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RESEARCH ARTICLE

Moments of integral-type downward functionals for single death processes

Jing WANG[1,2], **Yuhui ZHANG**[1]

1 School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China
2 School of Mathematics and Statistics, Yili Normal University, Yili 835000, China

Abstract We get an explicit and recursive representation for high order moments of integral-type downward functionals for single death processes. Meanwhile, main results are applied to more general integral-type downward functionals.

Keywords Single death process, integral-type functional, moment
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1 Introduction

Consider a continuous-time homogeneous Markov chain $\{X(t)\colon t \geqslant 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with transition probability matrix $P(t) = (p_{ij}(t))$ on a countable state space $E := \{0, 1, \dots\}$. We call $\{X(t)\colon t \geqslant 0\}$ a single death process if its transition rate matrix $Q = (q_{ij}\colon i, j \in E)$ is irreducible and satisfies that $q_{i,i-1} > 0$ for all $i \geqslant 1$ and $q_{i,i-j} = 0$ for all $i \geqslant j \geqslant 2$. Such a matrix $Q = (q_{ij})$ is called a single death Q-matrix. Assume that Q is conservative and totally stable (i.e., $q_i := -q_{ii} = \sum_{j\neq i} q_{ij} < \infty$ for all $i \in E$). Define the first hitting time for all $i \geqslant 0$:

$$\tau_i := \inf\{t > 0\colon X(t) = i\}.$$

Let V be a non-negative function and not identically equal to zero on E. Fix $i_0 \in E$. We consider the integral-type functional for single death processes in this paper:

$$\xi_{i_0} = \int_0^{\tau_{i_0}} V(X(t))\mathrm{d}t. \tag{1.1}$$

It is well known that the integral functional $Y(t) := \int_0^t f(X(s), s)\mathrm{d}s$, where f is a non-negative function, has attracted lots of attention, due to their

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Corresponding author: Yuhui ZHANG, E-mail: zhangyh@bnu.edu.cn

importance in practical applications. The integrals arise naturally in the theory of inventories and storage (see [6,7]). In an inventory system or storage reservoir, the integral represents the holding cost associated with the stock in the system over a particular period of time. The moments and distributions of the integrals were investigated. For queueing theory, particularly those involving automobile traffic such as traffic jams and intersection bottlenecks, the integrals up to dissipation are related to the cost of the the flow-stopping incident. See [2], where $f(X(s)) = sX(s)$. Some limit theorems were established. For biology, in the study of response of host to injection of virulent bacteria, $f(X(s)) = bX(s)$, with $b > 0$, could be regarded as a measure of the total amount of toxins produced by the bacteria during $(0,t)$, assuming a constant toxin-excretion rate per bacterium. Here, $X(t)$ denotes the number of live bacteria at time t, the growth of which is governed by a birth and death process. In [8], where $f(X(s)) = X(s)$, it was concerned with the joint distribution of $X(t)$ and $Y(t)$ and with its limiting forms. In [9], it established some limit theorems on branching processes concerning the behavior of the vector process $(X(t), Y(t), N(t))$, where $N(t)$ denotes the total number of particle-deaths occurring during $(0,t)$. More contents are seen in [10] and references therein.

In [12], for birth-death processes, the moments and distributions of integral-type functional (1.1) have been obtained. See [5] also. Wang [13,14] distinguished integral-type functional as upward or downward for birth-death processes. Wu [15] and Yang [16] obtained some intermediate results, where the question is investigated for a more general and general homogeneous denumerable Markov processes, respectively. Based on this, Hou and Guo [3] solved the problem completely by using the theory of the minimal non-negative solution of system of non-negative linear equations. See [4] for more general integral-type functionals.

For certain stochastic process and certain integral-type functional, we naturally want to get explicit results, like birth-death processes. In [17] and [18], the moments and distributions of integral-type upward functionals for single birth processes were obtained.

Without single-exit property of single birth processes, single death processes may be of infinite-exit so that is more difficult to deal with. In [11], on a finite state space, we have obtained the moments, the stay times, and the Laplace transform of the distribution of integral-type downward functionals for single death processes; further, the high order moments of integral-type functional in the case of $V \equiv 1$ on a countable state space has been solved by limitation approach, which was obtained firstly in [20] by different method. This paper is a continuation of [11,19,20]. We focus on obtaining some recursive representations of high order moments of integral-type downward functionals for single death processes.

The following sequences are used throughout this paper:

$$q_n^{(k)} = \sum_{j \geqslant k} q_{nj}, \quad k > n \geqslant 0,$$

and

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \ 1 \leqslant n < i.$$

It is easily known that

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \sum_{k=n}^{i-1} \frac{G_n^{(k)} q_k^{(i)}}{q_{k,k-1}}, \ 1 \leqslant n < i.$$

The main result is as follows.

Theorem 1.1 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding Q-process is recurrent. Give $i_0 \in E$ and a positive integer $n \geqslant 1$ arbitrarily. Then*

$$\mathbb{E}_i \xi_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0 \geqslant 0, \, n \geqslant 1.$$

This paper is organized as follows. The decomposition of moments of integral-type downward functional is given in the next section, which reveals the relations between moments from different starting states. Meanwhile, another explicit representation of integral-type functional is also presented in Section 2. Then Section 3 is devoted to the proof of Theorem 1.1. Finally, we apply Theorem 1.1 to more general integral-type downward functionals for single death processes.

2 Decomposition of moments of integral-type functional

Consider the moments of ξ_{i_0}. Set

$$m_{i,i_0}^{(n)} = \mathbb{E}_i \xi_{i_0}^n, \quad n \geqslant 1.$$

Note that

$$m_{i,i}^{(n)} = \mathbb{E}_i \xi_i^n = 0, \quad n \geqslant 1.$$

Set $m_{i,i_0}^{(0)} = 1$. Now, consider a decomposition of the moments of integral-type downward functional. Define

$$w_i^{(n)} := m_{i,i-1}^{(n)}, \quad i \geqslant 1, \, n \geqslant 0;$$

in particular, $w_i^{(0)} = 1$. We denote $w_i^{(1)}$ by w_i simply. By the Polynomial Theorem, we obtain the following assertions.

