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RESEARCH ARTICLE

High order moments of first hitting times for single death processes

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Abstract We present an explicit and recursive representation for high order moments of the first hitting times of single death processes. Based on that, some necessary or sufficient conditions of exponential ergodicity as well as a criterion on ℓ-ergodicity are obtained for single death processes, respectively.

Keywords Single death process, moment of the first hitting time, exponential ergodocity
MSC 60J60

1 Introduction

Consider a continuous-time homogeneous Markov chain $\{X(t)\colon t \geqslant 0\}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with transition probability matrix $P(t) = (p_{ij}(t))$ on a countable state space $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. We call $\{X(t)\colon t \geqslant 0\}$ a single death process if its transition rate matrix $Q = (q_{ij}\colon i, j \in \mathbb{Z}_+)$ is irreducible and satisfies that $q_{i,i-1} > 0$ for all $i \geqslant 1$ and $q_{i,i-j} = 0$ for all $i \geqslant j \geqslant 2$. Such a matrix $Q = (q_{ij})$ is called a single death Q-matrix. In the literature, the single death process is also called downwardly skip-free process.

Usually, single death processes are non-symmetric and hence can be regarded as the representative ones of non-symmetric processes. For general single death process, we only have some limited knowledge on stationary distribution and criterion on zero-entrance of them; refer to [3,7]. But as a special kind of single death process, the branching processes are fruitful and applicable intensively, on which one of the main tools used is the generation functions; see [1,2]. Note that the generation function is not valid for general single death process. Recently, in [9], we obtained an explicit representation of the first moments of hitting times for single death processes; further, presented a criterion on ergodicity and strong ergodicity. Meanwhile, some sufficient

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or necessary conditions for recurrence and exponential ergodicity as well as extinction probability for the single death processes were also derived in [9]. This paper is a continuous work of [9]. We focus on obtaining some recursive representations of high order moments of hitting times for single death processes.

Define the first hitting time

$$\tau_i := \inf\{t > 0\colon X_t = i\}, \quad \forall\, i \geqslant 0,$$

the first jumping time

$$\eta_1 := \inf\{t > 0\colon X_t \neq X_0\}, \tag{1.1}$$

and the first returning time

$$\sigma_i := \inf\{t > \eta_1\colon X_t = i\}, \quad \forall\, i \geqslant 0.$$

For birth-death process, define

$$a_i = q_{i,i-1},\ i \geqslant 1, \quad b_i = q_{i,i+1},\ i \geqslant 0,$$

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i},\ i \geqslant 1.$$

By [8], it is well known that

$$\mathbb{E}_i \tau_0^n = n \sum_{j=1}^{i} \frac{1}{\mu_j a_j} \sum_{k=j}^{\infty} \mu_k \mathbb{E}_k \tau_0^{n-1}, \quad i, n \geqslant 1.$$

The tool they used is the functional with integral type downward and approximation approach. In [10], we obtained a similar representation for birth-death processes on trees, using a different way. Although in [10], we used the symmetrizable property in our proof, we are conscious of the fact that the key property is the single death rather than the symmetrization. This is the motivation of this paper and the ideas or approaches are originated from [10].

Throughout the paper, we consider only totally stable and conservative single death Q-matrix:

$$q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty, \quad \forall\, i \in \mathbb{Z}_+.$$

The following sequences are used throughout this paper:

$$q_n^{(k)} = \sum_{j=k}^{\infty} q_{nj}, \quad k > n \geqslant 0,$$

and

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)},\ 1 \leqslant n < i.$$

It is easily known that

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \sum_{k=n}^{i-1} \frac{G_n^{(k)} q_k^{(i)}}{q_{k,k-1}}, \ 1 \leqslant n < i.$$

The main result of this paper is as follows.

Theorem 1.1 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Give $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then*

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0 + 1.$$

This paper is organized as follows. Some relations between moments from different starting states and another explicit representation of hitting times are given in the next section. Then Section 3 is devoted to the proof of Theorem 1.1. In Section 4, some explicit necessary or sufficient conditions of exponential ergodicity as well as a criterion on ℓ-ergodicity are obtained for single death processes, respectively.

2 Another representation of hitting times

First, we consider the relation between the n-th moments from different starting states. Define

$$m_i^{(n)} := \mathbb{E}_i \tau_{i-1}^n, \quad i \geqslant 1, \, n \geqslant 0;$$

in particular,

$$m_i^{(0)} = 1.$$

We denote $m_i^{(1)}$ by m_i simply. Recall that the Polynomial Theorem for the polynomial with ℓ variables and power n is described as follows:

$$(x_1 + x_2 + \cdots + x_\ell)^n = \sum_{(n_1, n_2, \ldots, n_\ell) \in \mathscr{A}_\ell} \frac{n!}{n_1! \, n_2! \cdots n_\ell!} x_1^{n_1} x_2^{n_2} \cdots x_\ell^{n_\ell},$$

where

$$\mathscr{A}_\ell = \{(k_1, k_2, \ldots, k_\ell) \in \mathbb{Z}_+^\ell \mid k_1 + k_2 + \cdots + k_\ell = n\}.$$

Analogously, we define the 'polynomial' with parentheses-power (n):

$$(m_i + m_{i+1} + \cdots + m_\ell)^{(n)} := \sum_{(n_i, n_{i+1}, \ldots, n_\ell) \in \mathscr{A}_{\ell-i+1}} \frac{n!}{n_i! \, n_{i+1}! \cdots n_\ell!} m_i^{(n_i)} \cdots m_\ell^{(n_\ell)}.$$

Then we obtain the following relation between the n-th moments from different starting states.

Theorem 2.1 *Under the conditions of Theorem* 1.1, *for any* $i_0 \in \mathbb{Z}_+$, *we have*

$$\mathbb{E}_i \tau_{i_0}^n = (m_{i_0+1} + m_{i_0+2} + \cdots + m_i)^{(n)}, \quad i > i_0,\, n \geqslant 1.$$

Proof It follows from the definition above that

$$(m_{i_0+1} + m_{i_0+2} + \cdots + m_i)^{(1)} = \sum_{k=i_0+1}^{i} m_k^{(1)} = \sum_{k=i_0+1}^{i} \mathbb{E}_k \tau_{k-1} = \mathbb{E}_i \tau_{i_0},$$

in which the last equality holds by the single death and strong Markov property. So the assertion holds in the case of $n = 1$.

Assume that the assertion holds until $n-1$. By the single death and strong Markov property, it is seen that for all $i > i_0 + 1$,

$$\begin{aligned}
\mathbb{E}_i \tau_{i_0}^n &= \mathbb{E}_i (\tau_{i-1} + \tau_{i_0} - \tau_{i-1})^n \\
&= \mathbb{E}_i \tau_{i-1}^n + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i ((\tau_{i_0} - \tau_{i-1})^s \tau_{i-1}^{n-s}) + \mathbb{E}_i (\tau_{i_0} - \tau_{i-1})^n \\
&= \mathbb{E}_i \tau_{i-1}^n + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i (\mathbb{E}_i ((\tau_{i_0} - \tau_{i-1})^s \tau_{i-1}^{n-s} \mid \mathscr{F}_{\tau_{i-1}})) \\
&\quad + \mathbb{E}_i (\mathbb{E}_i (\tau_{i_0} - \tau_{i-1})^n \mid \mathscr{F}_{\tau_{i-1}}) \\
&= \mathbb{E}_i \tau_{i-1}^n + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i \tau_{i-1}^{n-s} \mathbb{E}_{i-1} \tau_{i_0}^s + \mathbb{E}_{i-1} \tau_{i_0}^n.
\end{aligned}$$

Note that $\mathbb{E}_{i_0} \tau_{i_0}^s = 0$. Hence, the equality above is true for all $i > i_0$. Furthermore,

$$\begin{aligned}
\mathbb{E}_i \tau_{i_0}^n &= \mathbb{E}_i \tau_{i-1}^n + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_i \tau_{i-1}^{n-s} \mathbb{E}_{i-1} \tau_{i_0}^s + \mathbb{E}_{i-1} \tau_{i-2}^n \\
&\quad + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_{i-1} \tau_{i-2}^{n-s} \mathbb{E}_{i-2} \tau_{i_0}^s + \mathbb{E}_{i-2} \tau_{i_0}^n \\
&= \sum_{\ell=i-1}^{i} \mathbb{E}_\ell \tau_{\ell-1}^n + \sum_{\ell=i-1}^{i} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \tau_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \tau_{i_0}^s + \mathbb{E}_{i-2} \tau_{i_0}^n.
\end{aligned}$$

Hence, we can obtain recursively that

$$\mathbb{E}_i \tau_{i_0}^n = \sum_{\ell=i_0+1}^{i} \mathbb{E}_\ell \tau_{\ell-1}^n + \sum_{\ell=i_0+2}^{i} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \tau_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \tau_{i_0}^s.$$

