Front. Math. China 2019, 14(4): 833–854
https://doi.org/10.1007/s11464-019-0784-6

RESEARCH ARTICLE

Moments of first hitting times for birth-death processes on trees

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Abstract An explicit and recursive representation is presented for moments of the first hitting times of birth-death processes on trees. Based on that, the criteria on ergodicity, strong ergodicity, and ℓ-ergodicity of the processes as well as a necessary condition for exponential ergodicity are obtained.

Keywords Birth-death process on trees, ergodicity, hitting time, returning time
MSC 60J60

1 Introduction

The tree T we considered in this paper is a connected graph without cycles. We fix a point on T as the root, denoted by o. For any vertex $i \in T \setminus \{o\}$, there is a unique simple path from i to the root o. Denote $\mathscr{P}(i)$ the set of all the vertices on this path (the root o is excluded). The number of segments of this path is the length of i, denoted by $|i|$ and set $|o| = 0$. Define

$$E_n = \{i \in T \colon |i| = n\}, \quad \forall\, n \geqslant 0.$$

Two vertices i and j are called adjacent if they are joined by a segment, denoted by $i \sim j$. When $|j| = |i| + 1$ and $i \sim j$, j is called one offspring of i and the set of all the offsprings of i is denoted by $J(i)$. When $|j| = |i| - 1$ and $i \sim j$, j is called the father of i and denoted by i^*. Denote T_i the subtree with i as its root, including all the descendants of i. Thereafter, we assume that any vertex has finite offsprings.

We consider a birth-death process on this tree whose Q-matrix satisfies $q_{ij} > 0$ if and only if $i \sim j$, i.e., $j = i^*$, or $j \in J(i)$. In this paper, assume that

Received September 4, 2018; August 3, 2019
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the Q-matrix is totally stable and conservative, that is,

$$q_i := -q_{ii} = q_{ii^*} + \sum_{j \in J(i)} q_{ij} < \infty, \quad \forall\, i \in T.$$

Define a measure μ on T as follows:

$$\mu_o = 1, \quad \mu_i = \prod_{j \in \mathscr{P}(i)} \frac{q_{j^*j}}{q_{jj^*}},\ i \in T \setminus \{o\},$$

which is invariant with respect to Q. In fact, μ satisfies the so-called detailed balance equation:

$$\mu_i q_{ij} = \mu_j q_{ji}, \quad i \sim j. \tag{1.1}$$

For estimation of the spectral gap λ_1 on trees, Miclo [7] proposed a more explicit quantity defined in a recursive way to bound λ_1 by a factor 16. Note that Miclo's quantity is still quite difficult to be identified. Shao and Mao [9] obtained a variational formula for the Dirichlet eigenvalue. Ma [4] gave some explicit upper and lower bounds for the Dirichlet eigenvalue. Wang and Zhang [10] obtained three kinds of variational formulas for the Dirichlet eigenvalue. For finite trees, Liu et al. [3] obtained a two-sided variational estimate of λ_1 by a factor 2, with the method of Lyapunov test functions, and identified explicitly the Lipschitzian norm of the operator in appropriate functional space, which led to the identification of the best constant in the generalized Cheeger isoperimetric inequality and to the best constant of the transportation-information inequalities on the tree. Miclo [8] described the shapes of eigenfunctions associated to λ_1.

For some classic problems on trees such as uniqueness, recurrence, and ergodicity, Ma [4] obtained some computable sufficient conditions or necessary ones for that by constructing two corresponding birth-death processes on $\mathbb{Z}_+$.

In this paper, we are devoted to obtaining such an explicit representation for moments of the first hitting times that will lay the foundation or provide the preparation for further investigation on the ergodicity theory of birth-death processes on trees.

Denote the birth-death process on the tree T by $\{X_t \colon t \geqslant 0\}$. For all $i \in T$, define the first hitting time

$$\tau_i := \inf\{t > 0 \colon X_t = i\},$$

the first jumping time

$$\eta_1 := \inf\{t > 0 \colon X_t \neq X_0\},$$

and the first returning time

$$\sigma_i := \inf\{t > \eta_1 \colon X_t = i\}.$$

Our main result is presented as follows.

Theorem 1.1 *Assume that the process corresponding to the Q-matrix on the tree T is recurrent. Then the n-th moments of the first hitting time*

$$\mathbb{E}_i\tau_k^n = n \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell\in T_j} \mu_\ell \mathbb{E}_\ell \tau_k^{n-1}, \quad k\in\mathscr{P}(i),$$

and

$$\mathbb{E}_i\tau_o^n = n \sum_{j\in\mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell\in T_j} \mu_\ell \mathbb{E}_\ell \tau_o^{n-1}.$$

Essentially, by Theorem 1.1, we can get $\mathbb{E}_i\tau_j^n$ for any i and j in the tree when we choose one proper vertex as the corresponding root for this pair of vertices i and j.

This paper is organized as follows. We prove the result in the case of $n=1$ for Theorem 1.1 in the next section, from which the criteria on ergodicity and strong ergodicity of the process are obtained directly. Then Sections 3 is devoted to the proof of Theorem 1.1 by induction and a necessary condition for exponential ergodicity as well as a criterion on ℓ-ergodicity are presented there.

2 In the case of $n = 1$ for Theorem 1.1

Define

$$q_i^{(+)} = \sum_{j\in J(i)} q_{ij}, \quad \forall\, i\in T,$$

and

$$h_i = \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k, \quad i\in T\setminus\{o\}.$$

At first, we prove two important lemmas. The main idea comes from our study on the first moment of hitting times of single death processes. Refer to [11].

Lemma 2.1 $(h_i,\, i\in T\setminus\{o\})$ *is the minimal nonnegative solution of the following equations*:

$$y_i = \frac{q_i^{(+)}}{q_i} y_i + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i}, \quad i\in T\setminus\{o\}. \tag{2.1}$$

Proof Fix $N\in\mathbb{Z}_+$, and define Q-matrix $Q^{(N)} = (\widetilde{q}_{ij})$ on $E^{(N)} := \cup_{0\leqslant n\leqslant N} E_n$ as follows:

$$\widetilde{q}_{ij} = \begin{cases} q_{ij}, & |i|<N,\, |j|\leqslant N,\\ (q_i\vee N)G_N, & |i|=N,\, j=i^*,\\ 0, & \text{other cases of } j\neq i, \end{cases}$$

where

$$G_N = 1\vee \sum_{k\in E_N} \mu_k q_{kk^*},$$

which is finite by the assumption that any vertex has finite offsprings. Define a measure $\widetilde{\mu}$ on $E^{(N)}$ as follows:

$$\widetilde{\mu}_o = 1, \quad \widetilde{\mu}_i = \prod_{j\in\mathscr{P}(i)} \frac{\widetilde{q}_{j^*j}}{\widetilde{q}_{jj^*}}, \ i \in E^{(N)} \setminus \{o\},$$

and denote

$$\widetilde{q}_i^{(+)} = \sum_{j\in J(i)\cap E^{(N)}} \widetilde{q}_{ij}, \quad \forall\, i \in E^{(N)}. \tag{2.2}$$

Define

$$h_i^{(N)} = \frac{1}{\widetilde{\mu}_i \widetilde{q}_{ii^*}} \sum_{k\in T_i\cap E^{(N)}} \widetilde{\mu}_k, \quad i \in E^{(N)} \setminus \{o\}.$$