Theorem 2.1 *Under the conditions of Theorem* 1.1, *for any $i_0 \in E$, we have*

$$m_{i,i_0}^{(n)} = (w_{i_0+1} + \cdots + w_i)^{(n)}$$
$$=: \sum_{n_{i_0+1}+\cdots+n_i=n} \frac{n!}{n_{i_0+1}!\cdots n_i!} w_{i_0+1}^{(n_{i_0+1})} \cdots w_i^{(n_i)}, \quad i > i_0,\ n \geqslant 1.$$

Proof Denote $\int_u^v V(X(t))\mathrm{d}t$ by $\int_u^v$ simply. From the definition of ξ_{i_0} and the single death and strong Markov properties, it follows that

$$\begin{aligned}
m_{i,i_0}^{(1)} &= \mathbb{E}_i \int_0^{\tau_{i_0}} \\
&= \mathbb{E}_i\Big(\int_0^{\tau_{i-1}} + \int_{\tau_{i-1}}^{\tau_{i-2}} + \cdots + \int_{\tau_{i_0+1}}^{\tau_{i_0}}\Big) \\
&= \mathbb{E}_i \int_0^{\tau_{i-1}} + \mathbb{E}_i \int_{\tau_{i-1}}^{\tau_{i-2}} + \cdots + \mathbb{E}_i \int_{\tau_{i_0+1}}^{\tau_{i_0}} \\
&= w_i + \mathbb{E}_i\Big(\mathbb{E}_i\Big(\int_{\tau_{i-1}}^{\tau_{i-2}} \Big| \mathscr{F}_{\tau_{i-1}}\Big)\Big) + \cdots + \mathbb{E}_i\Big(\mathbb{E}_i\Big(\int_{\tau_{i_0+1}}^{\tau_{i_0}} \Big| \mathscr{F}_{\tau_{i_0+1}}\Big)\Big) \\
&= w_i + \mathbb{E}_{i-1} \int_0^{\tau_{i-2}} + \cdots + \mathbb{E}_{i_0+1} \int_0^{\tau_{i_0}} \\
&= (w_i + w_{i-1} + \cdots + w_{i_0+1})^{(1)}.
\end{aligned}$$

So the assertion holds in the case of $n = 1$.

Assume that the assertion holds until $n-1$. By the single death and strong Markov properties, it is seen that for all $i > i_0 + 1$,

$$\begin{aligned}
m_{i,i_0}^{(n)} &= \mathbb{E}_i\Big(\int_0^{\tau_{i-1}} + \int_{\tau_{i-1}}^{\tau_{i_0}}\Big)^n \\
&= m_{i,i-1}^{(n)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i\Big(\int_{\tau_{i-1}}^{\tau_{i_0}}\Big)^s \Big(\int_0^{\tau_{i-1}}\Big)^{n-s} + \mathbb{E}_i\Big(\int_{\tau_{i-1}}^{\tau_{i_0}}\Big)^n \\
&= m_{i,i-1}^{(n)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i\Big(\mathbb{E}_i\Big(\Big(\int_{\tau_{i-1}}^{\tau_{i_0}}\Big)^s \Big(\int_0^{\tau_{i-1}}\Big)^{n-s} \Big| \mathscr{F}_{\tau_{i-1}}\Big)\Big) \\
&\quad + \mathbb{E}_i\Big(\mathbb{E}_i\Big(\Big(\int_{\tau_{i-1}}^{\tau_{i_0}}\Big)^n \Big| \mathscr{F}_{\tau_{i-1}}\Big)\Big) \\
&= m_{i,i-1}^{(n)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i\Big(\int_0^{\tau_{i-1}}\Big)^{n-s} \mathbb{E}_{i-1}\Big(\int_0^{\tau_{i_0}}\Big)^s + \mathbb{E}_{i-1}\Big(\int_0^{\tau_{i_0}}\Big)^n \\
&= m_{i,i-1}^{(n)} + \sum_{s=1}^{n-1} \binom{n}{s} m_{i,i-1}^{(n-s)} m_{i-1,i_0}^{(s)} + m_{i-1,i_0}^{(n)}.
\end{aligned}$$

Note that $m^{(s)}_{i_0,i_0}=0$. Hence, the equality above is true for all $i>i_0$. Further,

$$\begin{aligned}m^{(n)}_{i,i_0}&=m^{(n)}_{i,i-1}+\sum_{s=1}^{n-1}\binom{n}{s}m^{(n-s)}_{i,i-1}m^{(s)}_{i-1,i_0}+m^{(n)}_{i-1,i-2}\\&\quad+\sum_{s=1}^{n-1}\binom{n}{s}m^{(n-s)}_{i-1,i-2}m^{(s)}_{i-2,i_0}+m^{(n)}_{i-2,i_0}\\&=\sum_{\ell=i-1}^{i}m^{(n)}_{\ell,\ell-1}+\sum_{\ell=i-1}^{i}\sum_{s=1}^{n-1}\binom{n}{s}m^{(n-s)}_{\ell,\ell-1}m^{(s)}_{\ell-1,i_0}+m^{(n)}_{i-2,i_0}.\end{aligned}$$

Hence, we can obtain recursively that

$$m^{(n)}_{i,i_0}=\sum_{\ell=i_0+1}^{i}m^{(n)}_{\ell,\ell-1}+\sum_{\ell=i_0+2}^{i}\sum_{s=1}^{n-1}\binom{n}{s}m^{(n-s)}_{\ell,\ell-1}m^{(s)}_{\ell-1,i_0}.$$

By the assumption, we know that

$$m^{(n)}_{i,i_0}=\sum_{\ell=i_0+1}^{i}w^{(n)}_{\ell}+\sum_{\ell=i_0+2}^{i}\sum_{s=1}^{n-1}\binom{n}{s}w^{(n-s)}_{\ell}(w_{i_0+1}+\cdots+w_{\ell-1})^{(s)}. \tag{2.1}$$

If $(w_{i_0+1}+\cdots+w_i)^{(n)}=\infty$, then there exists such $p\in[i_0+1,i]$ and $k\in[1,n]$ that $w^{(k)}_p=\infty$; further, $w^{(n)}_p=\infty$. By (2.1), we know that

$$m^{(n)}_{i,i_0}=\infty=(w_{i_0+1}+\cdots+w_i)^{(n)}.$$

If $(w_{i_0+1}+\cdots+w_i)^{(n)}<\infty$, then

$$(w_{i_0+1}+\cdots+w_\ell)^{(n)}<\infty,\quad w^{(n)}_\ell<\infty,\quad \ell\in[i_0+1,i].$$

Further, it follows that

$$\begin{aligned}&\sum_{\ell=i_0+2}^{i}\sum_{s=1}^{n-1}\binom{n}{s}w^{(n-s)}_{\ell}(w_{i_0+1}+\cdots+w_{\ell-1})^{(s)}\\&=\sum_{\ell=i_0+2}^{i}\left((w_{i_0+1}+\cdots+w_\ell)^{(n)}-(w_{i_0+1}+\cdots+w_{\ell-1})^{(n)}-w^{(n)}_\ell\right)\\&=(w_{i_0+1}+\cdots+w_i)^{(n)}-\sum_{\ell=i_0+1}^{i}w^{(n)}_\ell.\end{aligned}$$