By the assumption, we know that

$$\mathbb{E}_i \tau_{i_0}^n = \sum_{\ell=i_0+1}^{i} m_\ell^{(n)} + \sum_{\ell=i_0+2}^{i} \sum_{s=1}^{n-1} \binom{n}{s} m_\ell^{(n-s)} (m_{i_0+1} + m_{i_0+2} + \cdots + m_{\ell-1})^{(s)}. \quad (2.1)$$

If

$$(m_{i_0+1}+m_{i_0+2}+\cdots+m_i)^{(n)}=\infty,$$

then there exist $p\in[i_0+1,i]$ and $k\in[1,n]$ such that $m_p^{(k)}=\infty$; furthermore, $m_p^{(n)}=\infty$. By (2.1), we know that

$$\mathbb{E}_i\tau_{i_0}^n=\infty=(m_{i_0+1}+m_{i_0+2}+\cdots+m_i)^{(n)}.$$

If

$$(m_{i_0+1}+m_{i_0+2}+\cdots+m_i)^{(n)}<\infty,$$

then

$$(m_{i_0+1}+m_{i_0+2}+\cdots+m_\ell)^{(n)}<\infty,\quad m_\ell^{(n)}<\infty,\quad \forall\,\ell\in[i_0+1,i].$$

Furthermore, it follows that

$$\begin{aligned}
&\sum_{\ell=i_0+2}^{i}\sum_{s=1}^{n-1}\binom{n}{s}m_\ell^{(n-s)}(m_{i_0+1}+m_{i_0+2}+\cdots+m_{\ell-1})^{(s)}\\
&=\sum_{\ell=i_0+2}^{i}((m_{i_0+1}+m_{i_0+2}+\cdots+m_\ell)^{(n)}\\
&\qquad-(m_{i_0+1}+m_{i_0+2}+\cdots+m_{\ell-1})^{(n)}-m_\ell^{(n)})\\
&=(m_{i_0+1}+m_{i_0+2}+\cdots+m_i)^{(n)}-\sum_{\ell=i_0+1}^{i}m_\ell^{(n)}.
\end{aligned}$$

Hence, by the arguments above and (2.1), it holds that

$$\mathbb{E}_i\tau_{i_0}^n=(m_{i_0+1}+m_{i_0+2}+\cdots+m_i)^{(n)}.$$

That is, the assertion holds in the case of n. By induction, the assertion holds for all $n\geqslant 1$. The proof is finished. □

By Theorem 2.1, for representing $\mathbb{E}_i\tau_{i_0}^n$, it suffices to get the explicit formular of $\mathbb{E}_\ell\tau_{\ell-1}^{(s)}$. To do this, we need some notations as follows. Define

$$M_{ik}^{(n-1)}=\sum_{i+1\leqslant\ell\leqslant k}\sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_\ell\tau_{\ell-1}^{n-s}\mathbb{E}_{\ell-1}\tau_{i-1}^{s},\quad k\geqslant i+1,\ i\geqslant 1,$$

and $M_{ik}^{(n-1)}=0$ if $i\geqslant k\geqslant 1$. Define

$$M_i^{(n-1)}=m_i^{(n-1)}+\frac{1}{n}\sum_{k\geqslant i+1}q_{ik}M_{ik}^{(n-1)},\quad n,i\geqslant 1. \tag{2.2}$$

First, we introduce some properties about the definitions above.

Proposition 2.2 *Under the conditions of Theorem* 1.1, *the following assertions hold*:

$$\mathbb{E}_i\tau_{i_0}^n = \sum_{i_0+1\leqslant \ell\leqslant i} m_\ell^{(n)} + M_{i_0+1,i}^{(n-1)}, \quad i\geqslant i_0,\ n\geqslant 1, \tag{2.3}$$

and

$$M_{ik}^{(n-1)} = M_{ij}^{(n-1)} + M_{j+1,k}^{(n-1)} + \sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_k\tau_j^{n-s}\mathbb{E}_j\tau_{i-1}^s, \quad 1\leqslant i\leqslant j\leqslant k,\ n\geqslant 1. \tag{2.4}$$

Proof Obviously, (2.3) is obtained by (2.1) directly.

To prove (2.4), using the strong Markov property and single death property, for all $i\leqslant j\leqslant k$, we derive that

$$\begin{aligned}
M_{ik}^{(n-1)} &= M_{ij}^{(n-1)} + \sum_{j+1\leqslant \ell\leqslant k}\sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_\ell\tau_{\ell-1}^{n-s}\mathbb{E}_{\ell-1}\tau_{i-1}^s\\
&= M_{ij}^{(n-1)} + \sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_{j+1}\tau_j^{n-s}\mathbb{E}_j\tau_{i-1}^s\\
&\quad + \sum_{j+2\leqslant \ell\leqslant k}\sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_\ell\tau_{\ell-1}^{n-s}\mathbb{E}_{\ell-1}\tau_{i-1}^s\\
&= M_{ij}^{(n-1)} + \sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_{j+1}\tau_j^{n-s}\mathbb{E}_j\tau_{i-1}^s\\
&\quad + \sum_{j+2\leqslant \ell\leqslant k}\sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_\ell\tau_{\ell-1}^{n-s}\Big(\mathbb{E}_{\ell-1}\tau_j^s\\
&\quad + \sum_{1\leqslant u\leqslant s}\binom{s}{u}\mathbb{E}_{\ell-1}\tau_j^{s-u}\mathbb{E}_j\tau_{i-1}^u\Big)\\
&= M_{ij}^{(n-1)} + M_{j+1,k}^{(n-1)} + \sum_{1\leqslant s\leqslant n-1}\binom{n}{s}\mathbb{E}_{j+1}\tau_j^{n-s}\mathbb{E}_j\tau_{i-1}^s\\
&\quad + \sum_{j+2\leqslant \ell\leqslant k}\sum_{1\leqslant s\leqslant n-1}\mathbb{E}_\ell\tau_{\ell-1}^{n-s}\sum_{1\leqslant u\leqslant s}\binom{n}{u}\binom{n-u}{s-u}\mathbb{E}_{\ell-1}\tau_j^{s-u}\mathbb{E}_j\tau_{i-1}^u\\
&=: M_{ij}^{(n-1)} + M_{j+1,k}^{(n-1)} + \mathrm{I}.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathrm{I} = &\sum_{1\leqslant u\leqslant n-1}\binom{n}{u}\mathbb{E}_j\tau_{i-1}^u\sum_{j+2\leqslant \ell\leqslant k}\sum_{u\leqslant s\leqslant n-1}\binom{n-u}{s-u}\mathbb{E}_\ell\tau_{\ell-1}^{n-s}\mathbb{E}_{\ell-1}\tau_j^{s-u}\\
&+\sum_{1\leqslant u\leqslant n-1}\binom{n}{u}\mathbb{E}_{j+1}\tau_j^{n-u}\mathbb{E}_j\tau_{i-1}^u
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\leqslant u\leqslant n-1}\binom{n}{u}\mathbb{E}_j\tau_{i-1}^u\bigg(\sum_{j+2\leqslant \ell\leqslant k}\sum_{0\leqslant v\leqslant n-u-1}\binom{n-u}{v}\mathbb{E}_\ell\tau_{\ell-1}^{n-u-v}\mathbb{E}_{\ell-1}\tau_j^v \\
&\quad + \mathbb{E}_{j+1}\tau_j^{n-u}\bigg) \\
&= \sum_{1\leqslant u\leqslant n-1}\binom{n}{u}\mathbb{E}_j\tau_{i-1}^u\bigg(\sum_{j+2\leqslant \ell\leqslant k}\sum_{1\leqslant v\leqslant n-u-1}\binom{n-u}{v}\mathbb{E}_\ell\tau_{\ell-1}^{n-u-v}\mathbb{E}_{\ell-1}\tau_j^v \\
&\quad + \sum_{j+1\leqslant \ell\leqslant k}\mathbb{E}_\ell\tau_{\ell-1}^{n-u}\bigg) \\
&= \sum_{1\leqslant u\leqslant n-1}\binom{n}{u}\mathbb{E}_k\tau_j^{n-u}\mathbb{E}_j\tau_{i-1}^u,
\end{aligned}
$$

in which the last equality holds in virtue of (2.1). Hence, by the argument above, the proof of (2.4) is finished. □

Our result on the representation of $m_\ell^{(s)}$ by the above notations is presented as follows.