By the reversibility of (1.1), it is obtained that

$$\begin{aligned}
&\frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \sum_{j\in J(i)\cap E^{(N)}} \frac{\widetilde{q}_{ij}}{\widetilde{q}_i} h_j^{(N)} + \frac{1}{\widetilde{q}_i}\\
&\quad= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \frac{1}{\widetilde{\mu}_i \widetilde{q}_i} \sum_{j\in T_i\cap E^{(N)}\setminus\{i\}} \widetilde{\mu}_k + \frac{1}{\widetilde{q}_i}\\
&\quad= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \frac{1}{\widetilde{\mu}_i \widetilde{q}_i} \sum_{j\in T_i\cap E^{(N)}} \widetilde{\mu}_k\\
&\quad= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \frac{\widetilde{q}_{ii^*}}{\widetilde{q}_i} h_i^{(N)}\\
&\quad= h_i^{(N)}, \quad \forall\, i \in \bigcup_{1\leqslant n\leqslant N-1} E_n,
\end{aligned}$$

and

$$h_i^{(N)} = \frac{1}{\widetilde{q}_i}, \quad i \in E_N.$$

Hence, $(h_i^{(N)})$ is the unique solution (i.e., the minimal nonnegative solution) to the following equations:

$$y_i = \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i}\, y_i + \sum_{j\in J(i)\cap E^{(N)}} \frac{\widetilde{q}_{ij}}{\widetilde{q}_i} y_j + \frac{1}{\widetilde{q}_i}, \quad i \in E^{(N)} \setminus \{o\}. \tag{2.3}$$

Note that $\widetilde{q}_{ij} = q_{ij}$ for all $i \in E^{(N)} \setminus E_N$. Thus,

$$\widetilde{q}_i = q_i, \quad \widetilde{q}_i^{(+)} = q_i^{(+)}, \quad \forall\, i \in E^{(N)} \setminus E_N. \tag{2.4}$$

Rewrite (2.3) as

$$y_i = \begin{cases} \dfrac{1}{(q_i \vee N)G_N}, & i \in E_N, \\ \dfrac{q_i^{(+)}}{q_i}\, y_i + \displaystyle\sum_{j\in J(i)} \dfrac{q_{ij}}{q_i}\, y_j + \dfrac{1}{q_i}, & i \in \displaystyle\bigcup_{1\leqslant n\leqslant N-1} E_n. \end{cases} \tag{2.5}$$

From (2.1) and (2.5), by [1, Theorem 2.7], it follows that $(h_i^{(N)})$ is increasing to the minimal nonnegative solution of (2.1) as $N \to \infty$. Note that

$$\widetilde{\mu}_i = \begin{cases} \mu_i, & i \in E^{(N-1)}, \\ \dfrac{\mu_i q_{ii^*}}{(q_i \vee N)G_N}, & i \in E_N, \end{cases}$$

and

$$\begin{aligned} h_i^{(N)} &= \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i\cap E^{(N-1)}} \mu_k + \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i\cap E_N} \frac{\mu_k q_{kk^*}}{(q_k \vee N)G_N} \\ &\leqslant \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i\cap E^{(N-1)}} \mu_k + \frac{1}{\mu_i q_{ii^*} N}, \quad i \in E^{(N-1)} \setminus \{o\}. \end{aligned}$$

Thus, for all $i \in T \setminus \{o\}$, we see that

$$h_i^{(N)} \to \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k = h_i, \quad N \to \infty.$$

So it is proven that (h_i) is the minimal nonnegative solution of (2.1). □

Lemma 2.2 *Assume that the birth-death process on the tree T is recurrent. Fix $i_0 \in T$. Then*

$$\mathbb{E}_i \tau_{i_0} \leqslant \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k, \quad i \in T_{i_0}.$$

In particular,

$$\mathbb{E}_i \tau_{i^*} \leqslant \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k, \quad i \in T \setminus \{o\}.$$

Proof It is well known that $(\mathbb{E}_i \tau_{i_0},\, i \in T)$ is the minimal nonnegative solution to the following equations:

$$x_{i_0} = 0, \quad x_i = \sum_{j\neq i} \frac{q_{ij}}{q_i}\, x_j + \frac{1}{q_i},\ i \neq i_0;$$

refer to [5]. By the Localization Theorem (refer to [1, Theorem 2.13]) and the single death property, we obtain that $(\mathbb{E}_i \tau_{i_0},\, i \in T_{i_0})$ is the minimal nonnegative solution to the following equations:

$$x_{i_0} = 0, \quad x_i = \sum_{j\in T_{i_0}\setminus\{i_0, i\}} \frac{q_{ij}}{q_i}\, x_j + \frac{1}{q_i},\ i \in T_{i_0} \setminus \{i_0\}. \tag{2.6}$$

Rewrite the equation above as

$$x_i = \begin{cases} 0, & i = i_0, \\ \displaystyle\sum_{j\in J(i)} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, & i \in J(i_0), \\ \displaystyle\frac{q_{ii^*}}{q_i} x_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, & i \in T_{i_0} \setminus (\{i_0\} \cup J(i_0)). \end{cases}$$

Define

$$y_i = \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k, \quad i \in T_{i_0}.$$

Then, by (1.1), for all $i \in J(i_0)$, one gets that

$$\begin{aligned} \sum_{j\in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i} &= \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \sum_{\ell\in\mathscr{P}(j)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{q_i} \\ &= \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \left(\frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k + \frac{1}{\mu_j q_{jj^*}} \sum_{k\in T_j} \mu_k \right) + \frac{1}{q_i} \\ &= \frac{q_i^{(+)}}{q_i} \cdot \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \cdot \frac{1}{\mu_j q_{ji}} \sum_{k\in T_j} \mu_k + \frac{1}{q_i} \\ &= \frac{q_i^{(+)}}{q_i} \cdot \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k + \frac{1}{\mu_i q_i} \sum_{j\in J(i)} \sum_{k\in T_j} \mu_k + \frac{1}{q_i} \\ &= \frac{q_i^{(+)}}{q_i} \cdot \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k + \frac{1}{\mu_i q_i} \sum_{k\in T_i} \mu_k \\ &= \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k \\ &= y_i. \end{aligned}$$

For all $i \in T_{i_0} \setminus (\{i_0\} \cup J(i_0))$, it follows from (1.1) that

$$\begin{aligned} &\frac{q_{ii^*}}{q_i} y_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i} \\ &= \frac{q_{ii^*}}{q_i} y_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \sum_{\ell\in\mathscr{P}(j)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{q_i} \\ &= \frac{q_{ii^*}}{q_i} y_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \cdot \frac{1}{\mu_j q_{jj^*}} \sum_{k\in T_j} \mu_k + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{q_i} \\ &= \frac{q_{ii^*}}{q_i} y_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \cdot \frac{1}{\mu_j q_{ji}} \sum_{k\in T_j} \mu_k + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{q_i} \end{aligned}$$

$$= \frac{q_{ii^*}}{q_i}\, y_{i^*} + \frac{1}{\mu_i q_i} \sum_{j\in J(i)} \sum_{k\in T_j} \mu_k + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in \mathscr{P}(i)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{q_i}.$$

Furthermore, we obtain that

$$\begin{aligned}
&\frac{q_{ii^*}}{q_i}\, y_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i}\, y_j + \frac{1}{q_i}\\
&= \frac{q_{ii^*}}{q_i} \sum_{\ell\in \mathscr{P}(i^*)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{\mu_i q_i} \sum_{k\in T_i} \mu_k\\
&\quad + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in \mathscr{P}(i)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k\\
&= \frac{q_{ii^*}}{q_i} \left(\sum_{\ell\in \mathscr{P}(i^*)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k \right)\\
&\quad + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in \mathscr{P}(i)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k\\
&= \frac{q_{ii^*}}{q_i} \sum_{\ell\in \mathscr{P}(i)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in \mathscr{P}(i)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k\\
&= \sum_{\ell\in \mathscr{P}(i)\setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k\\
&= y_i.
\end{aligned}$$

So we have checked that $(y_i,\, i \in T_{i_0})$ is a nonnegative solution of (2.6). Hence, by the minimal property, one obtains that $\mathbb{E}_i \tau_{i_0} \leqslant y_i$ $(i \in T_{i_0})$. The first assertion holds. By the arbitrariness of i_0, it is easy to derive the second assertion. □

Then we present the corresponding result in the case of $n = 1$ for Theorem 1.1 as follows.