Hence, by the arguments above and (2.1), it holds that

$$m^{(n)}_{i,i_0}=(w_{i_0+1}+\cdots+w_i)^{(n)}.$$

That is, the assertion holds in the case of n. By induction, the assertion holds for all $n \geqslant 1$. The proof is finished. □

By Theorem 2.1, to represent $m_{i,i_0}^{(n)}$, we only need to get the explicit formula of $m_{\ell,\ell-1}^{(s)}$ (or $w_\ell^{(s)}$). To do this, we need some notations as follows. Define

$$M_{ik}^{(n-1)} = \sum_{i+1\leqslant \ell\leqslant k} \sum_{1\leqslant s\leqslant n-1} \binom{n}{s} m_{\ell,\ell-1}^{(n-s)} m_{\ell-1,i-1}^{(s)}, \quad i \geqslant 1.$$

Obviously, $M_{ik}^{(n-1)} = 0$ if $i \geqslant k \geqslant 1$ and $M_{ik}^{(0)} = 0$. Define

$$M_i^{(n-1)} = V(i) w_i^{(n-1)} + \frac{1}{n} \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(n-1)}, \quad n, i \geqslant 1. \tag{2.2}$$

Obviously, $M_i^{(0)} = V(i)$ for all $i \geqslant 1$.

First, we introduce some properties about the definitions above. The proof of the following proposition is similar to [20, Proposition 2.2], which is omitted here to save space.

Proposition 2.2 *Under the conditions of Theorem* 1.1, *the following assertions hold*:

$$m_{i,i_0}^{(n)} = \sum_{i_0+1\leqslant \ell\leqslant i} w_\ell^{(n)} + M_{i_0+1,i}^{(n-1)}, \quad i \geqslant i_0,\ n \geqslant 1, \tag{2.3}$$

$$M_{ik}^{(n-1)} = M_{ij}^{(n-1)} + M_{j+1,k}^{(n-1)} + \sum_{1\leqslant s\leqslant n-1} \binom{n}{s} m_{k,j}^{(n-s)} m_{j,i-1}^{(s)}, \quad 1 \leqslant i \leqslant j \leqslant k,\ n \geqslant 1. \tag{2.4}$$

Our result on the representation of $w_\ell^{(s)}$ by the above notations is presented as follows.

Theorem 2.3 *Under the conditions of Theorem* 1.1, *we have*

$$m_{i,i-1}^{(n)} = n \sum_{k\geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \quad i, n \geqslant 1.$$

Hence, combining Theorems 2.1 and 2.3, we obtain another representation of $m_{i,i_0}^{(n)}$ for all $i > i_0$.

To prove Theorem 2.3, we need three lemmas. The first lemma is presented as follows, which will be proven similarly as the proof of [20, Lemma 2.4].

Lemma 2.4 *Under the conditions of Theorem* 1.1, *for all* $n \geqslant 1$, $(m_{i,i-1}^{(n)},\ i \geqslant 1)$ *satisfies the following equation*:

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{\ell\geqslant i+1} \frac{q_i^{(\ell)}}{q_i} x_\ell + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geqslant 1. \tag{2.5}$$

Proof Denote $\int_u^v V(X(t))\mathrm{d}t$ by $\int_u^v$ simply. To prove the assertion, we use the induction. First, we prove the assertion in the case of $n=1$. Define the first jumping time

$$\eta_1 := \inf\{t>0\colon X(t)\neq X(0)\}.$$

By the strong Markov and the single death properties, it follows that

$$\begin{aligned}
w_i &= m_{i,i-1}\\
&= \mathbb{E}_i\biggl(\int_0^{\eta_1}+\int_{\eta_1}^{\tau_{i-1}}\biggr)\\
&= V(i)\mathbb{E}_i\eta_1+\mathbb{E}_i\int_{\eta_1}^{\tau_{i-1}}\\
&= \frac{V(i)}{q_i}+\mathbb{E}_i\biggl(\mathbb{E}_i\biggl(\int_{\eta_1}^{\tau_{i-1}}\Big|\mathscr{F}_{\eta_1}\biggr)\biggr)\\
&= \frac{V(i)}{q_i}+\frac{q_{i,i-1}}{q_i}\,\mathbb{E}_{i-1}\int_0^{\tau_{i-1}}+\sum_{k\geqslant i+1}\frac{q_{ik}}{q_i}\,\mathbb{E}_k\int_0^{\tau_{i-1}}\\
&= \frac{V(i)}{q_i}+\sum_{k\geqslant i+1}\frac{q_{ik}}{q_i}\sum_{\ell=i+1}^{k}m_{\ell,\ell-1}+\sum_{k\geqslant i+1}\frac{q_{ik}}{q_i}\,m_{i,i-1}\\
&= \frac{q_i^{(i+1)}}{q_i}\,w_i+\sum_{\ell\geqslant i+1}\frac{q_i^{(\ell)}}{q_i}\,w_\ell+\frac{V(i)}{q_i}.
\end{aligned}$$

So $(m_{i,i-1},\ i\geqslant 1)$ satisfies the following equation:

$$x_i=\frac{q_i^{(i+1)}}{q_i}\,x_i+\sum_{\ell\geqslant i+1}\frac{q_i^{(\ell)}}{q_i}\,x_\ell+\frac{V(i)}{q_i},\quad i\geqslant 1.$$

Hence, the assertion holds when $n=1$.

Assume that the assertion holds until $n-1$. By the strong Markov property, the single death property, and independence of η_1 and $X(\eta_1)$, we derive that

$$\begin{aligned}
m_{i,i-1}^{(n)} &= \mathbb{E}_i\biggl(\int_0^{\eta_1}+\int_{\eta_1}^{\tau_{i-1}}\biggr)^n\\
&= \mathbb{E}_i\biggl(V(i)\eta_1+\int_{\eta_1}^{\tau_{i-1}}\biggr)^n\\
&= \frac{n!V(i)^n}{q_i^n}+\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_i(V(i)\eta_1)^{n-s}\biggl(\int_{\eta_1}^{\tau_{i-1}}\biggr)^s+\mathbb{E}_i\biggl(\int_{\eta_1}^{\tau_{i-1}}\biggr)^n\\
&= \frac{n!V(i)^n}{q_i^n}+\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_i(V(i)\eta_1)^{n-s}\sum_{k\geqslant i+1}\frac{q_{ik}}{q_i}\,\mathbb{E}_k\biggl(\int_0^{\tau_{i-1}}\biggr)^s\\
&\quad+\sum_{k\geqslant i+1}\frac{q_{ik}}{q_i}\,\mathbb{E}_k\biggl(\int_0^{\tau_{i-1}}\biggr)^n
\end{aligned}$$

$$= \frac{n!V(i)^n}{q_i^n} + \sum_{s=1}^{n-1} \sum_{k\geqslant i+1} \frac{n!V(i)^{n-s}}{s!q_i^{n-s}} \frac{q_{ik}}{q_i} m_{k,i-1}^{(s)} + \sum_{k\geqslant i+1} \frac{q_{ik}}{q_i} m_{k,i-1}^{(n)}.$$