Theorem 2.3 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then*

$$\mathbb{E}_i\tau_{i-1}^n = n\sum_{k\geqslant i}\frac{G_i^{(k)}M_k^{(n-1)}}{q_{k,k-1}}, \quad i,n\geqslant 1.$$

Hence, combining Theorems 2.1 and 2.3, we obtain another representation of $\mathbb{E}_i\tau_{i_0}^n$ for $i > i_0$.

To prove Theorem 2.3, we need three lemmas. The first lemma is presented as follows.

Lemma 2.4 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then, for all $n \geqslant 1$, $(\mathbb{E}_i\tau_{i-1}^n,\ i\geqslant 1)$ satisfies the following equation*:

$$x_i = \frac{q_i^{(i+1)}}{q_i}x_i + \sum_{\ell\geqslant i+1}\frac{q_i^{(\ell)}}{q_i}x_\ell + \frac{n}{q_i}M_i^{(n-1)}, \quad i\geqslant 1. \tag{2.5}$$

Proof To prove the assertion, we use the induction. By [9, Remark 2.8], we know that $(\mathbb{E}_i\tau_{i-1},\ i\geqslant 1)$ satisfies the following equation:

$$x_i = \frac{q_i^{(i+1)}}{q_i}x_i + \sum_{\ell\geqslant i+1}\frac{q_i^{(\ell)}}{q_i}x_\ell + \frac{1}{q_i}, \quad i\geqslant 1.$$

Hence, the assertion holds when $n = 1$.

Assume that the assertion holds until $n-1$. Define the first jumping time η_1 as in (1.1). By the strong Markov and the single death properties, we derive that

$$\begin{aligned}\mathbb{E}_i\tau_{i-1}^n &= \mathbb{E}_i\eta_1^n + \sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_i((\tau_{i-1}-\eta_1)^s\eta_1^{n-s})) + \mathbb{E}_i(\tau_{i-1}-\eta_1)^n \\ &= \frac{n!}{q_i^n} + \sum_{s=1}^{n-1}\binom{n}{s}\sum_{k=i+1}^{\infty}\mathbb{E}_i(\eta_1^{n-s}\mathbb{1}_{\{X_{\eta_1}=k\}}\mathbb{E}_{X_{\eta_1}}\tau_{i-1}^s) \\ &\quad + \sum_{k=i+1}^{\infty}\mathbb{E}_i(\mathbb{1}_{\{X_{\eta_1}=k\}}\mathbb{E}_{X_{\eta_1}}\tau_{i-1}^n) \\ &= \frac{n!}{q_i^n} + \sum_{s=1}^{n-1}\binom{n}{s}\sum_{k=i+1}^{\infty}\frac{(n-s)!}{q_i^{n-s}}\cdot\frac{q_{ik}}{q_i}\,\mathbb{E}_k\tau_{i-1}^s + \sum_{k=i+1}^{\infty}\frac{q_{ik}}{q_i}\,\mathbb{E}_k\tau_{i-1}^n.\end{aligned}$$

It follows from (2.3) that

$$\begin{aligned}\mathbb{E}_i\tau_{i-1}^n &= \frac{n!}{q_i^n} + \sum_{s=1}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\sum_{k=i+1}^{\infty}q_{ik}\Big(\sum_{\ell=i}^{k}m_\ell^{(s)} + M_{ik}^{(s-1)}\Big) \\ &\quad + \sum_{k=i+1}^{\infty}\frac{q_{ik}}{q_i}\Big(\sum_{\ell=i}^{k}m_\ell^{(n)} + M_{ik}^{(n-1)}\Big) \\ &= \frac{n!}{q_i^n} + \sum_{s=1}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\Big(q_i^{(i+1)}m_i^{(s)} + \sum_{\ell=i+1}^{\infty}q_i^{(\ell)}m_\ell^{(s)} + \sum_{k=i+1}^{\infty}q_{ik}M_{ik}^{(s-1)}\Big) \\ &\quad + \frac{1}{q_i}\Big(q_i^{(i+1)}m_i^{(n)} + \sum_{\ell=i+1}^{\infty}q_i^{(\ell)}m_\ell^{(n)} + \sum_{k=i+1}^{\infty}q_{ik}M_{ik}^{(n-1)}\Big) \\ &=: \mathrm{I} + \frac{1}{q_i}\Big(q_i^{(i+1)}m_i^{(n)} + \sum_{\ell=i+1}^{\infty}q_i^{(\ell)}m_\ell^{(n)} + \sum_{k=i+1}^{\infty}q_{ik}M_{ik}^{(n-1)}\Big).\end{aligned}$$

By the assumption, we get that

$$\begin{aligned}\mathrm{I} &= \frac{n!}{q_i^n}\Big(q_i^{(i+1)}m_i^{(1)} + \sum_{\ell=i+1}^{\infty}q_i^{(\ell)}m_\ell^{(1)} + 1\Big) \\ &\quad + \sum_{s=2}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\Big(q_i^{(i+1)}m_i^{(s)} + \sum_{\ell=i+1}^{\infty}q_i^{(\ell)}m_\ell^{(s)} + \sum_{k=i+1}^{\infty}q_{ik}M_{ik}^{(s-1)}\Big) \\ &= \frac{n!}{q_i^{n-1}}\,m_i^{(1)} + \sum_{s=2}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\Big(q_i^{(i+1)}m_i^{(s)} + \sum_{\ell=i+1}^{\infty}q_i^{(\ell)}m_\ell^{(s)} + \sum_{k=i+1}^{\infty}q_{ik}M_{ik}^{(s-1)}\Big).\end{aligned}$$

Furthermore, by the assumption, it is derived that

$$
\begin{aligned}
\mathrm{I} = {} & \frac{n!}{2!\,q_i^{n-1}}\bigg(q_i^{(i+1)}m_i^{(2)} + \sum_{\ell=i+1}^{\infty} q_i^{(\ell)} m_\ell^{(2)} + 2m_i^{(1)} + \sum_{k=i+1}^{\infty} q_{ik}M_{ik}^{(1)}\bigg) \\
& + \sum_{s=3}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\bigg(q_i^{(i+1)}m_i^{(s)} + \sum_{\ell=i+1}^{\infty} q_i^{(\ell)} m_\ell^{(s)} + \sum_{k=i+1}^{\infty} q_{ik}M_{ik}^{(s-1)}\bigg) \\
= {} & \frac{n!}{2!\,q_i^{n-1}}\bigg(q_i^{(i+1)}m_i^{(2)} + \sum_{\ell=i+1}^{\infty} q_i^{(\ell)} m_\ell^{(2)} + 2M_i^{(1)}\bigg) \\
& + \sum_{s=3}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\bigg(q_i^{(i+1)}m_i^{(s)} + \sum_{\ell=i+1}^{\infty} q_i^{(\ell)} m_\ell^{(s)} + \sum_{k=i+1}^{\infty} q_{ik}M_{ik}^{(s-1)}\bigg) \\
= {} & \frac{n!}{2!\,q_i^{n-2}}\, m_i^{(2)} \\
& + \sum_{s=3}^{n-1}\frac{n!}{s!\,q_i^{n-s+1}}\bigg(q_i^{(i+1)}m_i^{(s)} + \sum_{\ell=i+1}^{\infty} q_i^{(\ell)} m_\ell^{(s)} + \sum_{k=i+1}^{\infty} q_{ik}M_{ik}^{(s-1)}\bigg).
\end{aligned}
$$

Recursively, it follows that

$$
\mathrm{I} = \frac{n!}{(n-1)!\,q_i}\, m_i^{(n-1)}.
$$

Furthermore,

$$
\begin{aligned}
\mathbb{E}_i\tau_{i-1}^n = {} & \frac{n!}{(n-1)!\,q_i}\, m_i^{(n-1)} \\
& + \frac{1}{q_i}\bigg(q_i^{(i+1)}m_i^{(n)} + \sum_{\ell=i+1}^{\infty} q_i^{(\ell)} m_\ell^{(n)} + \sum_{k=i+1}^{\infty} q_{ik}M_{ik}^{(n-1)}\bigg) \\
= {} & \frac{q_i^{(i+1)}}{q_i}\, m_i^{(n)} + \sum_{\ell=i+1}^{\infty} \frac{q_i^{(\ell)}}{q_i}\, m_\ell^{(n)} + \frac{n}{q_i}\, M_i^{(n-1)}.
\end{aligned}
$$

Hence, the assertion holds for n. By the induction, we know that $(m_i^{(n)},\, i \geqslant 1)$ satisfies equation (2.5) for all $n \geqslant 1$. □

The following lemma is the second one.