Theorem 2.3 *Assume that the birth-death process on the tree T is recurrent. Then*

$$\mathbb{E}_i \tau_k = \sum_{j\in \mathscr{P}(i)\setminus \mathscr{P}(k)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell\in T_j} \mu_\ell, \quad k \in \mathscr{P}(i),$$

and

$$\mathbb{E}_i \tau_o = \sum_{j\in \mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell\in T_j} \mu_\ell.$$

Proof Denote $\mathbb{E}_i \tau_{i^*}$ by m_i. By the strong Markov property, for all $i \in T \setminus \{o\}$, it holds that

$$\begin{aligned} m_i &= \mathbb{E}_i\eta_1 + \mathbb{E}_i(\mathbb{E}_i(\tau_{i^*} - \eta_1 \mid \mathscr{F}_{\eta_1})) \\ &= \frac{1}{q_i} + \mathbb{E}_i(\mathbb{E}_{X_{\eta_1}}\tau_{i^*}) \\ &= \frac{1}{q_i} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i}\,\mathbb{E}_j\tau_{i^*} \\ &= \frac{1}{q_i} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i}\,\mathbb{E}_j\tau_{j^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i}\,\mathbb{E}_i\tau_{i^*} \quad \text{(by (2.8))} \\ &= \frac{q_i^{(+)}}{q_i}\,m_i + \sum_{j\in J(i)} \frac{q_{ij}}{q_i}\,m_j + \frac{1}{q_i}. \end{aligned}$$

Hence, $(m_i,\ i\in T\backslash\{o\})$ satisfies (2.1). By Lemma 2.1 and the minimal property, we obtain $m_i \geqslant h_i$. And the inverse inequality $h_i \geqslant m_i$ is seen in Lemma 2.2. Thus, $m_i = h_i$ for all $i \in T\setminus\{o\}$. Hence, we have

$$\mathbb{E}_i\tau_{i^*} = \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k, \quad i \in T\setminus\{o\}. \tag{2.7}$$

By the strong Markov property, we know that for all $k \in \mathscr{P}(i)$,

$$\mathbb{E}_i\tau_k = \mathbb{E}_i\tau_{i^*} + \mathbb{E}_i(\mathbb{E}_i(\tau_k - \tau_{i^*} \mid \mathscr{F}_{\tau_{i^*}})) = \mathbb{E}_i\tau_{i^*} + \mathbb{E}_i(\mathbb{E}_{X_{\tau_{i^*}}}\tau_k) = \mathbb{E}_i\tau_{i^*} + \mathbb{E}_{i^*}\tau_k.$$

Recursively, it is derived that

$$\mathbb{E}_i\tau_k = \sum_{j\in\mathscr{P}(i)\backslash\mathscr{P}(k)} \mathbb{E}_j\tau_{j^*}, \quad k\in\mathscr{P}(i). \tag{2.8}$$

Similarly, it holds that

$$\mathbb{E}_i\tau_o = \sum_{j\in\mathscr{P}(i)} \mathbb{E}_j\tau_{j^*}. \tag{2.9}$$

From (2.7)–(2.9), the assertions are followed immediately. □

Remark 2.4 Under the conditions of Theorem 2.3, $(\mathbb{E}_i\tau_{i^*},\ i\in T\setminus\{o\})$ is the minimal nonnegative solution of (2.1) and satisfies

$$m_i = \frac{1}{q_{ii^*}}\bigg(1 + \sum_{j\in J(i)} q_{ij}m_j\bigg), \quad i\in T\setminus\{o\}.$$

For the first returning time σ_o of the root o, we have the following assertion.

Corollary 2.5 *Assume that the Q-matrix on the tree T is regular and the process is recurrent. Define $\mu = \sum_{i\in T}\mu_i$. Then $\mathbb{E}_o\sigma_o = \mu/q_o$.*

Proof It follows from Theorem 2.3 that

$$\begin{aligned}
\mathbb{E}_o\sigma_o &= \sum_{i\in J(o)} \frac{q_{oi}}{q_o}\,\mathbb{E}_i\tau_o + \frac{1}{q_o}\\
&= \sum_{i\in J(o)} \frac{q_{oi}}{q_o}\,\mathbb{E}_i\tau_{i^*} + \frac{1}{q_o}\\
&= \sum_{i\in J(o)} \frac{q_{oi}}{q_o}\cdot\frac{1}{\mu_i q_{ii^*}}\sum_{\ell\in T_i}\mu_\ell + \frac{1}{q_o}\\
&= \frac{1}{q_o}\sum_{i\in J(o)}\sum_{\ell\in T_i}\mu_\ell + \frac{1}{q_o}\\
&= \frac{\mu}{q_o}.
\end{aligned}$$

So the proof is finished. □

Based on the results and arguments above, we get the criteria on ergodicity and strong ergodicity directly as follows.

Corollary 2.6 *Assume that the Q-matrix on the tree T is regular. Then the process is ergodic if and only if $\mu<\infty$; the process is strongly ergodic if and only if*

$$\sup_{i\in T\setminus\{o\}}\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell<\infty. \tag{2.10}$$

Proof Under the assumption of uniqueness, if $\mu<\infty$, then $(\mu_i/\mu,\ i\in T)$ is the unique stationary distribution of the process; moreover, we know that the process is ergodic. Conversely, if the process is ergodic, then $\mathbb{E}_o\sigma_o<\infty$ (refer to [1, Theorem 4.44]); thus, it follows that $\mu<\infty$ from Corollary 2.5 immediately. The first assertion is proven.

If the process is strongly ergodic, then

$$\sup_{i\in T}\mathbb{E}_i\sigma_o<\infty$$

(refer to [1, Theorem 4.44] or [2]); furthermore,

$$\sup_{i\in T\setminus\{o\}}\mathbb{E}_i\tau_o<\infty.$$

By Theorem 2.3, we get that

$$\sup_{i\in T\setminus\{o\}}\mathbb{E}_i\tau_o=\sup_{i\in T\setminus\{o\}}\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell<\infty.$$

Conversely, if (2.10) holds, then

$$\max_{i\in J(o)}\sum_{\ell\in T_i}\mu_\ell<\infty.$$

Thus, $\mu < \infty$, which means that the process is ergodic. It follows from Theorem 2.3 and (2.10) that

$$\sup_{i\in T\setminus\{o\}} \mathbb{E}_i\tau_o < \infty.$$

Furthermore,

$$\max_{i\in J(o)} \mathbb{E}_i\tau_o < \infty,$$

which implies that $\mathbb{E}_o\sigma_o < \infty$. So it is seen that $\sup_{i\in T}\mathbb{E}_i\sigma_o < \infty$. Then by [1, Theorem 4.44], it is derived that the process is strongly ergodic. $\square$

Remark 2.7 Under the assumption of ergodicity, Martinez and Ycart [6] obtained the same result in Theorem 2.3 in a different approach but some mistakes were made in their proof. Although they took the same truncation of the state space but the approximation they used is of conditional expectation. The criterion on ergodicity is mentioned in [4, Remark 1.1] too.