From (2.3), it follows that

$$\begin{aligned}
& m_{i,i-1}^{(n)} \\
&\quad = \frac{n!V(i)^n}{q_i^n} + \sum_{s=1}^{n-1} \frac{n!V(i)^{n-s}}{s!q_i^{n-s+1}} \sum_{k\geqslant i+1} q_{ik} \left(\sum_{\ell=i}^{k} w_\ell^{(s)} + M_{ik}^{(s-1)} \right) \\
&\qquad + \sum_{k\geqslant i+1} \frac{q_{ik}}{q_i} \left(\sum_{\ell=i}^{k} w_\ell^{(n)} + M_{ik}^{(n-1)} \right) \\
&\quad = \frac{n!V(i)^n}{q_i^n} + \sum_{s=1}^{n-1} \frac{n!V(i)^{n-s}}{s!q_i^{n-s+1}} \left(q_i^{(i+1)} w_i^{(s)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(s)} + \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(s-1)} \right) \\
&\qquad + \frac{1}{q_i} \left(q_i^{(i+1)} w_i^{(n)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(n)} + \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(n-1)} \right) \\
&\quad =: \mathrm{I} + \frac{1}{q_i} \left(q_i^{(i+1)} w_i^{(n)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(n)} + \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(n-1)} \right).
\end{aligned}$$

By the assumption, we get that

$$\begin{aligned}
\mathrm{I} &= \frac{n!V(i)^{n-1}}{q_i^n} \left(q_i^{(i+1)} w_i^{(1)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(1)} + V(i) \right) \\
&\quad + \sum_{s=2}^{n-1} \frac{n!V(i)^{n-s}}{s!q_i^{n-s+1}} \left(q_i^{(i+1)} w_i^{(s)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(s)} + \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(s-1)} \right) \\
&= \frac{n!V(i)^{n-1}}{q_i^{n-1}} w_i^{(1)} \\
&\quad + \sum_{s=2}^{n-1} \frac{n!V(i)^{n-s}}{s!q_i^{n-s+1}} \left(q_i^{(i+1)} w_i^{(s)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(s)} + \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(s-1)} \right).
\end{aligned}$$

Further, by the assumption, it is derived recursively that

$$\begin{aligned}
\mathrm{I} &= \frac{n!V(i)^{n-2}}{2!q_i^{n-2}} w_i^{(2)} \\
&\quad + \sum_{s=3}^{n-1} \frac{n!}{s!q_i^{n-s+1}} \left(q_i^{(i+1)} w_i^{(s)} + \sum_{\ell\geqslant i+1} q_i^{(\ell)} w_\ell^{(s)} + \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(s-1)} \right) \\
&= \cdots
\end{aligned}$$

$$= \frac{n!V(i)}{(n-1)!q_i}\, w_i^{(n-1)}$$
$$= \frac{nV(i)}{q_i}\, w_i^{(n-1)}.$$

Hence,

$$\begin{aligned} m_{i,i-1}^{(n)} &= \frac{nV(i)}{q_i}\, w_i^{(n-1)} + \frac{1}{q_i}\left(q_i^{(i+1)} w_i^{(n)} + \sum_{\ell \geqslant i+1} q_i^{(\ell)} w_\ell^{(n)} + \sum_{k \geqslant i+1} q_{ik} M_{ik}^{(n-1)}\right) \\ &= \frac{q_i^{(i+1)}}{q_i}\, w_i^{(n)} + \sum_{\ell \geqslant i+1} \frac{q_i^{(\ell)}}{q_i}\, w_\ell^{(n)} + \frac{n}{q_i}\, M_i^{(n-1)}. \end{aligned}$$

Hence, the assertion holds for n. By the induction, we know that $(w_i^{(n)},\, i \geqslant 1)$ satisfies equation (2.5) for all $n \geqslant 1$. □

The second lemma is presented as follows, for which the proof parallel to the one of [20, Lemma 2.5] and is omitted here.

Lemma 2.5 *Under the conditions of Theorem* 1.1, *fix a positive integer* $n \geqslant 1$ *and define*

$$h_i = n \sum_{k \geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \quad i \geqslant 1. \tag{2.6}$$

Then $(h_i,\, i \geqslant 1)$ *is the minimal non-negative solution of equation* (2.5) *and satisfies*

$$h_i = \frac{1}{q_{i,i-1}}\left(\sum_{\ell \geqslant i+1} q_i^{(\ell)} h_\ell + n M_i^{(n-1)}\right). \tag{2.7}$$

The third lemma is presented as follows.

Lemma 2.6 *Under the conditions of Theorem* 1.1, *given* $i_0 \in E$ *arbitrarily, we have*

$$m_{i,i_0}^{(n)} \leqslant n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)}, \quad i \geqslant i_0.$$

Proof From [3, Theorem 9.5.2], we know that $(m_{i,i_0}^{(n)},\, i \geqslant 0)$ is the minimal non-negative solution to the following equation:

$$x_{i_0} = 0, \quad x_i = \sum_{j \neq i} \frac{q_{ij}}{q_i}\, x_j + \frac{nV(i)}{q_i}\, m_{i,i_0}^{(n-1)},\ i \neq i_0.$$

From [1, Theorem 2.13] (Localization Theorem) and single death property, it follows directly that $(m_{i,i_0}^{(n)},\, i \geqslant i_0)$ is the minimal non-negative solution to the following equation:

$$x_{i_0} = 0, \quad x_i = \sum_{j \neq i, j > i_0} \frac{q_{ij}}{q_i}\, x_j + \frac{nV(i)}{q_i}\, m_{i,i_0}^{(n-1)}, \quad i > i_0. \tag{2.8}$$

