=k}^{k+m-2} \frac{G_k^{(i)} q_i^{(k+m-2)}}{q_{i,i-2}} - \sum_{i=k}^{k+m-2} \frac{G_k^{(i)} q_{i,k+m-2}}{q_{i,i-2}} \\ &= \frac{G_k^{(k+m-1)} q_{k+m-1}^{(k+m-1)}}{q_{k+m-1,k+m-3}} + G_k^{(k+m-1)} - \sum_{i=k}^{k+m-2} \frac{G_k^{(i)} q_{i,k+m-2}}{q_{i,i-2}} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{G_k^{(k+m-1)} q_{k+m-1,k+m-2}}{q_{k+m-1,k+m-3}} - \sum_{i=k}^{k+m-2} \frac{G_k q_{i,k+m-2}}{q_{i,i-2}} \\
&= -\sum_{i=k}^{k+m-1} \frac{G_k^{(i)} q_{i,k+m-2}}{q_{i,i-2}}.
\end{aligned}
$$

Assume that these assertions hold until $n = m$. Now by the assumption we get

$$
\begin{aligned}
G_k^{(k+2m+2)} &= -\sum_{i=k}^{k+2m+1} \frac{G_k^{(i)} q_{i,k+2m}}{q_{i,i-2}} = -\sum_{\ell=0}^{m} \frac{G_k^{(k+2\ell)} q_{k+2\ell,k+2m}}{q_{k+2\ell,k+2\ell-2}} - \sum_{\ell=0}^{m} \frac{G_k^{(k+2\ell+1)} q_{k+2\ell+1,k+2m}}{q_{k+2\ell+1,k+2\ell-1}} \\
&\geqslant -\sum_{\ell=0}^{m} \frac{G_k^{(k+2\ell)} q_{k+2\ell,k+2m}}{q_{k+2\ell,k+2\ell-2}} =: -c_{k,m}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
c_{k,m} &= \sum_{0\leqslant \ell \leqslant m-1} \frac{G_k^{(k+2\ell)} q_{k+2\ell,k+2m}}{q_{k+2\ell,k+2\ell-2}} + \frac{G_k^{(k+2m)} q_{k+2m,k+2m}}{q_{k+2m,k+2m-2}} \\
&\leqslant \sum_{0\leqslant \ell \leqslant m-1} \frac{G_k^{(k+2\ell)} q_{k+2\ell,k+2m}}{q_{k+2\ell,k+2\ell-2}} - G_k^{(k+2m)} \\
&\leqslant \frac{q_{k,k+2m}}{q_{k,k-2}} + \sum_{1\leqslant \ell \leqslant m-1} \frac{G_k^{(k+2\ell)} q_{k+2\ell,k+2m}}{q_{k+2\ell,k+2\ell-2}} + c_{k,m-1} \\
&= \frac{1}{q_{k,k-2}} \sum_{m-1\leqslant s\leqslant m} q_{k,k+2s} + \sum_{1\leqslant \ell \leqslant m-2} \frac{G_k^{(k+2\ell)}}{q_{k+2\ell,k+2\ell-2}} \sum_{m-1\leqslant s\leqslant m} q_{k+2\ell,k+2s} \\
&\quad + \frac{G_k^{(k+2m-2)}}{q_{k+2m-2,k+2m-4}} \sum_{m-1\leqslant s\leqslant m} q_{k+2m-2,k+2s} \\
&\leqslant \frac{1}{q_{k,k-2}} \sum_{m-1\leqslant s\leqslant m} q_{k,k+2s} + \sum_{1\leqslant \ell \leqslant m-2} \frac{G_k^{(k+2\ell)}}{q_{k+2\ell,k+2\ell-2}} \sum_{m-1\leqslant s\leqslant m} q_{k+2\ell,k+2s} - G_k^{(k+2m-2)} \\
&\leqslant \frac{1}{q_{k,k-2}} \sum_{m-1\leqslant s\leqslant m} q_{k,k+2s} + \sum_{1\leqslant \ell \leqslant m-2} \frac{G_k^{(k+2\ell)}}{q_{k+2\ell,k+2\ell-2}} \sum_{m-1\leqslant s\leqslant m} q_{k+2\ell,k+2s} + c_{k,m-2}.
\end{aligned}
$$