3 Proof of Theorem 1.1

First, we introduce four lemmas as follows.

Lemma 3.1 *Assume that the Q-matrix on the tree T is regular and the process is recurrent. Fix $i_0 \in T$. Then*

$$\mathbb{E}_i\tau_{i_0}^n \leqslant n \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k\mathbb{E}_k\tau_{i_0}^{n-1}, \quad i\in T_{i_0}.$$

In particular,

$$\mathbb{E}_i\tau_{i^*}^n \leqslant \frac{n}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k\mathbb{E}_k\tau_{i^*}^{n-1}, \quad i\in T\setminus\{o\}.$$

Proof It is well known that $(\mathbb{E}_i\tau_{i_0}^n,\ i\in T)$ is the minimal nonnegative solution to the following equations:

$$x_{i_0} = 0, \quad x_i = \sum_{j\neq i} \frac{q_{ij}}{q_i}\, x_j + \frac{n}{q_i}\,\mathbb{E}_i\tau_{i_0}^{n-1},\ i\neq i_0;$$

refer to [5, Theorem 3.1]. By the Localization Theorem (refer to [1, Theorem 2.13]) and the single death property, it is obtained that $(\mathbb{E}_i\tau_{i_0}^n,\ i\in T_{i_0})$ is the minimal nonnegative solution to the following equations:

$$x_{i_0} = 0, \quad x_i = \sum_{j\in T_{i_0}\setminus\{i_0,i\}} \frac{q_{ij}}{q_i}\, x_j + \frac{n}{q_i}\,\mathbb{E}_i\tau_{i_0}^{n-1},\ i\in T_{i_0}\setminus\{i_0\}. \qquad (3.1)$$

Rewrite the equation above as

$$x_i = \begin{cases} 0, & i = i_0, \\ \displaystyle\sum_{j \in J(i)} \frac{q_{ij}}{q_i} x_j + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1}, & i \in J(i_0), \\ \displaystyle\frac{q_{ii^*}}{q_i} x_{i^*} + \sum_{j \in J(i)} \frac{q_{ij}}{q_i} x_j + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1}, & i \in T_{i_0} \setminus (\{i_0\} \cup J(i_0)). \end{cases}$$

Define

$$y_i = n \sum_{\ell \in \mathscr{P}(i) \setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k \in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1}, \quad i \in T_{i_0}.$$

Then, by (1.1), for all $i \in J(i_0)$, one gets that

$$\begin{aligned} &\sum_{j \in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\ &\quad = n \sum_{j \in J(i)} \frac{q_{ij}}{q_i} \sum_{\ell \in \mathscr{P}(j) \setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k \in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\ &\quad = n \sum_{j \in J(i)} \frac{q_{ij}}{q_i} \left(\frac{1}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{1}{\mu_j q_{jj^*}} \sum_{k \in T_j} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \right) + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\ &\quad = \frac{q_i^{(+)}}{q_i} \cdot \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \sum_{j \in J(i)} \frac{q_{ij}}{q_i} \cdot \frac{n}{\mu_j q_{ji}} \sum_{k \in T_j} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\ &\quad = \frac{q_i^{(+)}}{q_i} \cdot \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{\mu_i q_i} \sum_{j \in J(i)} \sum_{k \in T_j} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\ &\quad = \frac{q_i^{(+)}}{q_i} \cdot \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{\mu_i q_i} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\ &\quad = \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\ &\quad = y_i. \end{aligned}$$

For all $i \in T_{i_0} \setminus (\{i_0\} \cup J(i_0))$, it follows from (1.1) that

$$\begin{aligned} &\sum_{j \in J(i)} \frac{q_{ij}}{q_i} y_j \\ &= \sum_{j \in J(i)} \frac{q_{ij}}{q_i} \sum_{\ell \in \mathscr{P}(j) \setminus \mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k \in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\ &= \sum_{j \in J(i)} \frac{q_{ij}}{q_i} \cdot \frac{n}{\mu_j q_{jj^*}} \sum_{k \in T_j} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \sum_{j \in J(i)} \frac{q_{ij}}{q_i} \sum_{\ell \in \mathscr{P}(i) \setminus \mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k \in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \end{aligned}$$

$$= \sum_{j\in J(i)} \frac{q_{ij}}{q_i} \cdot \frac{n}{\mu_j q_{ji}} \sum_{k\in T_j} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1}$$

$$= \frac{n}{\mu_i q_i} \sum_{j\in J(i)} \sum_{k\in T_j} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1}.$$

Furthermore, it is derived that

$$\begin{aligned}
&\frac{q_{ii^*}}{q_i} y_{i^*} + \sum_{j\in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{n}{q_i} \mathbb{E}_i \tau_{i_0}^{n-1} \\
&\quad = \frac{q_{ii^*}}{q_i} \sum_{\ell\in\mathscr{P}(i^*)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{\mu_i q_i} \sum_{k\in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\
&\quad\quad + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\
&\quad = \frac{q_{ii^*}}{q_i} \Bigg(\sum_{\ell\in\mathscr{P}(i^*)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} + \frac{n}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \Bigg) \\
&\quad\quad + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\
&\quad = \frac{q_{ii^*}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\
&\quad\quad + \frac{q_i^{(+)}}{q_i} \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\
&\quad = \sum_{\ell\in\mathscr{P}(i)\setminus\mathscr{P}(i_0)} \frac{n}{\mu_\ell q_{\ell\ell^*}} \sum_{k\in T_\ell} \mu_k \mathbb{E}_k \tau_{i_0}^{n-1} \\
&\quad = y_i.
\end{aligned}$$

So it is verified that $(y_i,\, i \in T_{i_0})$ is a nonnegative solution of (3.1). Hence, by the minimal property, one obtains $\mathbb{E}_i \tau_{i_0}^n \leqslant y_i$ $(i \in T_{i_0})$. The first assertion holds. By the arbitrariness of i_0, it is easy to derive the second assertion. □

Lemma 3.2 *Assume that the Q-matrix on the tree T is regular and the process is recurrent. If there exist two vertices $i_0 \neq j_0$ in the tree and some positive integer $m \geqslant 1$ such that $\mathbb{E}_{i_0} \tau_{j_0}^m < \infty$, then $\mathbb{E}_i \tau_{j_0}^m < \infty$ for all $i \in T$. Moreover, $\mathbb{E}_i \tau_j^m < \infty$ for all $i, j \in T$ satisfying $j \neq o$ and $j_0 \in T_j$, or $J(j) \neq \emptyset$ and $j_0 \notin T_j$. In addition, if $j_0 \neq o$ and $\#\{J(o)\} > 1$, then $\mathbb{E}_i \tau_o^m < \infty$ for all $i \in T$.*

Proof For any $i \in T$, by the irreducibility of the process, we know that there exist some adjacent and distinct vertices $k_0 = i_0, k_1, \dots, k_n = i$ $(q_{k_\ell, k_{\ell+1}} > 0,\ \ell = 0, 1, \dots, n-1)$. Note that

$$\mathbb{E}_{i_0} \tau_{j_0}^m = \sum_{k\neq i_0} \frac{q_{i_0,k}}{q_{i_0}} \mathbb{E}_k \tau_{j_0}^m + \frac{m}{q_{i_0}} \mathbb{E}_{i_0} \tau_{j_0}^{m-1}.$$