Define

$$y_i = n\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)} = \sum_{i_0+1\leqslant k\leqslant i} h_k + M_{i_0+1,i}^{(n-1)}, \quad i\geqslant i_0,$$

where h_i is defined in (2.6). Note that

$$\begin{aligned}
&\sum_{j\neq i_0+1,\, j>i_0}\frac{q_{i_0+1,j}}{q_{i_0+1}}\,y_j + \frac{nV(i)}{q_{i_0+1}}\,m_{i_0+1,i_0}^{(n-1)}\\
&= \sum_{j\geqslant i_0+2}\frac{q_{i_0+1,j}}{q_{i_0+1}}\bigg(\sum_{i_0+1\leqslant k\leqslant j} h_k + M_{i_0+1,j}^{(n-1)}\bigg) + \frac{nV(i)}{q_{i_0+1}}\,w_{i_0+1}^{(n-1)}\\
&= \frac{1}{q_{i_0+1}}\bigg(q_{i_0+1}^{(i_0+2)}h_{i_0+1} + \sum_{k\geqslant i_0+2} q_{i_0+1}^{(k)}h_k + nM_{i_0+1}^{(n-1)}\bigg) \quad (\text{by } (2.2))\\
&= \frac{1}{q_{i_0+1}}\,(q_{i_0+1}^{(i_0+2)}h_{i_0+1} + q_{i_0+1,i_0}h_{i_0+1}) \quad (\text{by } (2.7))\\
&= h_{i_0+1}\\
&= y_{i_0+1}.
\end{aligned}$$

For all $i\geqslant i_0+2$, by the strong Markov property and the single death property, we can check easily that

$$m_{i,i_0}^{(n-1)} = m_{i,i-1}^{(n-1)} + \sum_{s=1}^{n-1}\binom{n-1}{s} m_{i,i-1}^{(n-1-s)} m_{i-1,i_0}^{(s)}. \tag{2.9}$$

Then it is derived that

$$\begin{aligned}
&\sum_{j\neq i,\, j>i_0}\frac{q_{ij}}{q_i}\,y_j + \frac{nV(i)}{q_i}\,m_{i,i_0}^{(n-1)}\\
&= \sum_{i_0+1\leqslant k\leqslant i-1} h_k + \frac{q_i^{(i+1)}}{q_i}\,h_i + \frac{1}{q_i}\sum_{k\geqslant i+1} q_i^{(k)}h_k + M_{i_0+1,i-1}^{(n-1)}\\
&\quad + \sum_{j\geqslant i+1}\frac{q_{ij}}{q_i}\,M_{ij}^{(n-1)} + \frac{nV(i)}{q_i}\,w_i^{(n-1)} + \sum_{j\geqslant i+1}\frac{q_{ij}}{q_i}\sum_{s=1}^{n-1}\binom{n}{s} m_{i-1,i_0}^{(s)} m_{j,i-1}^{(n-s)}\\
&\quad + \frac{nV(i)}{q_i}\sum_{s=1}^{n-1}\binom{n-1}{s} m_{i,i-1}^{(n-1-s)} m_{i-1,i_0}^{(s)} \quad (\text{by } (2.4),\ (2.9))\\
&= \frac{1}{q_i}\sum_{s=1}^{n-1}\binom{n}{s} m_{i-1,i_0}^{(s)}\bigg(\sum_{j\geqslant i+1} q_{ij}m_{j,i-1}^{(n-s)} + (n-s)V(i)m_{i,i-1}^{(n-s-1)}\bigg)\\
&\quad + \sum_{i_0+1\leqslant k\leqslant i} h_k + M_{i_0+1,i-1}^{(n-1)} \quad (\text{by } (2.2),\ (2.7))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_0+1\leqslant k\leqslant i} h_k + M_{i_0+1,i-1}^{(n-1)} + \sum_{s=1}^{n-1}\binom{n}{s} m_{i-1,i_0}^{(s)} m_{i,i-1}^{(n-s)} \quad \text{(by (2.3), Lemma 2.4)}\\
&= \sum_{i_0+1\leqslant k\leqslant i} h_k + M_{i_0+1,i}^{(n-1)} \quad \text{(by (2.4))}\\
&= y_i.
\end{aligned}
$$

Hence, we have checked that $(y_i,\, i \geqslant i_0)$ is a non-negative solution to equation (2.8). Further, it is followed that $m_{i,i_0}^{(n)} \leqslant y_i$ for all $i \geqslant i_0$ from the minimal property of $(m_{i,i_0}^{(n)},\, i \geqslant i_0)$ immediately. So the assertion is proven. □

Now, we prove the main result presented previously.

Proof of Theorem 2.3 On the one hand, by Lemmas 2.4, 2.5, and the minimal property, we obtain that $h_i \leqslant w_i^{(n)}$ for all $i \geqslant 1$, where h_i is defined in (2.6). On the other hand, by Lemma 2.6, it is seen that $w_i^{(n)} \leqslant h_i$ for all $i \geqslant 1$. Hence, it holds that $h_i = w_i^{(n)}$ for all $i \geqslant 1$. The assertion is proven. □

3 Proof of Theorem 1.1

To prove Theorem 1.1, we also need some lemmas. The first lemma is presented as follows.

Lemma 3.1 *Under the conditions of Theorem* 1.1, *we have*

$$
m_{i,i_0}^{(n)} \leqslant n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0 \geqslant 0,\, n \geqslant 1.
$$

Proof Fix i_0, $(m_{i,i_0}^{(n)},\, i \geqslant i_0)$ is the minimal non-negative solution to equation (2.8) as in the proof of Lemma 2.6. Now, rewrite (2.8) as follows:

$$
\begin{cases}
x_{i_0} = 0, \quad x_{i_0+1} = \displaystyle\sum_{j>i_0+1} \frac{q_{i_0+1,j}}{q_{i_0+1}} x_j + \frac{nV(i_0+1)}{q_{i_0+1}} m_{i_0+1,i_0}^{(n-1)},\\
x_s = \dfrac{q_{s,s-1}}{q_s} x_{s-1} + \displaystyle\sum_{j>s} \frac{q_{sj}}{q_s} x_j + \frac{nV(s)}{q_s} m_{s,i_0}^{(n-1)}, \quad s > i_0 + 1.
\end{cases}
$$

