Front. Math. China 2018, 13(5): 1215–1243
https://doi.org/10.1007/s11464-018-0722-z

RESEARCH ARTICLE

Criteria on ergodicity and strong ergodicity of single death processes

Yuhui ZHANG

School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, China

Abstract Based on an explicit representation of moments of hitting times for single death processes, the criteria on ergodicity and strong ergodicity are obtained. These results can be applied for an extended class of branching processes. Meanwhile, some sufficient and necessary conditions for recurrence and exponential ergodicity as well as extinction probability for the processes are presented.

Keywords Single death process, ergodicity, strong ergodicity, recurrence, moments of hitting times
MSC 60J60

1 Introduction

Consider a continuous-time homogeneous Markov chains $\{X(t)\colon t \geqslant 0\}$, on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, with transition probability matrix $P(t) = (p_{ij}(t))$ on a countable state space $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$. We call $\{X(t)\colon t \geqslant 0\}$ a single death process if its transition rate matrix $Q = (q_{ij}\colon i, j \in \mathbb{Z}_+)$ is irreducible and satisfies $q_{i,i-1} > 0$ for all $i \geqslant 1$ and $q_{i,i-j} = 0$ for all $i \geqslant j \geqslant 2$. Such a matrix $Q = (q_{ij})$ with $\sum_j q_{ij} = 0$ for every i (conservativeness) is called a single death Q-matrix. In the literature, the single death process is also called downwardly skip-free process.

Symmetrically, we can define single birth processes. The single birth process is nearly the largest class for which the explicit criteria on classical problems can be expected. Hence, the single birth process becomes a fundamental comparison tool in studying more complex processes, such as infinite-dimensional reaction-diffusion processes. Actually, the study on the process is quite fruited and relatively completed (cf. [2,4–6,11,15,17,18]).

Received June 29, 2018; accepted August 5, 2018
E-mail: zhangyh@bnu.edu.cn

Usually, the single birth process and the single death one are non-symmetric, and hence, they are regarded as the representative ones of the non-symmetric processes. For non-symmetric processes, comparing with the symmetric ones, our knowledge is much limited, except for single birth processes. For general single death process, we know some results on stationary distribution and criterion on zero-entrance of them. Refer to [1] and [13]. But as a special kind of single death process, the branching processes are fruitful and applicable intensively on which one of the main tools used is the generation functions. Although the generation function is not valid for general single death process, there exists some dual relations between single birth processes and single death ones. We expect to obtain some parallel or similar results for single death process on some classical problems, based on some ideas or approaches to study single birth processes and branching ones.

In this paper, we focus on obtaining some criteria on several ergodicity of single death processes. Define the hitting time

$$\tau_i := \inf\{t > 0\colon X_t = i\}, \quad i \geqslant 0,$$

the first jumping time

$$\eta_1 := \inf\{t > 0\colon X_t \neq X_0\},$$

and the first returning time

$$\sigma_i := \inf\{t > \eta_1\colon X_t = i\}, \quad i \geqslant 0.$$

Throughout the paper, we consider only totally stable and conservative single death Q-matrix:

$$q_i := -q_{ii} = \sum_{j\neq i} q_{ij} < +\infty, \quad i \in \mathbb{Z}_+.$$

The following sequences are used throughout this paper:

$$q_n^{(k)} = \sum_{j=k}^{+\infty} q_{nj}, \quad k > n \geqslant 0,$$

and

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \ 1 \leqslant n < i.$$

The main result is as follows.

Theorem 1.1 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then*

$$\mathbb{E}_n \sigma_0 = \sum_{1\leqslant k\leqslant n} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}, \quad n \geqslant 1,$$

and

$$\mathbb{E}_0\sigma_0 = \frac{1}{q_0} + \frac{1}{q_0}\sum_{k\geqslant 1} q_0^{(k)} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}.$$

Furthermore, the process is ergodic if and only if

$$D := \sum_{k\geqslant 1} q_0^{(k)} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < +\infty;$$

and it is strongly ergodic if and only if

$$S := \sum_{k\geqslant 1} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < +\infty.$$

Actually, for the last conclusion, the recurrence assumption can be replaced by the uniqueness one.