Then one gets that $\mathbb{E}_{k_1}\tau_{j_0}^m < \infty$ and

$$\mathbb{E}_{k_1}\tau_{j_0}^m = \sum_{k\neq k_1} \frac{q_{k_1,k}}{q_{k_1}}\,\mathbb{E}_k\tau_{j_0}^m + \frac{m}{q_{k_1}}\,\mathbb{E}_{k_1}\tau_{j_0}^{m-1}.$$

Hence, it derives that $\mathbb{E}_{k_2}\tau_{j_0}^m < \infty$ and so on. Inductively, we obtain that $\mathbb{E}_{k_n}\tau_{j_0}^m < \infty$, i.e.,

$$\mathbb{E}_i\tau_{j_0}^m < \infty, \quad \forall\, i \in T.$$

Give $j \in T$ arbitrarily with $j \neq j_0$. When $j_0 \in T_j$ with $j \neq o$, from the strong Markov property, it follows that

$$\begin{aligned}\infty &> \mathbb{E}_{j^*}\tau_{j_0}^m\\ &= \mathbb{E}_{j^*}(\tau_j + \tau_{j_0} - \tau_j)^m\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_{j^*}\tau_j^\ell\mathbb{E}_{j^*}(\mathbb{E}_{j^*}((\tau_{j_0}-\tau_j)^{m-\ell} \mid \mathscr{F}_{\tau_j}))\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_{j^*}\tau_j^\ell\mathbb{E}_{j^*}(\mathbb{E}_{X_{\tau_j}}\tau_{j_0}^{m-\ell})\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_{j^*}\tau_j^\ell\mathbb{E}_j\tau_{j_0}^{m-\ell}.\end{aligned}$$

Hence, we obtain $\mathbb{E}_{j^*}\tau_j^m < \infty$. Again by the irreducibility and the argument above, it is derived that $\mathbb{E}_i\tau_j^m < \infty$ for all $i \in T$.

When $j_0 \notin T_j$ with $J(j) \neq \emptyset$, take $k \in J(j)$. By the strong Markov property, one gets that

$$\begin{aligned}\infty &> \mathbb{E}_k\tau_{j_0}^m\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_k\tau_j^\ell\mathbb{E}_k(\mathbb{E}_k((\tau_{j_0}-\tau_j)^{m-\ell} \mid \mathscr{F}_{\tau_j}))\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_k\tau_j^\ell\mathbb{E}_j\tau_{j_0}^{m-\ell}.\end{aligned}$$

Hence, it holds that $\mathbb{E}_k\tau_j^m < \infty$. By the irreducibility, we know that $\mathbb{E}_i\tau_j^m < \infty$ for all $i \in T$.

In addition, assume that $j_0 \neq o$ and $\#\{J(o)\} > 1$. Then there exist $k \neq \ell$ such that $k, \ell \in J(o)$ and $j_0 \in T_\ell$. Note that

$$\begin{aligned}\infty &> \mathbb{E}_k\tau_{j_0}^m\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_k\tau_o^\ell\mathbb{E}_k(\mathbb{E}_k((\tau_{j_0}-\tau_o)^{m-\ell} \mid \mathscr{F}_{\tau_o}))\\ &= \sum_{\ell=0}^m \binom{m}{\ell}\mathbb{E}_k\tau_o^\ell\mathbb{E}_o\tau_{j_0}^{m-\ell}.\end{aligned}$$

Hence, from the strong Markov property, it follows that $\mathbb{E}_k\tau_o^m < \infty$. By the irreducibility, it is easy to seen that $\mathbb{E}_i\tau_o^m < \infty$ for all $i \in T$.

The proofs of all the assertions are finished. $\square$

Lemma 3.3 *Assume that the Q-matrix on the tree T is regular and the process is recurrent. Define*

$$f_i = \frac{1}{q_{ii^*}}\left(\mathbb{E}_i\tau_{i^*}^{n-1} + \sum_{j\in J(i)} q_{ij}\sum_{m=0}^{n-2}\binom{n-1}{m}\frac{1}{m+1}\,\mathbb{E}_j\tau_{j^*}^{m+1}\mathbb{E}_i\tau_{i^*}^{n-1-m}\right), \quad i \neq o.$$

Then $(\mathbb{E}_i\tau_{i^*}^n,\ i \neq o)$ *satisfies the following equation*:

$$y_i = \frac{q_i^{(+)}}{q_i}\,y_i + \sum_{j\in J(i)}\frac{q_{ij}}{q_i}\,y_j + \frac{nq_{ii^*}}{q_i}\,f_i, \quad i \neq o. \tag{3.2}$$

Moreover, $f_i < \infty$ for some i with $|i| > 1$ is equivalent to that $f_i < \infty$ for all i satisfying $|i| > 1$.

Proof Fix $i \neq o$. On the one hand, under the assumption that the process is recurrent, we know that $(\mathbb{E}_j\tau_{i^*}^n,\ j \in T)$ is the minimal nonnegative solution to the equation

$$x_{i^*} = 0, \quad x_j = \sum_{k\neq j}\frac{q_{jk}}{q_j}\,x_k + \frac{n}{q_j}\,\mathbb{E}_j\tau_{i^*}^{n-1},\ j \neq i^*.$$

So

$$\mathbb{E}_i\tau_{i^*}^n = \sum_{k\neq i}\frac{q_{ik}}{q_i}\,\mathbb{E}_k\tau_{i^*}^n + \frac{n}{q_i}\,\mathbb{E}_i\tau_{i^*}^{n-1} = \sum_{k\in J(i)}\frac{q_{ik}}{q_i}\,\mathbb{E}_k\tau_{i^*}^n + \frac{n}{q_i}\,\mathbb{E}_i\tau_{i^*}^{n-1}.$$

For $j \in J(i)$, by the strong Markov property, we have

$$\mathbb{E}_j\tau_{i^*}^n = \sum_{m=0}^{n}\binom{n}{m}\mathbb{E}_j\tau_{j^*}^m\mathbb{E}_i\tau_{i^*}^{n-m}.$$

Thus,

$$\begin{aligned}
\mathbb{E}_i\tau_{i^*}^n &= \frac{n}{q_i}\,\mathbb{E}_i\tau_{i^*}^{n-1} + \sum_{j\in J(i)}\frac{q_{ij}}{q_i}\sum_{m=0}^{n}\binom{n}{m}\mathbb{E}_j\tau_{j^*}^m\mathbb{E}_i\tau_{i^*}^{n-m} \\
&= \frac{q_i^{(+)}}{q_i}\,\mathbb{E}_i\tau_{i^*}^n + \sum_{j\in J(i)}\frac{q_{ij}}{q_i}\,\mathbb{E}_j\tau_{j^*}^n + \frac{n}{q_i}\,\mathbb{E}_i\tau_{i^*}^{n-1} \\
&\quad + \sum_{j\in J(i)}\frac{q_{ij}}{q_i}\sum_{m=1}^{n-1}\binom{n}{m}\mathbb{E}_j\tau_{j^*}^m\mathbb{E}_i\tau_{i^*}^{n-m} \\
&= \frac{q_i^{(+)}}{q_i}\,\mathbb{E}_i\tau_{i^*}^n + \sum_{j\in J(i)}\frac{q_{ij}}{q_i}\,\mathbb{E}_j\tau_{j^*}^n + \frac{n}{q_i}\,\mathbb{E}_i\tau_{i^*}^{n-1}
\end{aligned}$$

$$+\frac{n}{q_i}\sum_{j\in J(i)} q_{ij}\sum_{m=0}^{n-2}\binom{n-1}{m}\frac{1}{m+1}\,\mathbb{E}_j\tau_{j^*}^{m+1}\mathbb{E}_i\tau_{i^*}^{n-1-m}$$
$$=\frac{q_i^{(+)}}{q_i}\,\mathbb{E}_i\tau_{i^*}^n+\sum_{j\in J(i)}\frac{q_{ij}}{q_i}\,\mathbb{E}_j\tau_{j^*}^n+\frac{nq_{ii^*}}{q_i}\,f_i,\quad i\neq o.$$

Hence, $(\mathbb{E}_i\tau_{i^*}^n,\ i\neq o)$ satisfies (3.2), which means that the first assertion holds.