Eventually, we recursively obtain

$$
\begin{aligned}
c_{k,m} &\leqslant \frac{1}{q_{k,k-2}} \sum_{2\leqslant s\leqslant m} q_{k,k+2s} + \frac{G_k^{(k+2)}}{q_{k+2,k}} \sum_{2\leqslant s\leqslant m} q_{k+2,k+2s} + c_{k,1} \\
&= \frac{1}{q_{k,k-2}} \sum_{1\leqslant s\leqslant m} q_{k,k+2s} + \frac{G_k^{(k+2)}}{q_{k+2,k}} \sum_{1\leqslant s\leqslant m} q_{k+2,k+2s} \\
&\leqslant \frac{1}{q_{k,k-2}} \sum_{1\leqslant s\leqslant m} q_{k,k+2s} - G_k^{(k+2)} \\
&\leqslant \frac{1}{q_{k,k-2}} \sum_{1\leqslant s\leqslant m} q_{k,k+2s} + c_{k,0} \\
&= \frac{1}{q_{k,k-2}} \sum_{0\leqslant s\leqslant m} q_{k,k+2s} \\
&\leqslant \begin{cases} 0, & \text{if } k = 0; \\ -q_{10}, & \text{if } k = 1; \\ -1, & \text{if } k \geqslant 2. \end{cases}
\end{aligned}
$$

In the cases of $k = 0$ or $k = 1$ and $q_{10} = 0$, then the equalities in these inequalities above do not hold simultaneously, otherwise $\{k, k+2, \ldots, k+2m\}$ is closed which is a contradiction with the irreducibility. Hence whatever k is, it always holds that $c_{k,m} < 0$, further

$$
G_k^{(k+2m+2)} \geqslant -c_{k,m} > 0.
$$

Similarly, by the assumption we get

$$\begin{aligned}
G_k^{(k+2m+3)} &= -\sum_{\ell=0}^{m+1} \frac{G_k^{(k+2\ell)} q_{k+2\ell,k+2m+1}}{q_{k+2\ell,k+2\ell-2}} - \sum_{\ell=0}^{m} \frac{G_k^{(k+2\ell+1)} q_{k+2\ell+1,k+2m+1}}{q_{k+2\ell+1,k+2\ell-1}} \\
&\leqslant -\sum_{\ell=0}^{m} \frac{G_k^{(k+2\ell+1)} q_{k+2\ell+1,k+2m+1}}{q_{k+2\ell+1,k+2\ell-1}} =: -d_{k,m+1}.
\end{aligned}$$

Further,

$$\begin{aligned}
d_{k,m+1} &\geqslant \sum_{0\leqslant \ell\leqslant m-1} \frac{G_k^{(k+2\ell+1)} q_{k+2\ell+1,k+2m+1}}{q_{k+2\ell+1,k+2\ell-1}} - G_k^{(k+2m+1)} \\
&\geqslant \frac{G_k^{(k+1)} q_{k+1,k+2m+1}}{q_{k+1,k-1}} + \sum_{1\leqslant \ell\leqslant m-1} \frac{G_k^{(k+2\ell+1)} q_{k+2\ell+1,k+2m+1}}{q_{k+2\ell+1,k+2\ell-1}} + d_{k,m} \\
&\geqslant \frac{G_k^{(k+1)}}{q_{k+1,k-1}} \sum_{m-1\leqslant s\leqslant m} q_{k+1,k+2s+1} + \sum_{1\leqslant \ell\leqslant m-2} \frac{G_k^{(k+2\ell+1)}}{q_{k+2\ell+1,k+2\ell-1}} \sum_{m-1\leqslant s\leqslant m} q_{k+2\ell+1,k+2s+1} - G_k^{(k+2m-1)} \\
&\geqslant \frac{G_k^{(k+1)}}{q_{k+1,k-1}} \sum_{m-1\leqslant s\leqslant m} q_{k+1,k+2s+1} + \sum_{1\leqslant \ell\leqslant m-2} \frac{G_k^{(k+2\ell+1)}}{q_{k+2\ell+1,k+2\ell-1}} \sum_{m-1\leqslant s\leqslant m} q_{k+2\ell+1,k+2s+1} + d_{k,m-1}.
\end{aligned}$$