The original branching process can be described as follows. Let $\alpha > 0$, and let $(p_j\colon j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate

$$\alpha i p_0\colon i \to i-1, \quad i \geqslant 1,$$

and growth rate

$$\alpha i p_{k+1}\colon i \to i+k, \quad k \geqslant 1,\, i \in \mathbb{Z}_+.$$

Note that the process absorbs at state 0. In [7], an extended class of branching processes with the following Q-matrix is introduced:

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i-1,\, i \geqslant 1; \\ r_i p_{k+1}, & j = i+k,\, i,k \geqslant 1; \\ -r_i(1-p_1), & j = i \geqslant 1; \\ 0, & \text{else } i,j \in \mathbb{Z}_+. \end{cases} \tag{1.1}$$

where

$$r_i > 0,\, i \geqslant 1, \quad 0 < q_0 := \sum_{j\geqslant 1} q_{0j} < +\infty.$$

Define the convolution of two nonnegative vectors $\boldsymbol{a} = (a_n;\, n \geqslant 2)$ and $\boldsymbol{b} = (b_n;\, n \geqslant 2)$:

$$(\boldsymbol{a} * \boldsymbol{b})_n = \sum_{2\leqslant m\leqslant n} a_{n+2-m} b_m, \quad n \geqslant 2.$$

Define

$$f_n = \sum_{k \geqslant n} p_k,\ n \geqslant 0, \quad \boldsymbol{f} = (f_n;\, n \geqslant 2).$$

Denote the n-th convolution of $\boldsymbol{f}$ by $\boldsymbol{f}^{*n}$. In convention, $\boldsymbol{f}^{*1} = \boldsymbol{f}$.

The another main result is presented in the following.

Theorem 1.2 *Assume that the Q-matrix $Q = (q_{ij})$ given by (1.1) is irreducible and regular. Then the process is ergodic if and only if*

$$\sum_{\ell \geqslant 1} \frac{1}{r_\ell} \left(q_0^{(\ell)} + \sum_{1 \leqslant k \leqslant \ell-1} \frac{(\boldsymbol{f}^{*k} * \boldsymbol{q})_{\ell-k+1}}{p_0^k} \right) < +\infty;$$

it is strongly ergodic if and only if

$$\sum_{\ell \geqslant 1} \frac{1}{r_\ell} \left(1 + \sum_{1 \leqslant k \leqslant \ell-1} \frac{(\boldsymbol{f}^{*k} * \boldsymbol{1})_{\ell-k+1}}{p_0^k} \right) < +\infty,$$

where

$$\boldsymbol{1} = (1, 1, \ldots), \quad \boldsymbol{q} = (q_0^{(n-1)};\, n \geqslant 2).$$

This paper is organized as follows. The proof of Theorem 1.1 is given in the next section. Then Sections 3 is devoted to the proof of Theorem 1.2. In Section 4, an explicit sufficient condition for exponential ergodicity of single death processes is obtained. For recurrence of single death processes, we present some sufficient or necessary conditions, respectively, in Section 5, and the returning (extinction) probability of the process is obtained in this section.

2 Moments of hitting times, criteria on ergodicity and strong ergodicity

Let us begin with a simple result for the solution to a class of linear equations, which is an analogue of the results in [6, Section 2].

Lemma 2.1 *For given real numbers $(\alpha_{nk})_{n+1 \leqslant k \leqslant i}$ and $(f_n)_{1 \leqslant n \leqslant i}$, the solution $(g_n)_{1 \leqslant n \leqslant i}$ to the recursive inhomogeneous equations*

$$g_n = \sum_{n+1 \leqslant k \leqslant i} \alpha_{nk} g_k + f_n, \quad 1 \leqslant n \leqslant i, \tag{2.1}$$

can be represented as

$$g_n = \sum_{n \leqslant k \leqslant i} \gamma_{nk} f_k, \quad 1 \leqslant n \leqslant i, \tag{2.2}$$

where for fixed $k \geqslant 1$, $(\gamma_{nk})_{1 \leqslant n \leqslant k}$ with $\gamma_{kk} = 1$ is the solution to the recursive equations

$$\gamma_{nk} = \sum_{n+1 \leqslant j \leqslant k} \alpha_{nj} \gamma_{jk}, \quad 1 \leqslant n < k. \tag{2.3}$$