If $f_i<\infty$ for some i with $|i|>1$, we have to check that $f_k<\infty$ for other $k\neq i$ with $|k|>1$. Now, by the definition of f_i, it is seen that $\mathbb{E}_i\tau_{i^*}^{n-1}<\infty$. For $|k|>1$, by Lemma 3.2, it holds that $\mathbb{E}_k\tau_{k^*}^{n-1}<\infty$ and $\mathbb{E}_j\tau_{j^*}^{n-1}<\infty$ for all $j\in J(k)$. Hence, it is easy to be checked that $f_k<\infty$. The second assertion holds. □

The following lemma is a key result in the proof of Theorem 1.1.

Lemma 3.4 *Assume that the Q-matrix on the tree T is regular and the process is recurrent. Define*

$$h_i=\frac{n}{\mu_i q_{ii^*}}\sum_{k\in T_i}\mu_k q_{kk^*}f_k,\quad i\neq o,\tag{3.3}$$

where f_i is defined in Lemma 3.3. *Then $(h_i,\ i\neq o)$ is the minimal nonnegative solution of* (3.2).

Proof Fix a positive integer $N\geqslant 2$, and define Q-matrix $Q^{(N)}=(\widetilde{q}_{ij})$ on $E^{(N)}$:

$$\widetilde{q}_{ij}=\begin{cases} q_{ij}, & |i|<N,\ |j|\leqslant N,\\ \Big(\dfrac{q_i}{q_{ii^*}}\vee N\Big)G_N, & |i|=N,\ j=i^*,\\ 0, & \text{other cases of } j\neq i,\end{cases}$$

where

$$G_N=\begin{cases} 1\vee\displaystyle\sum_{k\in E_N}\mu_k q_{kk^*}f_k, & f_i<\infty \text{ for some } i \text{ with } |i|>1,\\ 1, & \text{otherwise.}\end{cases}$$

By Lemma 3.3, we know that $G_N<\infty$. Define a measure $\widetilde{\mu}$ on $E^{(N)}$ as follows:

$$\widetilde{\mu}_o=1,\quad \widetilde{\mu}_i=\prod_{j\in\mathscr{P}(i)}\frac{\widetilde{q}_{j^*j}}{\widetilde{q}_{jj^*}},\ i\in E^{(N)}\setminus\{o\},$$

and denote $\widetilde{q}_i^{(+)}$ as in (2.2). Define

$$h_i^{(N)}=\frac{n}{\widetilde{\mu}_i\widetilde{q}_{ii^*}}\bigg(\sum_{k\in T_i\cap E^{(N-1)}}\widetilde{\mu}_k\widetilde{q}_{kk^*}f_k+\sum_{k\in T_i\cap E_N}\widetilde{\mu}_k f_k\bigg),\quad i\in E^{(N)}\setminus\{o\}.$$

By the reversibility of (1.1), for all $i \neq o$, $|i| \leqslant N-1$, it holds that

$$\begin{aligned}
&\frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \sum_{j\in J(i)\cap E^{(N)}} \frac{\widetilde{q}_{ij}}{\widetilde{q}_i} h_j^{(N)} + \frac{n\widetilde{q}_{ii^*}}{\widetilde{q}_i} f_i \\
&= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \sum_{j\in J(i)\cap E^{(N)}} \frac{n}{\widetilde{\mu}_i\widetilde{q}_i} \Bigg(\sum_{k\in T_j\cap E^{(N-1)}} \widetilde{\mu}_k \widetilde{q}_{kk^*} f_k + \sum_{k\in T_j\cap E_N} \widetilde{\mu}_k f_k \Bigg) \\
&\quad + \frac{n\widetilde{q}_{ii^*}}{\widetilde{q}_i} f_i \\
&= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \frac{n}{\widetilde{\mu}_i\widetilde{q}_i} \Bigg(\sum_{k\in T_i\cap E^{(N-1)}\setminus\{i\}} \widetilde{\mu}_k \widetilde{q}_{kk^*} f_k + \sum_{k\in T_i\cap E_N} \widetilde{\mu}_k f_k \Bigg) + \frac{n\widetilde{q}_{ii^*}}{\widetilde{q}_i} f_i \\
&= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \frac{n}{\widetilde{\mu}_i\widetilde{q}_i} \Bigg(\sum_{k\in T_i\cap E^{(N-1)}} \widetilde{\mu}_k \widetilde{q}_{kk^*} f_k + \sum_{k\in T_i\cap E_N} \widetilde{\mu}_k f_k \Bigg) \\
&= \frac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} h_i^{(N)} + \frac{\widetilde{q}_{ii^*}}{\widetilde{q}_i} h_i^{(N)} \\
&= h_i^{(N)}.
\end{aligned}$$

Note that

$$h_i^{(N)} = \frac{n}{\widetilde{q}_{ii^*}} f_i, \quad i \in E_N.$$

Hence, $(h_i^{(N)}, i \in E^{(N)} \setminus \{o\})$ is the unique solution (i.e., the minimal nonnegative solution) to the following equation:

$$y_i = \begin{cases} \dfrac{\widetilde{q}_i^{(+)}}{\widetilde{q}_i} y_i + \displaystyle\sum_{j\in J(i)\cap E^{(N)}} \frac{\widetilde{q}_{ij}}{\widetilde{q}_i} y_j + \frac{n\widetilde{q}_{ii^*}}{\widetilde{q}_i} f_i, & i \neq o,\ |i| \leqslant N-1, \\ \dfrac{n}{\widetilde{q}_{ii^*}} f_i, & i \in E_N. \end{cases}$$

Note that $\widetilde{q}_{ij} = q_{ij}$ for all $i \in E^{(N)} \setminus E_N$. Thus, (2.4) holds. Rewrite the equation above as

$$y_i = \begin{cases} \dfrac{nf_i}{((q_i/q_{ii^*}) \vee N)G_N}, & i \in E_N, \\ \dfrac{q_i^{(+)}}{q_i} y_i + \displaystyle\sum_{j\in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{nq_{ii^*}}{q_i} f_i, & i \neq o,\ |i| \leqslant N-1. \end{cases} \tag{3.4}$$

From (3.2) and (3.4), by [1, Theorem 2.7], it follows that $(h_i^{(N)})$ is increasing to the minimal nonnegative solution of (3.2) as $N \to \infty$. Note that

$$\widetilde{\mu}_i = \begin{cases} \mu_i, & i \in E^{(N-1)}, \\ \dfrac{\mu_i q_{ii^*}}{((q_i/q_{ii^*}) \vee N)G_N}, & i \in E_N, \end{cases}$$

and

$$h_i^{(N)} = \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i \cap E^{(N-1)}} \mu_k q_{kk^*} f_k + \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i \cap E_N} \frac{\mu_k q_{kk^*} f_k}{((q_k/q_{kk^*}) \vee N) G_N}.$$

Thus, for all $i \in T \setminus \{o\}$, we see that

$$\begin{aligned}\lim_{N\to\infty} h_i^{(N)} &= \begin{cases} \dfrac{n}{\mu_i q_{ii^*}} \displaystyle\sum_{k \in T_i} \mu_k q_{kk^*} f_k, & f_j < \infty \text{ for some } j \text{ with } |j| > 1,\\ \infty, & \text{otherwise,}\end{cases}\\ &= \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k q_{kk^*} f_k\\ &= h_i.\end{aligned}$$

So it is verified that (h_i) is the minimal nonnegative solution of (3.2). The proof is finished. □

Proof of Theorem 1.1 We prove the theorem by the induction. By Theorem 2.3, we know that the assertion in Theorem 1.1 holds for $n = 1$. Assume that the assertion holds until $n - 1$. In the following arguments, we will prove the assertion for n.