Then by the same recursive procedure, we thus have

$$\begin{aligned}
d_{k,m+1} &\geqslant \frac{G_k^{(k+1)}}{q_{k+1,k-1}} \sum_{2\leqslant s\leqslant m} q_{k+1,k+2s+1} + \frac{G_k^{(k+3)}}{q_{k+3,k+1}} \sum_{2\leqslant s\leqslant m} q_{k+3,k+2s+1} + d_{k,2} \\
&\geqslant \frac{G_k^{(k+1)}}{q_{k+1,k-1}} \sum_{1\leqslant s\leqslant m} q_{k+1,k+2s+1} - G_k^{(k+3)} \\
&\geqslant \frac{G_k^{(k+1)}}{q_{k+1,k-1}} \sum_{1\leqslant s\leqslant m} q_{k+1,k+2s+1} + d_{k,1} \\
&= \frac{G_k^{(k+1)}}{q_{k+1,k-1}} \sum_{0\leqslant s\leqslant m} q_{k+1,k+2s+1} \\
&\geqslant \begin{cases} 0, & \text{if } k=0; \\ q_{10}, & \text{if } k=1; \\ 1+\frac{q_{k,k-1}}{q_{k,k-2}}, & \text{if } k\geqslant 2. \end{cases}
\end{aligned}$$

So $d_{k,m+1} \geqslant 0$. Further $G_k^{(k+2m+3)} \leqslant -d_{k,m+1} \leqslant 0$. Hence the assertions hold when $n = m+1$. From induction the conclusions are followed. The proof is finished. □

3. Proof of Theorem 1.1

By the equation of $\mu Q = 0$ we obtain that

$$\mu_i q_i = \sum_{0\leqslant j<i} \mu_j q_{ji} + \mu_{i+1} q_{i+1,i} + \mu_{i+2} q_{i+2,i}, \quad i\geqslant 0.$$

Summing up from 0 to n, then

$$\begin{aligned}
\sum_{i=0}^{n} \mu_i q_i &= \sum_{0<j<n} \mu_j \left(q_j^{(j+1)} - q_j^{(n+1)}\right) + \sum_{i=2}^{n+1} \mu_i (q_{i,i-1} + q_{i,i-2}) + \mu_1 q_{10} + \mu_{n+2} q_{n+2,n} \\
&= \sum_{i=0}^{n} \mu_i \left(q_i^{(i+1)} - q_i^{(n+1)}\right) + \sum_{i=2}^{n+1} \mu_i (q_{i,i-1} + q_{i,i-2}) + \mu_1 q_{10} + \mu_{n+2} q_{n+2,n} \\
&= \sum_{i=0}^{n} \mu_i q_i - \sum_{i=0}^{n+1} \mu_i q_i^{(n+1)} + \mu_{n+2} q_{n+2,n}, \quad n\geqslant 0.
\end{aligned}$$

Further,

$$\mu_{n+2} = \frac{1}{q_{n+2,n}} \sum_{i=0}^{n+1} \mu_i q_i^{(n+1)}, \quad n\geqslant 0.$$

We will check that

$$\mu_n = \frac{1}{q_{n,n-2}} \sum_{\ell=1}^{n-1} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) g_\ell^{(n-1)}, \quad n \geqslant 2.$$

Firstly,

$$\mu_2 = \frac{1}{q_{20}} \sum_{i=0}^{1} \mu_i q_i^{(1)} = \frac{1}{q_{20}} \left(\mu_0 q_0^{(1)} + \mu_1 q_1^{(1)}\right) = \frac{1}{q_{20}} \sum_{\ell=1}^{1} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) g_\ell^{(1)}.$$