Proof Use induction. For $n = i$, we have

$$g_i = f_i = \gamma_{ii} f_i = \sum_{i \leqslant k \leqslant i} \gamma_{ik} f_k.$$

Assume that (2.2) holds for all $m \leqslant n \leqslant i$. When $n = m - 1 \geqslant 1$, from (2.1), we see that

$$\begin{aligned} g_{m-1} &= \sum_{m \leqslant k \leqslant i} \alpha_{m-1,k}\, g_k + f_{m-1} \\ &= \sum_{m \leqslant k \leqslant i} \alpha_{m-1,k} \sum_{k \leqslant \ell \leqslant i} \gamma_{k\ell} f_\ell + f_{m-1} \\ &= \sum_{m \leqslant \ell \leqslant i} \bigg(\sum_{m \leqslant k \leqslant \ell} \alpha_{m-1,k} \gamma_{k\ell} \bigg) f_\ell + f_{m-1} \\ &= \sum_{m \leqslant \ell \leqslant i} \gamma_{m-1,\ell} f_\ell + f_{m-1} \\ &= \sum_{m-1 \leqslant \ell \leqslant i} \gamma_{m-1,\ell} f_\ell. \end{aligned}$$

Hence, (2.2) holds for $n = m - 1$. By induction, the representation (2.2) holds for all $n \geqslant 1$. □

Note that the coefficients (α_{nk}) are often fixed and so are (γ_{nk}). Then Lemma 2.1 says that once replacing (α_{nk}) by (γ_{nk}), the solution to (2.1) has a complete representation (2.2), mainly in terms of the inhomogeneous term (f_n) in (2.1).

Without condition $\gamma_{kk} = 1$, (2.3) is clearly homogeneous. However, it becomes inhomogeneous under condition $\gamma_{kk} \neq 0$ (then one may assume that $\gamma_{kk} = 1$):

$$\gamma_{nk} = \sum_{n+1 \leqslant j \leqslant k-1} \alpha_{nj} \gamma_{jk} + \alpha_{nk} \gamma_{kk}, \quad 1 \leqslant n \leqslant k-1,$$

provided $\alpha_{k-1,k} \neq 0$. Otherwise, once $\alpha_{k-1,k} = 0$, by induction, we actually have $\gamma_{nk} = 0$ for all $1 \leqslant n \leqslant k-1$. Thus, under condition $\gamma_{kk} = 1$, by Lemma 2.1 (for fixed k), we have the following alternative representation of (γ_{nk}):

$$\gamma_{nk} = \sum_{n \leqslant j \leqslant k-1} \gamma_{nj} \alpha_{jk}, \quad 1 \leqslant n \leqslant k-1.$$

In what follows, we will use the following variant of Lemma 2.1. Replacing the coefficient (α_{nk}) by $(\alpha_{nk}\beta_k)$, for some non-zero sequence (β_n), and set $h_n = g_n/\beta_n$ $(1 \leqslant n \leqslant i)$, we obtain the following result.

Corollary 2.2 *The solution $(h_n)_{1 \leqslant n \leqslant i}$ to the recursive equations*

$$h_n = \frac{1}{\beta_n} \bigg(\sum_{n+1 \leqslant k \leqslant i} \alpha_{nk} h_k + f_n \bigg), \quad 1 \leqslant n \leqslant i,$$

can be represented as

$$h_n = \sum_{n\leqslant k\leqslant i} \frac{\gamma_{nk}}{\beta_k} f_k, \quad 1 \leqslant n \leqslant i,$$

where for each fixed i, $(\gamma_{ni})_{1\leqslant n\leqslant i}$ *with* $\gamma_{ii} = 1$ *is the solution to the equations*

$$\gamma_{ni} = \frac{1}{\beta_n} \sum_{n+1\leqslant k\leqslant i} \alpha_{nk}\gamma_{ki}, \quad 1 \leqslant n < i.$$

Equivalently,

$$\gamma_{ii} = 1, \quad \gamma_{ni} = \sum_{n\leqslant k\leqslant i-1} \frac{\gamma_{nk}}{\beta_k} \alpha_{ki}, \quad 1 \leqslant n \leqslant i-1.$$

Specifying $\beta_n = q_{n,n-1}$ and $\alpha_{nk} = q_n^{(k)}$ and using the successive formula of $G_n^{(k)}$, we obtain the following result.