Define

$$
z_i = n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0.
$$

Note that

$$\sum_{j>i_0+1} \frac{q_{i_0+1,j}}{q_{i_0+1}} z_j$$

$$= n \sum_{j>i_0+1} \frac{q_{i_0+1,j}}{q_{i_0+1}} \sum_{i_0+1\leqslant k\leqslant j} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}$$

$$= n \sum_{j>i_0+1} \frac{q_{i_0+1,j}}{q_{i_0+1}} \bigg(\sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \sum_{i_0+1<k\leqslant j} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \bigg)$$

$$= \frac{n q_{i_0+1}^{(i_0+2)}}{q_{i_0+1}} \sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n}{q_{i_0+1}} \sum_{k>i_0+1} q_{i_0+1}^{(k)} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}$$

$$= \frac{n q_{i_0+1}^{(i_0+2)}}{q_{i_0+1}} \sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}$$

$$+ \frac{n}{q_{i_0+1}} \sum_{\ell>i_0+1} \frac{V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \sum_{i_0+2\leqslant k\leqslant \ell} q_{i_0+1}^{(k)} G_k^{(\ell)}$$

$$= \frac{n q_{i_0+1}^{(i_0+2)}}{q_{i_0+1}} \sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_{i_0+1,i_0}}{q_{i_0+1}} \sum_{\ell>i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}.$$

Hence, we obtain that

$$\sum_{j>i_0+1} \frac{q_{i_0+1,j}}{q_{i_0+1}} z_j + \frac{nV(i_0+1)}{q_{i_0+1}} m_{i_0+1,i_0}^{(n-1)}$$

$$= \frac{n q_{i_0+1}^{(i_0+2)}}{q_{i_0+1}} \sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_{i_0+1,i_0}}{q_{i_0+1}} \sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}$$

$$= n \sum_{\ell\geqslant i_0+1} \frac{G_{i_0+1}^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}$$

$$= z_{i_0+1}.$$

For $u > i_0 + 1$, it follows that

$$\sum_{j>u} \frac{q_{uj}}{q_u} z_j + \frac{nV(u)}{q_u} m_{u,i_0}^{(n-1)}$$

$$= n \sum_{j>u} \frac{q_{uj}}{q_u} \sum_{i_0+1\leqslant k\leqslant j} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{nV(u)}{q_u} m_{u,i_0}^{(n-1)}$$

$$= n \sum_{j>u} \frac{q_{uj}}{q_u} \sum_{u<k\leqslant j} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + n \sum_{j>u} \frac{q_{uj}}{q_u} \sum_{i_0+1\leqslant k\leqslant u} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}$$

$$
\begin{aligned}
&\quad + \frac{nV(u)}{q_u}\, m_{u,i_0}^{(n-1)} \\
&= \frac{n}{q_u} \sum_{k>u} q_u^{(k)} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_u^{(u+1)}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&\quad + \frac{nV(u)}{q_u}\, m_{u,i_0}^{(n-1)} \\
&= \frac{n}{q_u} \sum_{\ell>u} \frac{V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \sum_{u+1 \leqslant k \leqslant \ell} q_u^{(k)} G_k^{(\ell)} + \frac{n q_u^{(u+1)}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&\quad + \frac{nV(u)}{q_u}\, m_{u,i_0}^{(n-1)} \\
&= \frac{n q_{u,u-1}}{q_u} \sum_{\ell>u} \frac{G_u^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_u^{(u+1)}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&\quad + \frac{nV(u)}{q_u}\, m_{u,i_0}^{(n-1)} \\
&= \frac{n q_{u,u-1}}{q_u} \sum_{\ell \geqslant u} \frac{G_u^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_u^{(u+1)}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
&\frac{q_{u,u-1}}{q_u}\, z_{u-1} + \sum_{j>u} \frac{q_{uj}}{q_u}\, y_j + \frac{nV(u)}{q_u}\, m_{u,i_0}^{(n-1)} \\
&= \frac{n q_{u,u-1}}{q_u} \sum_{i+1 \leqslant k \leqslant u-1} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_{u,u-1}}{q_u} \sum_{\ell \geqslant u} \frac{G_u^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&\qquad + \frac{n q_u^{(u+1)}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&= \frac{n q_{u,u-1}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} + \frac{n q_u^{(u+1)}}{q_u} \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&= n \sum_{i+1 \leqslant k \leqslant u} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} \\
&= z_u.
\end{aligned}
$$

Hence, $(z_i,\, i \geqslant i_0)$ is a non-negative solution of equation (2.8). Meanwhile, it is obtained that $m_{i,i_0}^{(n)} \leqslant z_i$ for all $i \geqslant i_0$ from the minimal property of $m_{i,i_0}^{(n)}$ immediately. So the assertion is proven. □

The second lemma is presented as follows.

Lemma 3.2 *Under the conditions of Theorem* 1.1, *we have*

$$m_{i,i_0}=\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}},\quad i\geqslant i_0\geqslant 0.$$

Proof On the one hand, by Lemma 2.5, when $n=1$, it holds that

$$h_k=\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}},\quad k\geqslant 1,$$

and $(h_k,\ k\geqslant 1)$ is the minimal non-negative solution of the following equation:

$$x_k=\frac{q_k^{(k+1)}}{q_k}\,x_k+\sum_{\ell\geqslant k+1}\frac{q_k^{(\ell)}}{q_k}\,x_\ell+\frac{V(k)}{q_k},\quad k\geqslant 1.$$

By Lemma 2.4, it is seen that $(m_{k,k-1},\ k\geqslant 1)$ satisfies the equation above. Hence, $h_k\leqslant m_{k,k-1}$ for all $k\geqslant 1$. Then

$$\sum_{i_0+1\leqslant k\leqslant i}h_k\leqslant\sum_{i_0+1\leqslant k\leqslant i}m_{k,k-1}=m_{i,i_0},\quad k\geqslant 1.$$

On the other hand, by Lemmas 3.1 and 2.5, when $n=1$, it holds that

$$m_{i,i_0}\leqslant\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}}=\sum_{i_0+1\leqslant k\leqslant i}h_k,\quad i\geqslant i_0\geqslant 0.$$

So the assertion is proven. □

The third lemma is taken from [11, Corollary 1.2] directly. In the following, the definition of $\xi_{i_0}^{(N)}$ is similar to ξ_{i_0}.

Lemma 3.3 *Assume that the single death Q-matrix $Q^{(N)}$ is totally stable, conservative, and irreducible on the finite state space $E_N:=\{0,1,\ldots,N\}$ $(N\geqslant 2)$. Set*

$$m_{i,i_0}^{(N,n)}=\mathbb{E}_i(\xi_{i_0}^{(N)})^n.$$

Then

$$\begin{cases}m_{i,i_0}^{(N,n)}=n\displaystyle\sum_{k=i_0+1}^{i}\sum_{\ell=k}^{N}\frac{G_k^{(\ell)}V(\ell)m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}}, & 0\leqslant i_0<i\leqslant N,\\ m_{i_0,i_0}^{(N,n)}=0.\end{cases}$$

Now, we prove the main result.