Lemma 2.5 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Fix a positive integer $n \geqslant 1$. Define*

$$
h_i = n\sum_{k\geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \quad i \geqslant 1. \tag{2.6}
$$

Then $(h_i,\, i \geqslant 1)$ is the minimal nonnegative solution of equation (2.5) and satisfies

$$
h_i = \frac{1}{q_{i,i-1}}\bigg(\sum_{\ell\geqslant i+1} q_i^{(\ell)} h_\ell + nM_i^{(n-1)}\bigg). \tag{2.7}
$$

Proof Fix a positive integer $N \geqslant 2$, and define a Q-matrix $Q^{(N)} = (\widetilde{q}_{ij})$ on $\{0, 1, \ldots, N\}$ as follows:

$$\widetilde{q}_{ij} = \begin{cases} q_{ij}, & i, j < N, \\ q_i^{(N)}, & i < N,\, j = N, \\ (q_N \vee N)(1 + nG^{(N)} a_N), & i = N,\, j = N-1, \\ -(q_N \vee N)(1 + nG^{(N)} a_N), & i = j = N, \\ 0, & i = N,\, j < N-1. \end{cases}$$

where

$$G^{(N)} = \max_{1 \leqslant i \leqslant N} G_i^{(N)}, \quad a_N = \begin{cases} M_N^{(n-1)}, & M_N^{(n-1)} < \infty, \\ 1, & M_N^{(n-1)} = \infty. \end{cases}$$

Define

$$\widetilde{q}_n^{(k)} = \sum_{j=k}^{N} \widetilde{q}_{nj}, \quad 0 \leqslant n < k \leqslant N,$$

and

$$\widetilde{G}_i^{(i)} = 1, \quad \widetilde{G}_n^{(i)} = \frac{1}{\widetilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \widetilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \quad 1 \leqslant n < i \leqslant N.$$

It is easy to check that

$$h_i^{(N)} := n \sum_{k=i}^{N} \frac{\widetilde{G}_i^{(k)} M_k^{(n-1)}}{\widetilde{q}_{k,k-1}}, \quad 1 \leqslant i \leqslant N, \tag{2.8}$$

is a unique solution (the minimal non-negative solution) to the following equation:

$$x_i = \frac{\widetilde{q}_i^{(i+1)}}{\widetilde{q}_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{\widetilde{q}_i^{(\ell)}}{\widetilde{q}_i} \cdot x_\ell + \frac{n}{\widetilde{q}_i} M_i^{(n-1)}, \quad 1 \leqslant i \leqslant N. \tag{2.9}$$

Note that

$$\widetilde{q}_i := -\widetilde{q}_{ii} = -q_{ii} = q_i,\ 0 \leqslant i < N, \quad \widetilde{q}_i^{(k)} = q_i^{(k)},\ 0 \leqslant i < k \leqslant N.$$

Furthermore,

$$\widetilde{G}_j^{(i)} = G_j^{(i)}, \quad 1 \leqslant j \leqslant i \leqslant N.$$

Hence, we can rewrite (2.9) as

$$x_i = \begin{cases} \dfrac{q_i^{(i+1)}}{q_i} \cdot x_i + \displaystyle\sum_{\ell=i+1}^{N} \frac{q_i^{(\ell)}}{q_i} \cdot x_\ell + \frac{n}{q_i} M_i^{(n-1)}, & 1 \leqslant i \leqslant N-1, \\ \dfrac{n M_N^{(n-1)}}{(q_N \vee N)(1 + nG^{(N)} a_N)}, & i = N. \end{cases} \tag{2.10}$$

On the one hand, from (2.5) and (2.10), by [5, Theorem 2.7], we know that $(h_i^{(N)})$ is increasing to the minimal non-negative solution of (2.5) as $N \to \infty$. On the other hand, it follows from (2.8) that

$$h_i^{(N)} = n \sum_{k=i}^{N-1} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}} + \frac{n G_i^{(N)} M_N^{(n-1)}}{(q_N \vee N)(1 + n G^{(N)} a_N)}.$$

Combining the equality with the definition of a_N, it is not difficult to check that

$$\lim_{N\to\infty} h_i^{(N)} = \begin{cases} n \sum_{k=i}^{\infty} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, & M_N^{(n-1)} < \infty, \forall N \geqslant 2, \\ \infty, & \text{otherwise}, \end{cases}$$

i.e.,

$$\lim_{N\to\infty} h_i^{(N)} = h_i, \quad \forall i \geqslant 1.$$

So it has proven that $(h_i,\, i \geqslant 1)$ is the minimal non-negative solution of (2.8). Finially, it is not difficult to check that $(h_i,\, i \geqslant 1)$ satisfies the equality (2.7). The proof of the assertions are finished. □

The third lemma is presented as follows.

Lemma 2.6 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Give $i_0 \in E$ arbitrarily. Then*

$$\mathbb{E}_i \tau_{i_0}^n \leqslant n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)}, \quad i \geqslant i_0.$$

Proof It is well known that $(\mathbb{E}_i \tau_{i_0}^n,\, i \geqslant 0)$ is the minimal non-negative solution to the following equation:

$$x_{i_0} = 0, \quad x_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1}, \; i \neq i_0.$$

From [5, Theorem 2.13] (Localization Theorem) and the single death property, it follows directly that $(\mathbb{E}_i \tau_{i_0}^n,\, i \geqslant i_0)$ is the minimal non-negative solution to the following equation:

$$x_{i_0} = 0, \quad x_i = \sum_{j \neq i, j > i_0} \frac{q_{ij}}{q_i} \cdot x_j + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1}, \; i > i_0. \tag{2.11}$$

Define

$$y_i = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)} = \sum_{i_0+1 \leqslant k \leqslant i} h_k + M_{i_0+1,i}^{(n-1)}, \quad i \geqslant i_0,$$

where h_i is defined in (2.6). Note that

$$\begin{aligned}
&\sum_{j>i_0+1}\frac{q_{i_0+1,j}}{q_{i_0+1}}\cdot y_j+\frac{n}{q_{i_0+1}}\,\mathbb{E}_{i_0+1}\tau_{i_0}^{n-1}\\
&\quad=\sum_{j\geqslant i_0+2}\frac{q_{i_0+1,j}}{q_{i_0+1}}\Big(\sum_{i_0+1\leqslant k\leqslant j}h_k+M_{i_0+1,j}^{(n-1)}\Big)+\frac{n}{q_{i_0+1}}\,m_{i_0+1}^{(n-1)}\\
&\quad=\frac{1}{q_{i_0+1}}\Big(q_{i_0+1}^{(i_0+2)}h_{i_0+1}+\sum_{k\geqslant i_0+2}q_{i_0+1}^{(k)}h_k+nM_{i_0+1}^{(n-1)}\Big)\quad(\text{by }(2.2))\\
&\quad=\frac{1}{q_{i_0+1}}\,(q_{i_0+1}^{(i_0+2)}h_{i_0+1}+q_{i_0+1,i_0}h_{i_0+1})\quad(\text{by }(2.7))\\
&\quad=h_{i_0+1}\\
&\quad=y_{i_0+1}.
\end{aligned}$$

For all $i\geqslant i_0+2$, by the strong Markov property and the single death property, we can check easily that

$$\mathbb{E}_i\tau_{i_0}^{n-1}=\mathbb{E}_i\tau_{i-1}^{n-1}+\sum_{s=1}^{n-1}\binom{n-1}{s}\mathbb{E}_i\tau_{i-1}^{n-1-s}\mathbb{E}_{i-1}\tau_{i_0}^{s}. \tag{2.12}$$

Then it is derived that

$$\begin{aligned}
&\sum_{j\neq i,\,j>i_0}\frac{q_{ij}}{q_i}\cdot y_j+\frac{n}{q_i}\,\mathbb{E}_i\tau_{i_0}^{n-1}\\
&=\sum_{i_0+1\leqslant k\leqslant i-1}h_k+\frac{q_i^{(i+1)}}{q_i}\,h_i+\frac{1}{q_i}\sum_{k\geqslant i+1}q_i^{(k)}h_k+M_{i_0+1,i-1}^{(n-1)}\\
&\quad+\sum_{j\geqslant i+1}\frac{q_{ij}}{q_i}\,M_{ij}^{(n-1)}+\frac{n}{q_i}\,m_i^{(n-1)}+\sum_{j\geqslant i+1}\frac{q_{ij}}{q_i}\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{i-1}\tau_{i_0}^{s}\mathbb{E}_j\tau_{i-1}^{n-s}\\
&\quad+\frac{n}{q_i}\sum_{s=1}^{n-1}\binom{n-1}{s}\mathbb{E}_i\tau_{i-1}^{n-1-s}\mathbb{E}_{i-1}\tau_{i_0}^{s}\quad(\text{by }(2.4),\,(2.12))\\
&=\frac{1}{q_i}\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{i-1}\tau_{i_0}^{s}\Big(\sum_{j\geqslant i+1}q_{ij}\mathbb{E}_j\tau_{i-1}^{n-s}+(n-s)\mathbb{E}_i\tau_{i-1}^{n-s-1}\Big)\\
&\quad+\sum_{i_0+1\leqslant k\leqslant i}h_k+M_{i_0+1,i-1}^{(n-1)}\quad(\text{by }(2.2),\,(2.7))\\
&=\sum_{i_0+1\leqslant k\leqslant i}h_k+M_{i_0+1,i-1}^{(n-1)}+\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{i-1}\tau_{i_0}^{s}\mathbb{E}_i\tau_{i-1}^{n-s}\quad(\text{by }(2.3),\text{ Lemma 2.4})\\
&=\sum_{i_0+1\leqslant k\leqslant i}h_k+M_{i_0+1,i}^{(n-1)}\quad(\text{by }(2.4))\\
&=y_i.
\end{aligned}$$

Hence, we have checked that $(y_i,\, i \geqslant i_0)$ is a non-negative solution to (2.11). Furthermore, it is followed that $\mathbb{E}_i\tau_{i_0}^n \leqslant y_i$ for all $i \geqslant i_0$ from the minimal property of $(\mathbb{E}_i\tau_{i_0}^n,\, i \geqslant i_0)$ immediately. So the assertion is proven. □

Now, we prove the main result presented previously.