Assume that the assertion holds up to $n+1$. Using Lemma 2.1 now we have

$$\begin{aligned}
\mu_{n+2} &= \frac{1}{q_{n+2,n}} \left(\mu_0 q_0^{(n+1)} + \mu_1 q_1^{(n+1)} + \sum_{i=2}^{n+1} \mu_i q_i^{(n+1)}\right) \\
&= \frac{1}{q_{n+2,n}} \left(\mu_0 q_0^{(n+1)} + \mu_1 q_1^{(n+1)} + \sum_{i=2}^{n+1} \frac{1}{q_{i,i-2}} \sum_{\ell=1}^{i-1} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) g_\ell^{(i-1)} q_i^{(n+1)}\right) \\
&= \frac{1}{q_{n+2,n}} \left(\mu_0 q_0^{(n+1)} + \mu_1 q_1^{(n+1)} + \sum_{\ell=1}^{n} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) \sum_{i=\ell+1}^{n+1} \frac{g_\ell^{(i-1)} q_i^{(n+1)}}{q_{i,i-2}}\right) \\
&= \frac{1}{q_{n+2,n}} \left(\mu_0 q_0^{(n+1)} + \mu_1 q_1^{(n+1)} + \sum_{\ell=1}^{n} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) g_\ell^{(n+1)}\right) \\
&= \frac{1}{q_{n+2,n}} \sum_{\ell=1}^{n+1} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) g_\ell^{(n+1)} \\
&= \frac{1}{q_{n+2,n}} \sum_{\ell=1}^{n+1} \left(\mu_0 q_0^{(\ell)} + \mu_1 q_1^{(\ell)}\right) G_{\ell+1}^{(n+2)} = \frac{1}{q_{n+2,n}} \left(\mu_0 G_0^{(n)} + \mu_1 G_1^{(n)}\right), \quad n \geqslant 0,
\end{aligned}$$

where the last identity is due to (1.1). Besides, via Lemma 2.2, (1.2) could be naturally derived by the positive property of the invariant measures. For the second assertion, if the process is ergodic, then

$$\mu_1 = \inf_{m\geqslant 1} \left(\frac{G_0^{(2m+1)}}{G_1^{(2m+1)}}\right) = \sup_{m\geqslant 1} \left(\frac{G_0^{(2\ m)}}{G_1^{(2\ m)}}\right)$$

and $\sum_{n\geqslant 0} \mu_n < \infty$. On the other hand,

$(\pi_n)_{n\geqslant 0} := (\mu_n / \sum_n \mu_n)_{n\geqslant 0}$ is the unique stationary distribution for the process, therefore it is ergodic. □

4. Applications

In this section we demonstrate the validity of our results by showing three concrete examples as follows, which are taken from Zhang (2023).

Example 4.1. Let $q_{i,i-2} = 1$ for all $i \geqslant 2, q_{10} = 1, q_{i,i+1} = 1$ for all $i \geqslant 0$ and $q_{ij} = 0$ for other $j \neq i$. Then $q_i^{(i)} = -1$ for all $i \geqslant 1, q_i^{(i+1)} = 1$ and $q_i^{(k)} = 0$ for all $k \geqslant i+2 \geqslant 2$. Hence

$$g_n^{(n)} = 1 (n \geqslant 1), \quad g_{n-1}^{(n)} = -1, \quad g_{i-1}^{(n)} = g_{i+1}^{(n)} - g_i^{(n)}, \quad 2 \leqslant i < n.$$

Further

$$g_i^{(n)} = (-1)^{n-i} F_{n-i}, \quad 1 \leqslant i \leqslant n,$$

where $\{F_n\}$ is the Fibonacci sequence:

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right) =: \frac{1}{\sqrt{5}} \left(A^{n+1} - (-B)^{n+1}\right), \quad n \geqslant 0.$$

Then

$$\mu_2 = \mu_0 - \mu_1, \quad \mu_n = \mu_2 g_1^{(n-1)} + \mu_1 g_2^{(n-1)}, \quad n \geqslant 3.$$

So

$$\mu_n = \mu_2 (-1)^{n-2} F_{n-2} + \mu_1 (-1)^{n-3} F_{n-3} = \mu_0 (-1)^{n-2} F_{n-2} + \mu_1 (-1)^{n-3} F_{n-1}, \quad n \geqslant 3.$$

Let $\mu_0 = 1$. For the positive property of μ_n, on the one hand, for all $n = 2k$,

$$\mu_1 < \frac{A^{2k-1} - (-B)^{2k-1}}{A^{2k} - (-B)^{2k}} = -A + \frac{A+B}{1 - (-B/A)^{2k}} \downarrow B.$$

On the other hand for all $n = 2k+1$,

$$\mu_1 > \frac{A^{2k} - (-B)^{2k}}{A^{2k+1} - (-B)^{2k+1}} = \frac{1 + A^2}{A + B(-B/A)^{2k}} - A \uparrow B.$$

Hence, $\mu_1 = B$ and further $\mu_2 = 1 - B = B^2$ and

$$\mu_n = (-1)^{n-2}\left(F_{n-2} - BF_{n-1}\right) = B^n, \quad n \geqslant 3.$$

So $\mu_n = B^n$ for all $n \geqslant 0$ and $\mu := \sum_{n\geqslant 0} \mu_n = A^2$. Then the stationary distribution is followed by $\pi_n = \mu_n/\mu = B^{n+2}$ for any $n \geqslant 0$.