Proof of Theorem 1.1 In the following, we prove the assertion by combining induction with limitation approximation. For this, fix $n\geqslant 1$ and $N\geqslant 2$. We copy the approximating construction in the proof of [20, Lemma 2.5] completely, and define the transition rate matrix $Q^{(N)} = (q_{ij}^{(N)}, 0 \leqslant i, j \leqslant N)$ on E_N as follows:

$$q_{ij}^{(N)} = \begin{cases} q_{ij}, & i, j < N, \\ q_i^{(N)}, & i < N,\ j = N, \\ (q_N \vee N)(1 + nG^{(N)} a_N), & i = N,\ j = N-1, \\ -(q_N \vee N)(1 + nG^{(N)} a_N), & i = j = N, \\ 0, & i = N,\ j < N-1, \end{cases}$$

where

$$G^{(N)} = \max_{1 \leqslant i \leqslant N} G_i^{(N)}, \quad a_N = \begin{cases} M_N^{(n-1)}, & M_N^{(n-1)} < \infty, \\ 1, & M_N^{(n-1)} = \infty. \end{cases}$$

Define

$$q_n^{(N,k)} = \sum_{j=k}^{N} q_{nj}^{(N)}, \quad 0 \leqslant n < k \leqslant N,$$

and

$$G_i^{(N,i)} = 1, \quad G_n^{(N,j)} = \frac{1}{q_{n,n-1}^{(N)}} \sum_{n+1 \leqslant k \leqslant i} q_n^{(N,k)} G_k^{(N,i)}, \quad 1 \leqslant n < i \leqslant N.$$

Note that

$$q_{ii}^{(N)} = q_{ii},\ 0 \leqslant i < N, \quad q_i^{(N,k)} = q_i^{(k)},\ 0 \leqslant i < k \leqslant N.$$

Further,

$$G_j^{(N,i)} = G_j^{(i)},\ 1 \leqslant j \leqslant i \leqslant N, \quad V^{(N)}(k) = V(k),\ 1 \leqslant k \leqslant N. \tag{3.1}$$

Now, by Lemma 3.3 and (3.1), for any $n \geqslant 1$, we have

$$\begin{cases} m_{i,i_0}^{(N,n)} = n \displaystyle\sum_{i_0+1 \leqslant k \leqslant i} \sum_{k \leqslant \ell \leqslant N} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}}, & 0 \leqslant i_0 < i \leqslant N, \\ m_{i_0,i_0}^{(N,n)} = 0. \end{cases} \tag{3.2}$$

Now, we will prove two facts as follows. The first fact is

$$m_{i,i_0}^{(N,n)} \uparrow n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \quad N \uparrow \infty. \tag{3.3}$$

The second fact is

$$n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}} = m_{i,i_0}^{(n)}. \tag{3.4}$$

At first, when $n=1$, by (3.2), we obtain that

$$m_{i,i_0}^{(N)}=\sum_{i_0+1\leqslant k\leqslant i}\sum_{k\leqslant \ell\leqslant N}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}^{(N)}}=:\sum_{i_0+1\leqslant k\leqslant i}h_k^{(N)}.$$

Under the assumption of the theorem, similar to the proof of [20, Lemma 2.5], we can get

$$h_k^{(N)}\uparrow\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}},\quad N\uparrow\infty.$$

Further, we derive that the following facts hold:

$$m_{i,i_0}^{(N)}\uparrow\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}},\quad N\uparrow\infty,$$

and

$$\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}}=m_{i,i_0},$$

where the last equality is held by Lemma 3.2. Hence, both (3.3) and (3.4) hold in the case of $n=1$.

Assume that both (3.3) and (3.4) hold until $n-1$. Denote

$$\nu_{i,i_0}^{(n)}:=n\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}.$$

Give k satisfying $i_0+1\leqslant k\leqslant i$. Fix $\ell\geqslant k$. We investigate the monotonicity of the following series:

$$\left\{\frac{G_k^{(\ell)}V(\ell)m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}}\cdot\mathbb{1}_{\{\ell\leqslant N\}}\right\}_{N\geqslant 2}. \tag{3.5}$$

From the construction of $Q^{(N+1)}$ and the assumption above, we know that

$$\begin{aligned}\frac{G_k^{(\ell)}V(\ell)m_{\ell,i_0}^{(N+1,n-1)}}{q_{\ell,\ell-1}^{(N+1)}}\cdot\mathbb{1}_{\{\ell\leqslant N+1\}}&\geqslant\frac{G_k^{(\ell)}V(\ell)m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}^{(N+1)}}\cdot\mathbb{1}_{\{\ell\leqslant N\}}\\&=\frac{G_k^{(\ell)}V(\ell)m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}}\cdot\mathbb{1}_{\{\ell\leqslant N\}}.\end{aligned}$$

Due to

$$q_{N,N-1}^{(N)}\geqslant q_N\geqslant q_{N,N-1},\quad q_{\ell,\ell-1}^{(N)}=q_{\ell,\ell-1},\ 1\leqslant\ell\leqslant N-1,$$

i.e.,

$$q_{\ell,\ell-1}^{(N)} \geqslant q_{\ell,\ell-1}, \quad 1 \leqslant \ell \leqslant N,$$

by the arguments above, we derive that

$$\frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(N+1,n-1)}}{q_{\ell,\ell-1}^{(N+1)}} \cdot \mathbb{1}_{\{\ell \leqslant N+1\}} \geqslant \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} \cdot \mathbb{1}_{\{\ell \leqslant N\}},$$

which means that series (3.5) is increasing. Hence, combining these facts with the assumptions above, we get

$$\frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} \cdot \mathbb{1}_{\{\ell \leqslant N\}} \uparrow \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \quad N \uparrow \infty.$$

Further, by the monotone convergence theorem and (3.2), it is obtained that

$$m_{i,i_0}^{(N,n)} = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) m_{\ell,i_0}^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} \cdot \mathbb{1}_{\{\ell \leqslant N\}} \uparrow \nu_{i,i_0}^{(n)}, \quad N \uparrow \infty,$$

which implies that (3.3) holds for n.

Now, we come to prove (3.4) for n: $m_{i,i_0}^{(n)} = \nu_{i,i_0}^{(n)}$. By Lemma 3.1, one gets that $m_{i,i_0}^{(n)} \leqslant \nu_{i,i_0}^{(n)}$. So we only need to prove the inverse inequality.