Proof of Theorem 2.3 On the one hand, by Lemmas 2.4, 2.5, and the minimal property, we obtain that $h_i \leqslant m_i^{(n)}$ for all $i \geqslant 1$, where h_i is defined in (2.6). On the other hand, by Lemma 2.6, it is seen that $m_i^{(n)} \leqslant h_i$ for all $i \geqslant 1$. Hence, it holds that $h_i = m_i^{(n)}$ for all $i \geqslant 1$. The assertion is proven. □

3 Proof of Theorem 1.1

To prove Theorem 1.1, we still need some lemmas. The first lemma is taken from [9, Proposition 2.4] directly.

Lemma 3.1 *For all* $1 \leqslant i \leqslant v < u$, *the following relation holds*:

$$G_i^{(u)} = \sum_{i \leqslant k \leqslant v} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell=v+1}^{u} q_k^{(\ell)} G_\ell^{(u)}.$$

The second lemma is presented as follows.

Lemma 3.2 *Assume that the single death Q-matrix is regular and the process is recurrent. Then, for any nonnegative sequence* $\{a_m\}$ *and* $n \geqslant 1$, *the following equality holds*:

$$\sum_{k \geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1 \geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1}\tau_{\ell_1-1}^n \sum_{\ell_2=k}^{\ell_1-1} a_{\ell_2} = n \sum_{u \geqslant i+1} \frac{G_i^{(u)} M_u^{(n-1)}}{q_{u,u-1}} \sum_{\ell_2=i}^{u-1} a_{\ell_2}.$$

Proof Denote the left-hand side of the equality above by I. By Theorem 2.3 and Lemma 3.1, it is seen that

$$\begin{aligned}
\mathrm{I} &= n \sum_{k \geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1 \geqslant k+1} q_k^{(\ell_1)} \sum_{u \geqslant \ell_1} \frac{G_{\ell_1}^{(u)} M_u^{(n-1)}}{q_{u,u-1}} \sum_{\ell_2=k}^{\ell_1-1} a_{\ell_2} \\
&= n \sum_{u \geqslant i+1} \frac{M_u^{(n-1)}}{q_{u,u-1}} \sum_{\ell_2=i}^{u-1} a_{\ell_2} \sum_{k=i}^{\ell_2} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1=\ell_2+1}^{u} q_k^{(\ell_1)} G_{\ell_1}^{(u)} \\
&= n \sum_{u \geqslant i+1} \frac{G_i^{(u)} M_u^{(n-1)}}{q_{u,u-1}} \sum_{\ell_2=i}^{u-1} a_{\ell_2}.
\end{aligned}$$

The assertion is proven. □

The third lemma is an important relation formula.

Lemma 3.3 *Assume that the single death Q-matrix is regular and the process is recurrent. Then, for any $1 \leqslant i \leqslant k < \ell_1$ and $n \geqslant 2$, the following assertion holds*:

$$M_{k,\ell_1-1}^{(n-1)} + \sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-s}\mathbb{E}_{k-1}\tau_{i-1}^{s} = \sum_{\ell_2=k}^{\ell_1-1}\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s}\mathbb{E}_{\ell_2-1}\tau_{i-1}^{s}.$$

Proof When $k=i$, it is obvious that both sides of the above formula are equal to $M_{k,\ell_1-1}^{(n-1)}$. So the assertion holds.

Assume that $i<k<\ell_1$. Denote the right-hand side of the equality above by I. Note that

$$\begin{aligned}
\mathrm{I} = & \sum_{\ell_2=k}^{\ell_1-1}\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s}\sum_{s_1=1}^{s}\binom{s}{s_1}\mathbb{E}_{\ell_2-1}\tau_{k-1}^{s-s_1}\mathbb{E}_{k-1}\tau_{i-1}^{s_1} \\
& + \sum_{\ell_2=k}^{\ell_1-1}\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s}\mathbb{E}_{\ell_2-1}\tau_{k-1}^{s} \\
= & \sum_{\ell_2=k}^{\ell_1-1}\sum_{s_1=1}^{n-1}\binom{n}{s_1}\mathbb{E}_{k-1}\tau_{i-1}^{s_1}\sum_{s=s_1}^{n-1}\binom{n-s_1}{s-s_1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s}\mathbb{E}_{\ell_2-1}\tau_{k-1}^{s-s_1} + M_{k,\ell_1-1}^{(n-1)} \\
= & M_{k,\ell_1-1}^{(n-1)} + \sum_{s_1=1}^{n-1}\binom{n}{s_1}\mathbb{E}_{k-1}\tau_{i-1}^{s_1}\sum_{\ell_2=k}^{\ell_1-1}\sum_{s=0}^{n-s_1-1}\binom{n-s_1}{s}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s_1-s}\mathbb{E}_{\ell_2-1}\tau_{k-1}^{s}.
\end{aligned}$$

If $\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1}<\infty$, then $\mathbb{E}_{\ell_2}\tau_{k-1}^{n-1}<\infty$ for all $k \leqslant \ell_2 \leqslant \ell_1-1$. In fact, by the strong Markov property and the single death property, we see that

$$\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1} = \sum_{s=0}^{n-2}\binom{n-1}{s}\mathbb{E}_{\ell_1-1}\tau_{\ell_1-2}^{n-1-s}\mathbb{E}_{\ell_1-2}\tau_{k-1}^{s} + \mathbb{E}_{\ell_1-2}\tau_{k-1}^{n-1}; \tag{3.1}$$

so $\mathbb{E}_{\ell_1-2}\tau_{k-1}^{n-1}<\infty$. Recursively, it is derived that $\mathbb{E}_{\ell_2}\tau_{k-1}^{n-1}<\infty$ $(k \leqslant \ell_2 \leqslant \ell_1-1)$. Hence,

$$\begin{aligned}
\mathrm{I} &= M_{k,\ell_1-1}^{(n-1)} + \sum_{s_1=1}^{n-1}\binom{n}{s_1}\mathbb{E}_{k-1}\tau_{i-1}^{s_1}\sum_{\ell_2=k}^{\ell_1-1}(\mathbb{E}_{\ell_2}\tau_{k-1}^{n-s_1} - \mathbb{E}_{\ell_2-1}\tau_{k-1}^{n-s_1}) \\
&= M_{k,\ell_1-1}^{(n-1)} + \sum_{s_1=1}^{n-1}\binom{n-1}{s_1}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-s_1}\mathbb{E}_{k-1}\tau_{i-1}^{s_1}.
\end{aligned}$$

If $\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1}=\infty$, then, by (3.1), either there exists such $s\in[1,n-1]$ that $\mathbb{E}_{\ell_1-1}\tau_{\ell_1-2}^{s}=\infty$, or $\mathbb{E}_{\ell_1-2}\tau_{k-1}^{n-1}=\infty$. For the former case, both sides of the above formulae are ∞ and hence the assertion holds. For the latter case, then either there exists such $s\in[1,n-1]$ that $\mathbb{E}_{\ell_1-2}\tau_{\ell_1-3}^{s}=\infty$, or $\mathbb{E}_{\ell_1-3}\tau_{k-1}^{n-1}=\infty$. The argument goes on. Recursively, in the end either there exists such $s \in [1, n-1]$ that $\mathbb{E}_{k+1}\tau_k^s = \infty$, or $\mathbb{E}_k\tau_{k-1}^{n-1} = \infty$. Now, both sides of the above formula are ∞, which implies that the assertion holds. The proof is finished. $\square$

By Theorem 2.3, Lemmas 3.2 and 3.3, we can obtain the following result immediately.