Example 4.2. Let $Q = \left(q_{ij}\right)$ be a regular and irreducible Q-matrix satisfying

$$q_{i,i-2} = c > 0, \quad q_{i,i-1} = a \geqslant 0, \quad i \geqslant 2; \quad q_{10} = d \geqslant 0; \quad q_{i,i+1} = b > 0, \quad i \geqslant 0,$$

and $q_{ij} = 0$ for other $i, j \geqslant 1$ with $i \neq j$. Assume that the process is recurrent, *i.e.*, $a + 2c \geqslant b$. Then $q_1^{(1)} = -d$, $q_i^{(i)} = -(a+c)$ for all $i \geqslant 2$, $q_i^{(i+1)} = b$ and $q_i^{(k)} = 0$ for all $k > i+1 \geqslant 1$. Hence

$$g_n^{(n)} = 1 (n \geqslant 1), \quad g_{n-1}^{(n)} = -\frac{a+c}{c}(n \geqslant 2),$$

and

$$g_{i-1}^{(n)} = -\frac{a+c}{c} g_i^{(n)} + \frac{b}{c} g_{i+1}^{(n)}, \quad 2 \leqslant i \leqslant n-1.$$

By the theory of difference equations, we obtain

$$g_i^{(n)} = \frac{(-1)^{n-i}}{\lambda_1 - \lambda_2}\left(\lambda_1^{n-i+1} - \lambda_2^{n-i+1}\right), \quad 1 \leqslant i \leqslant n,$$

where

$$\lambda_1 = \frac{a + c + \sqrt{(a+c)^2 + 4bc}}{2c}, \quad \lambda_2 = \frac{a + c - \sqrt{(a+c)^2 + 4bc}}{2c}.$$

Then we have

$$\mu_2 = \frac{1}{c}\left(b\mu_0 - d\mu_1\right), \quad \mu_n = \mu_2 g_1^{(n-1)} + \frac{b}{c}\mu_1 g_2^{(n-1)}, \quad n \geqslant 3.$$

So for $n \geqslant 2$,

$$\mu_n = \mu_0 \frac{b}{c}\frac{(-1)^{n-2}}{\lambda_1 - \lambda_2}\left(\lambda_1^{n-1} - \lambda_2^{n-1}\right) + \mu_1 \frac{(-1)^{n-1}}{\lambda_1 - \lambda_2}\left(\lambda_1^{n-1}\left(\frac{d}{c} - \lambda_2\right) - \lambda_2^{n-1}\left(\frac{d}{c} - \lambda_1\right)\right).$$

Let $\mu_0 = 1$. For the positive property of μ_n, on the one hand, for all $n = 2k$,

$$\mu_1 < \frac{b}{d - c\lambda_2}\left(1 + \frac{\lambda_1 - \lambda_2}{(d/c - \lambda_1) + \left(\lambda_1/\left(-\lambda_2\right)\right)^{2k-1}(d/c - \lambda_2)}\right) \downarrow \frac{b}{d - c\lambda_2}.$$

On the other hand for all $n = 2k+1$,

$$\mu_1 > \frac{b}{d - c\lambda_2}\left(1 + \frac{\lambda_1 - \lambda_2}{(d/c - \lambda_1) - \left(\lambda_1/\lambda_2\right)^{2k}(d/c - \lambda_2)}\right) \uparrow \frac{b}{d - c\lambda_2}.$$

Hence,

$$\mu_1 = \frac{b}{d - c\lambda_2} = \frac{2b}{2d + \sqrt{(a+c)^2 + 4bc} - (a+c)}.$$