To do that, we assume $m_{i,i_0}^{(n)} < \infty$. Otherwise, if $m_{i,i_0}^{(n)} = \infty$, then the inverse inequality holds obviously. Applying Lemma 2.6 and (3.1) on the finite state space E_N directly, we obtain that

$$m_{i,i_0}^{(N,n)} \leqslant n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{k \leqslant \ell \leqslant N} \frac{G_k^{(\ell)} M_\ell^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} + M_{i_0+1,i}^{(N,n-1)}, \quad i_0 \leqslant i \leqslant N, \tag{3.6}$$

where

$$M_{\ell k}^{(N,n-1)} = \sum_{\ell+1 \leqslant u \leqslant k} \sum_{1 \leqslant s \leqslant n-1} \binom{n}{s} m_{u,u-1}^{(N,n-s)} m_{u-1,\ell-1}^{(N,s)}$$

and

$$M_\ell^{(N,n-1)} = V(\ell) m_{\ell,\ell-1}^{(N,n-1)} + \frac{1}{n} \sum_{\ell+1 \leqslant k \leqslant N} q_{\ell k}^{(N)} M_{\ell k}^{(N,n-1)}, \quad \ell \geqslant 1.$$

By the assumption above, it holds that $m_{u,j}^{(N,s)} \uparrow m_{u,j}^{(s)}$ as $N \uparrow \infty$ for all $u \geqslant j \geqslant 0$ and $1 \leqslant s \leqslant n-1$, which implies that

$$M_{\ell k}^{(N,n-1)} \uparrow \sum_{\ell+1 \leqslant u \leqslant k} \sum_{1 \leqslant s \leqslant n-1} \binom{n}{s} m_{u,u-1}^{(n-s)} m_{u-1,\ell-1}^{(s)} = M_{\ell k}^{(n-1)}.$$

Due to

$$\infty > m_{i,i_0}^{(n)} = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)},$$

where the equality is followed from Proposition 2.2 and Theorem 2.3 immediately, by the assumption above, it is seen that

$$\begin{aligned}
\sum_{k \geqslant \ell+1} q_{\ell k}^{(N)} M_{\ell k}^{(N,n-1)} \cdot \mathbb{1}_{\{k \leqslant N\}} &= \sum_{k \geqslant \ell+1} q_{\ell k}^{(N)} M_{\ell k}^{(N,n-1)} \cdot \mathbb{1}_{\{k < N\}} + q_{\ell N}^{(N)} M_{\ell N}^{(N,n-1)} \\
&\leqslant \sum_{k \geqslant \ell+1} q_{\ell k} M_{\ell k}^{(N,n-1)} \cdot \mathbb{1}_{\{k < N\}} + \sum_{k \geqslant N} q_{\ell k} M_{\ell k}^{(N,n-1)} \\
&\leqslant \sum_{k \geqslant \ell+1} q_{\ell k} M_{\ell k}^{(n-1)} \\
&< \infty
\end{aligned}$$

and

$$M_\ell^{(N,n-1)} \leqslant V(\ell) m_{\ell,\ell-1}^{(n-1)} + \frac{1}{n} \sum_{k \geqslant \ell+1} q_{\ell k} M_{\ell k}^{(n-1)} = M_\ell^{(n)}$$

for all $\ell \geqslant i_0 + 1$. Further, by the dominated convergence theorem, we derive that

$$\begin{aligned}
\lim_{N \to \infty} M_\ell^{(N,n-1)} &= \lim_{N \to \infty} V(\ell) m_{\ell,\ell-1}^{(N,n-1)} + \lim_{N \to \infty} \frac{1}{n} \sum_{k \geqslant \ell+1} q_{\ell k}^{(N)} M_{\ell k}^{(N,n-1)} \cdot \mathbb{1}_{\{k \leqslant N\}} \\
&= V(\ell) m_{\ell,\ell-1}^{(n-1)} + \frac{1}{n} \sum_{k \geqslant \ell+1} q_{\ell k} M_{\ell k}^{(n-1)} \\
&= M_\ell^{(n-1)}.
\end{aligned}$$

Now,

$$\sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} \cdot \mathbb{1}_{\{\ell \leqslant N\}} \leqslant \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}^{(N)}} \cdot \mathbb{1}_{\{\ell \leqslant N\}} \uparrow \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} < \infty$$

as $N \uparrow \infty$ due to $m_{i,i_0}^{(n)} < \infty$. Here, the proof for the increasing property of the series $\{\mathbb{1}_{\{\ell \leqslant N\}} / q_{\ell,\ell-1}^{(N)}\}$ is similar to the one for (3.5).

By the arguments above and the dominated convergence theorem, taking the limitation for both sides of (3.6), one obtains that

$$\nu_{i,i_0}^{(n)} \leqslant n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)} = m_{i,i_0}^{(n)}.$$

Hence, we get the inverse inequality holds. So $m_{i,i_0}^{(n)} = \nu_{i,i_0}^{(n)}$ is proven, i.e., (3.4) holds for n.

By induction, (3.3) and (3.4) hold for all $n \geqslant 1$. The proof of the assertion is finished. □

At the end, we apply Theorem 1.1 to more general integral-type downward functionals for single death processes. Let r be such a non-negative function on $\mathbb{R}_+$ that its differential function $r' \geqslant 0$. Fix $i_0 \in E$. Define

$$\xi_{i_0}(r) = \int_0^{\tau_{i_0}} r(t)V(X(t))\mathrm{d}t, \quad \widetilde{V}(i) = \mathbb{E}_i \xi_{i_0}(r'), \quad \overline{V} = \widetilde{V} + r(0)V.$$

From [4, Lemma 2], it follows immediately that

$$\mathbb{E}_i \xi_{i_0}(r) = \mathbb{E}_i \int_0^{\tau_{i_0}} \overline{V}(X(t))\mathrm{d}t, \quad i > i_0.$$

Hence, by Theorem 1.1, it is derived that

$$\mathbb{E}_i \xi_{i_0}(r) = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \overline{V}(\ell)}{q_{\ell,\ell-1}}, \quad i \geqslant i_0 \geqslant 0.$$

In particular, when $r(t) = t^n$, where n is a positive integer, define

$$\xi_{i_0}^{(n)} = \int_0^{\tau_{i_0}} t^n V(X(t))\mathrm{d}t, \quad n \geqslant 0.$$

Note that $\xi_{i_0}^{(0)} = \xi_{i_0}$ and $r(0) = 0$. Then

$$\widetilde{V}(i) = \overline{V}(i) = n\mathbb{E}_i \xi_{i_0}^{(n-1)}.$$

Hence,

$$\mathbb{E}_i \xi_{i_0}^{(n)} = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \xi_{i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0 \geqslant 0,\ n \geqslant 1,$$

and

$$\mathbb{E}_i \xi_{i_0}^{(0)} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}}, \quad i \geqslant i_0 \geqslant 0.$$

The results above can be used to investigate polynomial ergodicity and Central Limit Theorem for single death processes. Refer to our subsequent work on these topics.

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