Lemma 3.4 *Assume that the single death Q-matrix is regular and the process is recurrent. Then, for all* $1 \leqslant s_1 < n$, *the following equality holds*:

$$\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\left(M_{k,\ell_1-1}^{(s_1-1)}+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\mathbb{E}_{k-1}\tau_{i-1}^{s_2}\right)$$
$$=(n-s_1)\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}M_u^{(n-s_1-1)}M_{i,u-1}^{(s_1-1)}.$$

Now, we prove the main result.

Proof of Theorem 1.1 At first, we will prove

$$h_i = n\sum_{k\geqslant i}\frac{G_i^{(k)}\mathbb{E}_k\tau_{i-1}^{n-1}}{q_{k,k-1}},\quad i\geqslant 1,$$

where h_i is defined in Lemma 2.5 (see (2.6)).

By [9, Remark 2.8], we know that the assertion holds for $n=1$. For all $n\geqslant 2$, we obtain that

$$\begin{aligned}
&\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_0\geqslant k+1}q_{k\ell_0}M_{k\ell_0}^{(n-1)}\\
&=\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_0\geqslant k+1}q_{k\ell_0}\sum_{\ell_1=k+1}^{\ell_0}\sum_{1\leqslant s_1\leqslant n-1}\binom{n}{s_1}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1}\\
&=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{\ell_2=i}^{u-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-1}\\
&\quad+n\sum_{1\leqslant s_1\leqslant n-2}\binom{n-1}{s_1}\sum_{u\geqslant i+1}\frac{G_i^{(u)}M_u^{(n-s_1-1)}}{q_{u,u-1}}\sum_{\ell_2=i}^{u-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_1}\\
&\quad+\sum_{2\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}M_{k,\ell_1-1}^{(s_1-1)}.
\end{aligned}$$

From Lemma 3.2, it follows that

$$h_i = n\sum_{u\geqslant i}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{\ell_2=i}^{u}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-1}$$

$$
\begin{aligned}
&+ n \sum_{1\leqslant s_1\leqslant n-2} \binom{n-1}{s_1} \sum_{u\geqslant i+1} \frac{G_i^{(u)} M_u^{(n-s_1-1)}}{q_{u,u-1}} \sum_{\ell_2=i}^{u-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_1} \\
&+ \sum_{2\leqslant s_1\leqslant n-1} \binom{n}{s_1} \sum_{k\geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^{n-s_1} M_{k,\ell_1-1}^{(s_1-1)}. \\
=:&\ \mathrm{I} + \mathrm{V} + \mathrm{IV}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\mathrm{V} =&\ n \sum_{1\leqslant s_1\leqslant n-2} \binom{n-1}{s_1} \sum_{u\geqslant i+1} \frac{G_i^{(u)} \mathbb{E}_u \tau_{u-1}^{n-s_1-1}}{q_{u,u-1}} \sum_{\ell_2=i}^{u-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_1} \\
&+ \sum_{1\leqslant s_1\leqslant n-2} \binom{n}{s_1} \sum_{u\geqslant i+1} \frac{G_i^{(u)}}{q_{u,u-1}} \sum_{j\geqslant u+1} q_{uj} M_{uj}^{(n-s_1-1)} \sum_{\ell_2=i}^{u-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_1} \\
=&\ n \sum_{u\geqslant i+1} \frac{G_i^{(u)}}{q_{u,u-1}} \sum_{1\leqslant s_1\leqslant n-2} \binom{n-1}{s_1} \mathbb{E}_u \tau_{u-1}^{n-1-s_1} \sum_{\ell_2=i}^{u-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_1} + \sum_{2\leqslant s_1\leqslant n-1} \binom{n}{s_1} \\
&\times \sum_{k\geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^{n-s_1} \sum_{s_2=1}^{s_1-1} \binom{s_1}{s_2} \mathbb{E}_{\ell_1-1} \tau_{k-1}^{s_1-s_2} \sum_{i\leqslant \ell_2\leqslant k-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_2} \\
=:&\ \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Then, by Lemma 3.4, it is derived that

$$
\begin{aligned}
\mathrm{III} + \mathrm{IV} =&\ \mathbb{1}_{\{n\geqslant 3\}} \binom{n}{2} \sum_{k\geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^{n-2} \big(2\mathbb{E}_{\ell_1-1} \tau_{k-1} \mathbb{E}_{k-1} \tau_{i-1} \\
&+ M_{k,\ell_1-1}^{(1)}\big) + \sum_{3\leqslant s_1\leqslant n-1} \binom{n}{s_1} \sum_{k\geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^{n-s_1} \\
&\times \bigg(\sum_{s_2=1}^{s_1-1} \binom{s_1}{s_2} \mathbb{E}_{\ell_1-1} \tau_{k-1}^{s_1-s_2} \sum_{i\leqslant \ell_2\leqslant k-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_2} + M_{k,\ell_1-1}^{(s_1-1)} \bigg) \\
=&\ \binom{n}{2} (n-2) \sum_{u\geqslant i+1} \frac{G_i^{(u)} M_u^{(n-3)}}{q_{u,u-1}} M_{i,u-1}^{(1)} \\
&+ \sum_{3\leqslant s_1\leqslant n-1} \binom{n}{s_1} \sum_{k\geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^{n-s_1} \\
&\times \bigg(\sum_{s_2=1}^{s_1-1} \binom{s_1}{s_2} \mathbb{E}_{\ell_1-1} \tau_{k-1}^{s_1-s_2} \sum_{\ell_2=i}^{k-1} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{s_2} + M_{k,\ell_1-1}^{(s_1-1)} \bigg) \\
=&\ n \binom{n-1}{2} \sum_{u\geqslant i+1} \frac{G_i^{(u)}}{q_{u,u-1}} \mathbb{E}_u \tau_{u-1}^{n-3} M_{i,u-1}^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbb{1}_{\{n\geqslant 4\}}\binom{n}{2}\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{j\geqslant u+1}q_{uj}M_{uj}^{(n-3)}M_{i,u-1}^{(1)}\\
&+\sum_{3\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\\
&\times\bigg(\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)\\
=:\ & n\binom{n-1}{2}\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\mathbb{E}_u\tau_{u-1}^{n-3}M_{i,u-1}^{(1)}+\mathrm{J};
\end{aligned}
$$

furthermore,

$$
\begin{aligned}
\mathrm{J}=\ &\mathbb{1}_{\{n\geqslant 4\}}\binom{n}{2}\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{\ell_1\geqslant u+1}q_u^{(\ell_1)}\sum_{s_1=1}^{n-3}\binom{n-2}{s_1}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-2-s_1}\\
&\times\mathbb{E}_{\ell_1-1}\tau_{u-1}^{s_1}M_{i,u-1}^{(1)}+\sum_{3\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\\
&\times\bigg(\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)\\
=\ &\sum_{3\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}\\
&+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg).
\end{aligned}
$$

It follows from Lemma 3.4 that

$$
\begin{aligned}
\mathrm{J}=\ &\sum_{4\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}\\
&+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)+\mathbb{1}_{\{n\geqslant 4\}}\binom{n}{3}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\\
&\times\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-3}\bigg(\sum_{s_2=1}^{2}\binom{3}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{3-s_2}\mathbb{E}_{k-1}\tau_{i-1}^{s_2}+M_{k,\ell_1-1}^{(2)}\bigg)\\
=\ &\sum_{4\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}\\
&+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)
\end{aligned}
$$

$$+n\binom{n-1}{3}\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}M_u^{(n-4)}M_{i,u-1}^{(2)}.$$

Hence, we obtain that

$$\begin{aligned}
&\mathrm{III}+\mathrm{IV}\\
&=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{3}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}\\
&\quad+\sum_{4\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}\\
&\quad+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)\\
&\quad+\mathbb{1}_{\{n\geqslant 5\}}\binom{n}{3}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{j\geqslant k+1}q_{kj}M_{kj}^{(n-4)}M_{i,k-1}^{(2)}\\
&=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{3}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}\\
&\quad+\sum_{4\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}\\
&\quad+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)+\mathbb{1}_{\{n\geqslant 5\}}\binom{n}{3}\\
&\quad\times\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\sum_{s_1=4}^{n-1}\binom{n-3}{s_1-3}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-3}M_{i,k-1}^{(2)}.
\end{aligned}$$