Denote $\mathbb{E}_i \tau_{i^*}^n$ by m_i. On the one hand, by Lemmas 3.3, 3.4, and the minimal property, we obtain that $h_i \leqslant m_i$ for all $i \neq o$, where h_i is defined in (3.3). On the other hand, we can prove that

$$h_i = \frac{n}{\mu_i q_{ii^*}} \sum_{s \in T_i} \mu_s \mathbb{E}_s \tau_{i^*}^{n-1}, \quad i \neq o. \tag{3.5}$$

In fact, if there exists some $s \in T_i$ such that $\mathbb{E}_s \tau_{i^*}^{n-1} = \infty$, then by Lemma 3.2, we know that $\mathbb{E}_k \tau_{i^*}^{n-1} = \infty$ for all $k \in T$; furthermore $f_i = \infty$. Hence,

$$\frac{n}{\mu_i q_{ii^*}} \sum_{s \in T_i} \mu_s \mathbb{E}_s \tau_{i^*}^{n-1} = \infty = h_i.$$

If $\mathbb{E}_s \tau_{i^*}^{n-1} < \infty$ for all $s \in T_i$, then by Lemma 3.2, we know that $\mathbb{E}_s \tau_{k^*}^{n-1} < \infty$ and $\mathbb{E}_s \tau_k^{n-1} < \infty$ for all $k \in T_i$ and $s \in T_k$. Then by the definition of f_i, we have

$$h_i = \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \left(\mathbb{E}_k \tau_{k^*}^{n-1} + \sum_{j \in J(k)} q_{kj} \sum_{\ell=0}^{n-2} \binom{n-1}{\ell} \frac{1}{\ell+1} \mathbb{E}_j \tau_{j^*}^{\ell+1} \mathbb{E}_k \tau_{k^*}^{n-1-\ell} \right).$$

By the assumption that the assertion in Theorem 1.1 holds until $n-1$, we have

$$\mathbb{E}_j \tau_{j^*}^{\ell+1} = \frac{\ell+1}{\mu_j q_{jj^*}} \sum_{s \in T_j} \mu_s \mathbb{E}_s \tau_{j^*}^{\ell}, \quad 0 \leqslant \ell \leqslant n-2.$$

Hence,

$$h_i = \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \bigg(\mathbb{E}_k \tau_{k^*}^{n-1} + \sum_{j \in J(k)} q_{kj} \sum_{\ell=0}^{n-2} \binom{n-1}{\ell} \frac{1}{\mu_j q_{jj^*}} \cdot \sum_{s \in T_j} \mu_s \mathbb{E}_s \tau_{j^*}^{\ell} \mathbb{E}_k \tau_{k^*}^{n-1-\ell} \bigg)$$

$$= \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \bigg(\mu_k \mathbb{E}_k \tau_{k^*}^{n-1} + \sum_{j \in J(k)} \mu_k q_{kj} \sum_{\ell=0}^{n-2} \binom{n-1}{\ell} \frac{1}{\mu_j q_{jk}} \cdot \sum_{s \in T_j} \mu_s \mathbb{E}_s \tau_{j^*}^{\ell} \mathbb{E}_k \tau_{k^*}^{n-1-\ell} \bigg)$$

$$= \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \bigg(\mu_k \mathbb{E}_k \tau_{k^*}^{n-1} + \sum_{j \in J(k)} \sum_{s \in T_j} \sum_{\ell=0}^{n-2} \binom{n-1}{\ell} \mu_s \mathbb{E}_s \tau_k^{\ell} \mathbb{E}_k \tau_{k^*}^{n-1-\ell} \bigg)$$

$$= \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \sum_{s \in T_k} \sum_{\ell=0}^{n-2} \binom{n-1}{\ell} \mu_s \mathbb{E}_s \tau_k^{\ell} \mathbb{E}_k \tau_{k^*}^{n-1-\ell}.$$

Moreover, we have

$$h_i = \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \sum_{s \in T_k} \mu_s \big(\mathbb{E}_s \tau_{k^*}^{n-1} - \mathbb{E}_s \tau_k^{n-1} \big)$$

$$= \frac{n}{\mu_i q_{ii^*}} \sum_{s \in T_i} \mu_s \sum_{k \in \mathscr{P}(s) \setminus \mathscr{P}(i^*)} \big(\mathbb{E}_s \tau_{k^*}^{n-1} - \mathbb{E}_s \tau_k^{n-1} \big)$$

$$= \frac{n}{\mu_i q_{ii^*}} \sum_{s \in T_i} \mu_s \mathbb{E}_s \tau_{i^*}^{n-1}.$$

Hence, (3.5) holds. From (3.5) and Lemma 3.1, it follows that $m_i \leqslant h_i$ for all $i \neq o$ immediately.

Summing up the arguments above, we obtain that

$$\mathbb{E}_i \tau_{i^*}^n = m_i = h_i = \frac{n}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k \mathbb{E}_k \tau_{i^*}^{n-1}, \quad i \neq o.$$

From the strong Markov property, it follows that, for any $k \in \mathscr{P}(i) \setminus \{i\}$,

$$\mathbb{E}_i \tau_k^n = \sum_{\ell=0}^{n} \binom{n}{\ell} \mathbb{E}_i \tau_{i^*}^{\ell} \cdot \mathbb{E}_{i^*} \tau_k^{n-\ell} = \mathbb{E}_i \tau_{i^*}^n + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \mathbb{E}_i \tau_{i^*}^{\ell} \cdot \mathbb{E}_{i^*} \tau_k^{n-\ell} + \mathbb{E}_{i^*} \tau_k^n.$$

Then we inductively obtain that

$$\mathbb{E}_i \tau_k^n = \sum_{j \in \mathscr{P}(i) \setminus \mathscr{P}(k)} \mathbb{E}_j \tau_{j^*}^n + \sum_{j \in \mathscr{P}(i) \setminus \mathscr{P}(k)} \sum_{\ell=1}^{n-1} \binom{n}{\ell} \mathbb{E}_j \tau_{j^*}^{\ell} \cdot \mathbb{E}_{j^*} \tau_k^{n-\ell}$$