Corollary 2.3 *For given* f, *the sequence* (h_n) *defined successively by*

$$h_n = \frac{1}{q_{n,n-1}} \bigg(\sum_{n+1\leqslant k\leqslant i} q_n^{(k)} h_k + f_n \bigg), \quad 1 \leqslant n \leqslant i,$$

has a unified expression as follows:

$$h_n = \sum_{k=n}^{i} \frac{G_n^{(k)}}{q_{k,k-1}} f_k, \quad 1 \leqslant n \leqslant i.$$

In particular, the sequence $\big(G_n^{(k)}\big)$ *has the following expression*:

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \sum_{k=n}^{i-1} \frac{G_n^{(k)} q_k^{(i)}}{q_{k,k-1}}, \; 1 \leqslant n \leqslant i-1.$$

Before moving further, let us mention a comparison result for different γ_{nj}, which is useful elsewhere.

Proposition 2.4 *For each triple* $1 \leqslant n \leqslant i < j$, *the following assertion holds*:

$$\gamma_{nj} = \sum_{n\leqslant k\leqslant i} \frac{\gamma_{nk}}{\beta_k} \sum_{i+1\leqslant \ell\leqslant j} \alpha_{k\ell}\gamma_{\ell j}. \tag{2.4}$$

Furthermore, if $\alpha_{nk} \geqslant 0$ *and* $\beta_n > 0$ *for all* $1 \leqslant n < k$, *then* $\gamma_{ni}\gamma_{ij} \leqslant \gamma_{nj}$ *for all* $1 \leqslant n \leqslant i \leqslant j$.

Proof The first assertion is simply a consequence of Corollary 2.2. In fact, for fixed $i > j$, take

$$f_n = \sum_{i+1\leqslant \ell\leqslant j} \alpha_{n\ell}\gamma_{\ell j}, \quad 1 \leqslant n \leqslant i.$$

Then, for $1 \leqslant n \leqslant i$,

$$\gamma_{nj} = \frac{1}{\beta_n}\bigg[\sum_{n+1\leqslant \ell\leqslant i} \alpha_{n\ell}\gamma_{\ell j} + \sum_{i+1\leqslant \ell\leqslant j} \alpha_{n\ell}\gamma_{\ell j}\bigg] = \frac{1}{\beta_n}\bigg[\sum_{n+1\leqslant \ell\leqslant i} \alpha_{n\ell}\gamma_{\ell j} + f_n\bigg].$$

Hence, by Corollary 2.2, we get

$$\gamma_{nj} = \sum_{n\leqslant k\leqslant i} \frac{\gamma_{nk}}{\beta_k} f_k = \sum_{n\leqslant k\leqslant i} \frac{\gamma_{nk}}{\beta_k} \sum_{i+1\leqslant \ell\leqslant j} \alpha_{k\ell}\, \gamma_{\ell j}, \quad 1 \leqslant n \leqslant i.$$

If $\alpha_{nk} \geqslant 0$ and $\beta_n > 0$ for all n and k, then from (2.4), it follows that for all $1 \leqslant n < i < j$,

$$\gamma_{nj} = \gamma_{ni}\gamma_{ij} + \sum_{n\leqslant k\leqslant i-1} \frac{\gamma_{nk}}{\beta_k} \sum_{i+1\leqslant \ell\leqslant j} \alpha_{k\ell}\gamma_{\ell j} \geqslant \gamma_{ni}\gamma_{ij}.$$

In the cases of $n = i$ or $i = j$, the conclusion is trivial. □

To prove Theorem 1.1, we first prove the following proposition.

Proposition 2.5 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then*

$$\mathbb{E}_n\tau_{n-1} = \sum_{k\geqslant n} \frac{G_n^{(k)}}{q_{k,k-1}}, \quad n \geqslant 1.$$

As prepration for the proof of Proposition 2.5, we need to check two lemmas.