So, by Lemma 3.4, we get that

$$\begin{aligned}
&\mathrm{III}+\mathrm{IV}\\
&=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{3}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}+\sum_{4\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\\
&\quad\times\sum_{\ell_1\geqslant k+1}q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}+\binom{s_1}{3}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-3}M_{i,k-1}^{(2)}\\
&\quad+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)\\
&=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{3}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}+\mathbb{1}_{\{n\geqslant 5\}}\binom{n}{4}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}
\end{aligned}$$

$$\times \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-4}\bigg(\sum_{s_2=1}^{3}\binom{4}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{4-s_2}\mathbb{E}_{k-1}\tau_{i-1}^{s_2}+M_{k,\ell_1-1}^{(3)}\bigg)$$

$$+\sum_{5\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}$$

$$+\binom{s_1}{3}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-3}M_{i,k-1}^{(2)}+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)$$

$$=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{3}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}+n\binom{n-1}{4}\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}$$

$$\times M_u^{(n-5)}M_{i,u-1}^{(3)}+\sum_{5\leqslant s_1\leqslant n-1}\binom{n}{s_1}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}$$

$$\times\bigg(\binom{s_1}{2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-2}M_{i,k-1}^{(1)}+\binom{s_1}{3}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-3}M_{i,k-1}^{(2)}$$

$$+\sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(s_1-1)}\bigg)$$

$$=\cdots$$

$$=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{n-3}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}+n\binom{n-1}{n-2}\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}$$

$$\times M_u^{(1)}M_{i,u-1}^{(n-3)}+n\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}$$

$$\times\bigg(\sum_{s_2=1}^{n-3}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}M_{i,k-1}^{(s_2-1)}$$

$$+\sum_{s_2=1}^{n-2}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(n-2)}\bigg),$$

where the last equality is derived recursively. Hence, by Lemma 3.4, it is seen that

$$\text{III}+\text{IV}$$

$$=n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{n-2}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}$$

$$+n\binom{n-1}{n-2}\frac{1}{2}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{j\geqslant k+1}q_{kj}M_{kj}^{(1)}M_{i,k-1}^{(n-3)}$$

$$
\begin{aligned}
&+ n\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}\bigg(\sum_{s_2=1}^{n-3}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}M_{i,k-1}^{(s_2-1)}\\
&+\sum_{s_2=1}^{n-2}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(n-2)}\bigg)\\
=\,& n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{n-2}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)}\\
&+ n\binom{n-1}{n-2}\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}\mathbb{E}_{\ell_1-1}\tau_{k-1}M_{i,k-1}^{(n-3)}\\
&+ n\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}\bigg(\sum_{s_2=1}^{n-3}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}M_{i,k-1}^{(s_2-1)}\\
&+\sum_{s_2=1}^{n-2}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}\sum_{\ell_2=i}^{k-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_2}+M_{k,\ell_1-1}^{(n-2)}\bigg)\\
=\,& n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=2}^{n-2}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)} + n\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\\
&\times\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}\bigg(\sum_{s_2=1}^{n-2}\binom{n-1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-1-s_2}\mathbb{E}_{k-1}\tau_{i-1}^{s_2}+M_{k,\ell_1-1}^{(n-2)}\bigg)\\
=\,& n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=1}^{n-2}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}M_{i,u-1}^{(s_1-1)} + n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}M_{i,u-1}^{(n-2)}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\mathrm{V}+\mathrm{IV} &= \mathrm{II}+\mathrm{III}+\mathrm{IV}\\
&= n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\sum_{s_1=1}^{n-2}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}\bigg(\sum_{\ell_2=i}^{u-1}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{s_1}+M_{i,u-1}^{(s_1-1)}\bigg)\\
&\quad + n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}M_{i,u-1}^{(n-2)};
\end{aligned}
$$

furthermore,

$$
\begin{aligned}
\mathrm{V}+\mathrm{IV} &= n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}\bigg(\sum_{s_1=1}^{n-2}\binom{n-1}{s_1}\mathbb{E}_u\tau_{u-1}^{n-1-s_1}\mathbb{E}_{u-1}\tau_{i-1}^{s_1}+M_{i,u-1}^{(n-2)}\bigg)\\
&= n\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}M_{iu}^{(n-2)}.
\end{aligned}
$$

So

$$h_i = \mathrm{I} + \mathrm{V} + \mathrm{IV} = n \sum_{u \geqslant i} \frac{G_i^{(u)}}{q_{u,u-1}} \left(\sum_{\ell_2 = i}^{u} \mathbb{E}_{\ell_2} \tau_{\ell_2 - 1}^{n-1} + M_{iu}^{(n-2)} \right) = n \sum_{u \geqslant i} \frac{G_i^{(u)} \mathbb{E}_u \tau_{i-1}^{n-1}}{q_{u,u-1}}.$$

Summing up the arguments above, we obtain that

$$\mathbb{E}_i \tau_{i-1}^n = h_i = n \sum_{k \geqslant i} \frac{G_i^{(k)} \mathbb{E}_k \tau_{i-1}^{n-1}}{q_{k,k-1}}, \quad i, n \geqslant 1.$$

For any $0 \leqslant k < i$, we see that

$$\mathbb{E}_i \tau_k^n = \sum_{j=k+1}^{i} \left(\mathbb{E}_j \tau_{j-1}^n + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_j \tau_{j-1}^s \cdot \mathbb{E}_{j-1} \tau_k^{n-s} \right).$$

Furthermore, we obtain that

$$\begin{aligned}
\mathbb{E}_i \tau_k^n &= \sum_{k+1 \leqslant j \leqslant i} \left(n \sum_{\ell \geqslant j} \frac{G_j^{(\ell)} \mathbb{E}_\ell \tau_{j-1}^{n-1}}{q_{\ell,\ell-1}} + \sum_{s=1}^{n-1} \binom{n}{s} \cdot s \sum_{\ell \geqslant j} \frac{G_j^{(\ell)} \mathbb{E}_\ell \tau_{j-1}^{s-1}}{q_{\ell,\ell-1}} \cdot \mathbb{E}_{j-1} \tau_k^{n-s} \right) \\
&= n \sum_{k+1 \leqslant j \leqslant i} \sum_{\ell \geqslant j} \frac{G_j^{(\ell)}}{q_{\ell,\ell-1}} \left(\mathbb{E}_\ell \tau_{j-1}^{n-1} + \sum_{s=1}^{n-1} \binom{n-1}{s-1} \mathbb{E}_\ell \tau_{j-1}^{s-1} \cdot \mathbb{E}_{j-1} \tau_k^{n-s} \right) \\
&= n \sum_{k+1 \leqslant j \leqslant i} \sum_{\ell \geqslant j} \frac{G_j^{(\ell)}}{q_{\ell,\ell-1}} \left(\mathbb{E}_\ell \tau_{j-1}^{n-1} + \sum_{s=0}^{n-2} \binom{n-1}{s} \mathbb{E}_\ell \tau_{j-1}^{s} \cdot \mathbb{E}_{j-1} \tau_k^{n-1-s} \right) \\
&= n \sum_{k+1 \leqslant j \leqslant i} \sum_{\ell \geqslant j} \frac{G_j^{(\ell)}}{q_{\ell,\ell-1}} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \mathbb{E}_\ell \tau_{j-1}^{\ell} \cdot \mathbb{E}_{j-1} \tau_k^{n-1-\ell} \\
&= n \sum_{k+1 \leqslant j \leqslant i} \sum_{\ell \geqslant j} \frac{G_j^{(\ell)} \mathbb{E}_\ell \tau_k^{n-1}}{q_{\ell,\ell-1}}, \quad 0 \leqslant k < i,\ n \geqslant 1.
\end{aligned}$$

So the assertion holds for all $n \geqslant 1$. The proof of Theorem 1.1 is finished. □

In the end of this section, let us consider the following example, which comes from [9].