$$
\begin{aligned}
&= \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \frac{n}{\mu_j q_{jj^*}} \sum_{s\in T_j} \mu_s \mathbb{E}_s \tau_{j^*}^{n-1} + \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \sum_{\ell=1}^{n-1} \binom{n}{\ell} \frac{\ell}{\mu_j q_{jj^*}} \\
&\quad \cdot \sum_{s\in T_j} \mu_s \mathbb{E}_s \tau_{j^*}^{\ell-1} \cdot \mathbb{E}_{j^*} \tau_k^{n-\ell} \\
&= \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \frac{n}{\mu_j q_{jj^*}} \sum_{s\in T_j} \mu_s \left(\mathbb{E}_s \tau_{j^*}^{n-1} + \sum_{\ell=1}^{n-1} \binom{n-1}{\ell-1} \mathbb{E}_s \tau_{j^*}^{\ell-1} \cdot \mathbb{E}_{j^*} \tau_k^{n-\ell} \right) \\
&= \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \frac{n}{\mu_j q_{jj^*}} \sum_{s\in T_j} \mu_s \left(\mathbb{E}_s \tau_{j^*}^{n-1} + \sum_{\ell=0}^{n-2} \binom{n-1}{\ell} \mathbb{E}_s \tau_{j^*}^{\ell} \cdot \mathbb{E}_{j^*} \tau_k^{n-1-\ell} \right) \\
&= \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \frac{n}{\mu_j q_{jj^*}} \sum_{s\in T_j} \mu_s \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \mathbb{E}_s \tau_{j^*}^{\ell} \cdot \mathbb{E}_{j^*} \tau_k^{n-1-\ell} \\
&= n \sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)} \frac{1}{\mu_j q_{jj^*}} \sum_{s\in T_j} \mu_s \mathbb{E}_s \tau_k^{n-1}, \quad k \in \mathscr{P}(i),
\end{aligned}
$$

and

$$\mathbb{E}_i \tau_o^n = n \sum_{j\in\mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{s\in T_j} \mu_s \mathbb{E}_s \tau_o^{n-1}.$$

So the assertion in Theorem 1.1 holds for n. By the induction, we know that the assertion in Theorem 1.1 holds for all $n \geqslant 1$. The proof is finished. □

Remark 3.5 Under the conditions of Theorem 1.1, $(\mathbb{E}_i \tau_{i^*}^n,\ i \in T\setminus\{o\})$ is the minimal nonnegative solution of (3.2) and satisfies

$$m_i = \frac{1}{q_{ii^*}} \left(n q_{ii^*} f_i + \sum_{j\in J(i)} q_{ij} m_j \right), \quad i \in T\setminus\{o\}.$$

By Theorem 1.1, we can get a necessary condition for the exponential ergodicy of the process as follows.

Corollary 3.6 *Assume that the Q-matrix on the tree T is regular and the process is exponentially ergodic. Then*

$$\delta := \sup_{i\in T\setminus\{o\}} \sum_{j\in\mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell\in T_i} \mu_\ell < \infty.$$

Proof By the exponential ergodicy of the process, we see that the process is recurrent. Then

$$\mathbb{E}_i \tau_o^n = n \sum_{j\in\mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell\in T_j} \mu_\ell \mathbb{E}_\ell \tau_o^{n-1}, \quad i \in T,\ n \geqslant 1.$$

By the fact that $\mathbb{E}_i \tau_o^{n-1} \leqslant \mathbb{E}_\ell \tau_o^{n-1}$ for all $\ell \in T_i$ and Theorem 1.1, one gets that

$$\begin{aligned}\mathbb{E}_i\tau_o^n &\geqslant n\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\mathbb{E}_\ell\tau_o^{n-1}\\ &\geqslant n\bigg(\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\bigg)\mathbb{E}_i\tau_o^{n-1}\\ &\geqslant\cdots\\ &\geqslant n!\bigg(\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\bigg)^n,\quad n\geqslant 1.\end{aligned}$$

By the exponential ergodicity and [1, Theorem 4.44 (2)], there exists a λ with $0<\lambda<q_i$ for all i such that $\mathbb{E}_o\mathrm{e}^{\lambda\sigma_o}<\infty$. Then, by [1, p. 148], it holds that $\mathbb{E}_i\mathrm{e}^{\lambda\sigma_o}<\infty$, i.e., $\mathbb{E}_i\mathrm{e}^{\lambda\tau_o}<\infty$ for all $i\neq o$. The Taylor expansion

$$\infty>\mathbb{E}_i\mathrm{e}^{\lambda\tau_o}=\sum_{n\geqslant 0}\frac{\lambda^n}{n!}\mathbb{E}_i\tau_o^n$$

leads us to

$$\infty>\sum_{n\geqslant 0}\bigg(\lambda\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\bigg)^n,$$

which implies that

$$\lambda\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell<1.$$

Taking the supremum over $i\neq o$, we obtain

$$\delta=\sup_{i\in T\setminus\{o\}}\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\leqslant\frac{1}{\lambda}<\infty.$$

The proof is finished. □

Now, we present a remark about exponential moment as follows.

Remark 3.7 Under the conditions of Theorem 1.1, we obtain that

$$\mathbb{E}_i\mathrm{e}^{\lambda\sigma_o}=1+\lambda\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell\mathbb{E}_\ell\mathrm{e}^{\lambda\tau_o},\quad i\neq o.$$

In the end, we consider ℓ-ergodicity of birth-death processes on trees. Given a positive integer ℓ, a recurrent Q-process is called to be ℓ-ergodic provided that $\mathbb{E}_j\sigma_j^\ell<\infty$ for some (equivalently, all) j. Refer to [5]. Then we obtain the following result.

Corollary 3.8 *Assume that the Q-matrix on the tree T is regular and the process is recurrent. Then the process is ℓ-ergodic if and only if*

$$d(\ell):=\sum_{k\in J(o)}\sum_{j\in T_k}\mu_j\mathbb{E}_j\tau_o^{\ell-1}<\infty.$$

Proof By the strong Markov property and the single death property, we derive that

$$
\begin{aligned}
\mathbb{E}_o\sigma_o^n &= \mathbb{E}_o(\eta_1 + \sigma_o - \eta_1)^n \\
&= \mathbb{E}_o\eta_1^n + \sum_{s=1}^{n}\binom{n}{s}\mathbb{E}_o((\sigma_o-\eta_1)^s\eta_1^{n-s})) \\
&= \frac{n!}{q_o^n} + \sum_{s=1}^{n}\binom{n}{s}\sum_{k\in J(o)}\mathbb{E}_o(\eta_1^{n-s}\mathbb{1}_{\{X_{\eta_1}=k\}}\mathbb{E}_{X_{\eta_1}}\tau_o^s) \\
&= \frac{n!}{q_o^n} + \sum_{s=1}^{n}\binom{n}{s}\sum_{k\in J(o)}\frac{(n-s)!}{q_o^{n-s}}\cdot\frac{q_{ok}}{q_o}\mathbb{E}_k\tau_o^s \\
&= \frac{n!}{q_o^n} + \sum_{s=1}^{n}\binom{n}{s}\sum_{k\in J(o)}\frac{(n-s)!}{q_o^{n-s}}\cdot\frac{q_{ok}}{q_o}\cdot\frac{s}{\mu_k q_{ko}}\sum_{\ell\in T_k}\mu_\ell\mathbb{E}_\ell\tau_o^{s-1} \\
&= \frac{n!}{q_o^n} + \sum_{s=1}^{n}\frac{n!}{(s-1)!\,q_o^{n-s+1}}\sum_{k\in J(o)}\sum_{\ell\in T_k}\mu_\ell\mathbb{E}_\ell\tau_o^{s-1}.
\end{aligned}
$$

By the argument above, it is easy to check that $\mathbb{E}_o\sigma_o^n$ is finite if and only if $\sum_{k\in J(o)}\sum_{\ell\in T_k}\mu_\ell\mathbb{E}_\ell\tau_o^{n-1} < \infty$. Hence, the assertion holds. □

Acknowledgements The author acknowledges the constructive discussion with Professors Mu-Fa Chen, Yong-Hua Mao, and Yutao Ma, and thanks the anonymous referees for their very valuable suggestions and careful reading of the draft, which greatly improved the quality of the paper. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11571043, 11771047, 11871008).

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