Further it is derived that

$$\mu_n = \mu_1\left(-\lambda_2\right)^{n-1}, \quad n \geqslant 1.$$

So

$$\mu := \sum_{n\geqslant 0} \mu_n = 1 + \mu_1 \sum_{n\geqslant 1}\left(-\lambda_2\right)^{n-1} = 1 + \mu_1 \sum_{n\geqslant 0}\left(-\lambda_2\right)^{n} < +\infty$$

if and only if $-\lambda_2 < 1$, equivalently, $a + 2c > b$. Hence, the process is ergodic if and only if $a + 2c > b$. Now in the ergodic case, $\mu = 1 + \mu_1/\left(1 + \lambda_2\right)$ and the stationary distribution is followed by

$$\pi_0 = \frac{1 + \lambda_2}{1 + \lambda_2 + \mu_1}, \quad \pi_n = \frac{\left(1 + \lambda_2\right)\mu_1}{1 + \lambda_2 + \mu_1} \cdot \left(-\lambda_2\right)^{n-1}, \quad n \geqslant 1.$$

Example 4.3. Let $q_{i,i-2} = 1$ for all $i \geqslant 2, q_{i,i+2} = 1$ for all $i \geqslant 0, q_{01} = q_{10} = 1$ and $q_{ij} = 0$ for other $i, j \geqslant 1$ with $j \neq i$. Then $q_i^{(i)} = -1$ for all $i \geqslant 1, q_i^{(i+1)} = q_i^{(i+2)} = 1$ and $q_i^{(k)} = 0$ for all $k > i+2 > 2$. Meanwhile $q_0^{(1)} = 2, q_0^{(2)} = 1$ and $q_0^{(\ell)} = 0$ for any $\ell > 2$. Hence

$$g_n^{(n)} = 1 (n \geqslant 1), \quad g_{n-1}^{(n)} = -1, \quad g_{n-2}^{(n)} = -g_{n-1}^{(n)} + g_n^{(n)} = 2,$$

and

$$g_{i-1}^{(n)} = -g_i^{(n)} + g_{i+1}^{(n)} + g_{i+2}^{(n)}, \quad 2 \leqslant i < n-1.$$

Further

$$g_i^{(n)} = (-1)^{n-i}\left(\left[\frac{n-i}{2}\right]+1\right), \quad 1 \leqslant i \leqslant n,$$

where $[n]$ represents the largest integer less than or equal to n. Hence

$$\mu_2 = \left(2-\mu_1\right) g_1^{(1)} = 2-\mu_1, \quad \mu_3 = \left(2-\mu_1\right) g_1^{(2)} + \left(1+\mu_1\right) g_2^{(2)} = -1+2\mu_1,$$

and for $n \geqslant 4$,

$$\begin{aligned}\mu_n &= \left(2-\mu_1\right) g_1^{(n-1)} + \left(1+\mu_1\right) g_2^{(n-1)} + \mu_1 g_3^{(n-1)} \\ &= 2(-1)^{n-2}\left(\left[\frac{n-2}{2}\right]+1\right) + (-1)^{n-3}\left(\left[\frac{n-3}{2}\right]+1\right) + \mu_1(-1)^{n-1}\left(\left[\frac{n-1}{2}\right]+1\right).\end{aligned}$$

In the case of $n = 2k+1$, then

$$\mu_{2k+1} = -2k + k + (k+1)\mu_1 = -k + (k+1)\mu_1.$$

Further, for all $k > 0$, $\mu_{2k+1} > 0$ if and only if $\mu_1 > k/(k+1)$, which implies that $\mu_1 \geqslant 1$. In the case of $n = 2k+2$, then

$$\mu_{2k+2} = 2(k+1) - k - (k+1)\mu_1 = k + 2 - (k+1)\mu_1.$$

Therefore, for all $k \geqslant 0$, $\mu_{2k+2} > 0$ if and only if $\mu_1 < (k+2)/(k+1)$, which implies that $\mu_1 \leqslant 1$. Hence $\mu_1 = 1$ and further $\mu_n \equiv 1$ is the unique invariant measure of the process. Then the process is recurrent but not ergodic. In fact, by rearranging the states, it can be regarded as a simple random walk or a symmetric birth–death process on $\mathbb{Z}$.

Data availability

No data was used for the research described in the article.

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