Example 3.5 Given a constant $b > 2$ (for regularity, we only need that $b > 1$). Define a totally stable, conservative, and irreducible single death Q-matrix $Q = (q_{ij})$ as follows:

$$q_{ij} = \frac{b-1}{b^{j-i+2}}, \quad j \geqslant i+1; \quad q_{i,i-1} = \frac{b-1}{b}, \quad q_i = -q_{ii} = \frac{b^2 - b + 1}{b^2}, \quad i \geqslant 1;$$
$$q_{0j} = \frac{b-1}{b^{j+1}}, \quad j \geqslant 1; \quad q_0 = -q_{00} = \frac{1}{b}.$$

By [9], we know that the corresponding process is exponentially ergodic but not strongly ergodic. Here,

$$q_n^{(k)} = \frac{1}{b^{k-n+1}},\ 1 \leqslant n < k, \quad q_0^{(k)} = \frac{1}{b^k},\ k \geqslant 1,$$
$$G_i^{(i)} = 1, \quad G_n^{(i)} = \frac{1}{b(b-1)^{i-n}}, \quad 1 \leqslant n < i,$$

and

$$\mathbb{E}_i \tau_{i_0} = (i - i_0)\frac{b-1}{b-2}, \quad i > i_0.$$

For $1 \leqslant k < \ell$, we know that

$$M_{k\ell}^{(1)} = (\ell - k)(\ell - k + 1)\frac{(b-1)^2}{(b-2)^2}, \quad M_k^{(1)} = \frac{b^2 - 3b + 3}{(b-2)^2};$$

furthermore, for $i \geqslant 1$,

$$\mathbb{E}_i \tau_{i-1}^2 = 2\sum_{k \geqslant i} \frac{G_i^{(k)} M_k^{(1)}}{q_{k,k-1}} = \frac{2(b-1)(b^2-3b+3)}{(b-2)^3} = 2\sum_{k \geqslant i} \frac{G_i^{(k)} \mathbb{E}_k \tau_{i-1}}{q_{k,k-1}}.$$

From here, we have checked Theorems 2.3 and 1.1 in the case of $n = 2$, by the last part in the proof of Theorem 1.1. By the way, we have

$$\mathbb{E}_\ell \tau_k^2 = \frac{(\ell-k)(\ell-k+1)(b-1)^2}{(b-2)^2} + \frac{2(\ell-k)(b-1)}{(b-2)^3}, \quad 0 \leqslant k < \ell.$$

4 Exponential ergodicity and ℓ-ergodicity

By Theorem 1.1, we can get one necessary condition for the exponential ergodicy of the single death processes as follows.

Corollary 4.1 *Assume that the single death Q-matrix is regular, irreducible, and the corresponding process is exponentially ergodic. Then*

$$\delta := \sup_{i \geqslant 1} \sum_{1 \leqslant k \leqslant i} \sum_{\ell \geqslant i} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

Proof By the exponential ergodicy of the process, we see that the process is recurrent. Then

$$\mathbb{E}_i \tau_0^n = n \sum_{1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_0^{n-1}}{q_{\ell,\ell-1}}, \quad i \geqslant 0,\ n \geqslant 1.$$

By the fact that $\mathbb{E}_i\tau_0^{n-1} \leqslant \mathbb{E}_\ell\tau_0^{n-1}$ for all $\ell \geqslant i$ and Theorem 1.1, one gets that

$$\begin{aligned}
\mathbb{E}_i\tau_0^n &\geqslant n \sum_{1\leqslant k\leqslant i}\sum_{\ell\geqslant i}\frac{G_k^{(\ell)}\mathbb{E}_\ell\tau_0^{n-1}}{q_{\ell,\ell-1}}\\
&\geqslant n\bigg(\sum_{1\leqslant k\leqslant i}\sum_{\ell\geqslant i}\frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}\bigg)\mathbb{E}_i\tau_0^{n-1}\\
&\geqslant \cdots\\
&\geqslant n!\bigg(\sum_{1\leqslant k\leqslant i}\sum_{\ell\geqslant i}\frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}\bigg)^n, \quad n\geqslant 1.
\end{aligned}$$

From the exponential ergodicity, by [5, Theorem 4.44 (2)], there exists some λ with $0 < \lambda < q_i$ for all i such that $\mathbb{E}_0\mathrm{e}^{\lambda\sigma_0} < \infty$. Then, by [5, p. 148], it holds that $\mathbb{E}_i\mathrm{e}^{\lambda\sigma_0} < \infty$, i.e., $\mathbb{E}_i\mathrm{e}^{\lambda\tau_0} < \infty$ for all $i \geqslant 1$. The Taylor expansion

$$\infty > \mathbb{E}_i\mathrm{e}^{\lambda\tau_0} = \sum_{n\geqslant 0}\frac{\lambda^n}{n!}\,\mathbb{E}_i\tau_0^n$$

leads us to

$$\infty > \sum_{n\geqslant 0}\bigg(\lambda\sum_{1\leqslant k\leqslant i}\sum_{\ell\geqslant i}\frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}\bigg)^n,$$

which implies that

$$\lambda\sum_{1\leqslant k\leqslant i}\sum_{\ell\geqslant i}\frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < 1.$$

Taking the supremum over $i \geqslant 1$, we obtain

$$\delta = \sup_{i\geqslant 1}\sum_{1\leqslant k\leqslant i}\sum_{\ell\geqslant i}\frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

The proof is finished. □

Given a positive integer ℓ. Another application of Theorem 1.1 is a criterion on ℓ-ergodicity of single death processes. A recurrent Q-process is called to be ℓ-ergodic provided that $\mathbb{E}_j\sigma_j^\ell < \infty$ for some (equivalently, all) $j \in \mathbb{Z}_+$; refer to [6]. Then we obtain the following result.

Corollary 4.2 *Assume that the single death Q-matrix is regular, irreducible, and the corresponding process is recurrent. Then the process is ℓ-ergodic if and only if*

$$d(\ell) := \sum_{k\geqslant 1} q_0^{(k)}\sum_{j\geqslant k}\frac{G_k^{(j)}}{q_{j,j-1}}\mathbb{E}_j\tau_0^{\ell-1} < \infty.$$

Proof By the strong Markov and the single death properties, we derive that

$$\begin{aligned}\mathbb{E}_0\sigma_0^n &= \mathbb{E}_0(\eta_1+\sigma_0-\eta_1)^n\\ &= \mathbb{E}_0\eta_1^n + \sum_{s=1}^n \binom{n}{s}\mathbb{E}_0((\sigma_0-\eta_1)^s\eta_1^{n-s}))\\ &= \frac{n!}{q_0^n} + \sum_{s=1}^n \binom{n}{s}\sum_{k=1}^\infty \mathbb{E}_0(\eta_1^{n-s}\mathbb{1}_{\{X_{\eta_1}=k\}}\mathbb{E}_{X_{\eta_1}}\tau_0^s)\\ &= \frac{n!}{q_0^n} + \sum_{s=1}^n \binom{n}{s}\sum_{k=1}^\infty \frac{(n-s)!}{q_0^{n-s}}\cdot\frac{q_{0k}}{q_0}\,\mathbb{E}_k\tau_0^s\\ &= \frac{n!}{q_0^n} + \sum_{s=1}^n \binom{n}{s}\sum_{k=1}^\infty \frac{(n-s)!}{q_0^{n-s}}\cdot\frac{q_{0k}}{q_0}\cdot s\sum_{1\leqslant i\leqslant k}\sum_{\ell\geqslant i}\frac{G_i^{(\ell)}\mathbb{E}_\ell\tau_0^{s-1}}{q_{\ell,\ell-1}}\\ &= \frac{n!}{q_0^n} + \sum_{s=1}^n \frac{n!}{(s-1)!\,q_0^{n-s+1}}\sum_{i\geqslant 1}q_0^{(i)}\sum_{\ell\geqslant i}\frac{G_i^{(\ell)}\mathbb{E}_\ell\tau_0^{s-1}}{q_{\ell,\ell-1}}.\end{aligned}$$

By the argument above, it is easy to check that $\mathbb{E}_0\sigma_0^n$ is finite if and only if

$$\sum_{i\geqslant 1}q_0^{(i)}\sum_{\ell\geqslant i}\frac{G_i^{(\ell)}\mathbb{E}_\ell\tau_0^{n-1}}{q_{\ell,\ell-1}} < \infty.$$

Hence, the assertion holds. □

By the way, we can get a sufficient condition of exponential ergodicity for singled death processes as follows.

Proposition 4.3 *Let the single death Q-matrix be regular and irreducible. Assume that*

$$\sum_{k\geqslant 1}\frac{q_0^{(k)}}{G_1^{(k)}} < \infty.$$

If

$$q := \inf_{n\geqslant 0} q_n > 0, \quad M := \sup_{n\geqslant 1}\Big(\sum_{k=1}^n \frac{1}{G_1^{(k)}}\Big)\Big(\sum_{\ell=n}^\infty \frac{G_1^{(\ell)}}{q_{\ell,\ell-1}}\Big) < \infty,$$

then the process is exponentially ergodic.

The condition above is a little different from the one in [9] but the proof of them are similar. The main construction idea of test functions comes from [4]. So we omit the detailed proof here.

By Lemma 3.1, it is easy to check that $G_1^{(k)}G_k^{(\ell)} \leqslant G_1^{(\ell)}$ for all $1\leqslant k\leqslant \ell$. So it is obvious that $M \geqslant \delta$.

Come back to Example 3.5. Now, we know that

$$\sum_{k\geqslant 1}\frac{q_0^{(k)}}{G_1^{(k)}} = \frac{b^2-b+1}{b}, \quad \delta = \frac{b^2-3b+3}{(b-2)^2}, \quad M = \frac{b(b-1)}{(b-2)^2}.$$

So the process is exponentially ergodic. Obviously, $M > \delta$ because $b > 2$ here. Note that for this example, the quantity of M is equal to the corresponding one in [9].

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