Lemma 2.6 *Define*

$$h_n := \sum_{k\geqslant n} \frac{G_n^{(k)}}{q_{k,k-1}}, \quad n \geqslant 1.$$

Then $(h_n;\, n \geqslant 1)$ is the minimal nonnegative solution of the following equation:

$$x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{\ell\geqslant i+1} \frac{q_i^{(\ell)}}{q_i} \cdot x_\ell + \frac{1}{q_i}, \quad i \geqslant 1. \tag{2.5}$$

Moreover, $(h_n;\, n \geqslant 1)$ satisfies the following relations:

$$h_n = \frac{1}{q_{n,n-1}}\bigg(1 + \sum_{k=n+1}^{+\infty} q_n^{(k)} h_k\bigg), \quad n \geqslant 1. \tag{2.6}$$

Proof Fix $N \in \mathbb{Z}_+$, and define Q-matrix $Q^{(N)} = (\widetilde{q}_{ij})$ on $\{0, 1, \dots, N\}$:

$$\widetilde{q}_{ij} = \begin{cases} q_{ij}, & i, j < N; \\ q_i^{(N)}, & i < N,\, j = N; \\ (q_N \vee N)(1 + G^{(N)}), & i = N,\, j = N-1; \\ -(q_N \vee N)(1 + G^{(N)}), & i = j = N; \\ 0, & i = N,\, j < N-1, \end{cases}$$

where $G^{(N)} = \max_{1\leqslant n\leqslant N} G_n^{(N)}$.

Define

$$\widetilde{q}_n^{(k)} = \sum_{j=k}^{N} \widetilde{q}_{nj}, \quad 0 \leqslant n < k \leqslant N,$$

and

$$\widetilde{G}_i^{(i)} = 1, \quad \widetilde{G}_n^{(i)} = \frac{1}{\widetilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \widetilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \ 1 \leqslant n < i \leqslant N.$$

It is easy to check that

$$h_n^{(N)} := \sum_{k=n}^{N} \frac{\widetilde{G}_n^{(k)}}{\widetilde{q}_{k,k-1}}, \quad 1 \leqslant n \leqslant N, \tag{2.7}$$

is a unique solution (the minimal non-negative solution) to the following equations:

$$x_i = \frac{\widetilde{q}_i^{(i+1)}}{\widetilde{q}_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{\widetilde{q}_i^{(\ell)}}{\widetilde{q}_i} \cdot x_\ell + \frac{1}{\widetilde{q}_i}, \quad 1 \leqslant i \leqslant N. \tag{2.8}$$

Note that $\widetilde{q}_i := -\widetilde{q}_{ii} = -q_{ii} = q_i$ for all $i < N$ and $\widetilde{q}_n^{(k)} = q_n^{(k)}$ for all $n < k \leqslant N$. Furthermore, $\widetilde{G}_n^{(i)} = G_n^{(i)}$ for all $n \leqslant i \leqslant N$. Hence, we can rewrite (2.8) as

$$x_N = \frac{1}{(q_N \vee N)(1 + G^{(N)})}, \quad x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{q_i^{(\ell)}}{q_i} \cdot x_\ell + \frac{1}{q_i}, \quad 1 \leqslant i < N. \tag{2.9}$$

On the one hand, from Equations (2.5) and (2.9), by [5, Theorem 2.7], we know that $(h_n^{(N)})$ is increasing to the minimal non-negative solution of Equation (2.5) as $N \to +\infty$. On the other hand, from (2.7), it follows that

$$h_n^{(N)} = \sum_{k=n}^{N-1} \frac{G_n^{(k)}}{q_{k,k-1}} + \frac{G_n^{(N)}}{(q_N \vee N)(1 + G^{(N)})} \longrightarrow \sum_{k=n}^{+\infty} \frac{G_n^{(k)}}{q_{k,k-1}} = h_n, \quad N \to +\infty,$$

for all $n \geqslant 1$. So it has proven that (h_n) is the minimal non-negative solution of Equation (2.5).

It is not difficult to check that $(h_n; n \geqslant 1)$ satisfies equality (2.6). The proof of assertions are finished. □

Lemma 2.7 *Assume that the single death Q-matrix $Q = (q_{ij})$ is irreducible and corresponding process is recurrent. Give $i_0 \in \mathbb{Z}_+$ arbitrarily. Then*

$$\mathbb{E}_i \tau_{i_0} \leqslant \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0.$$

Proof It is well known that $(\mathbb{E}_i\tau_{i_0};\ i\geqslant 0)$ is the minimal non-negative solution to the following equation:

$$x_{i_0}=0,\quad x_i=\sum_{j\neq i}\frac{q_{ij}}{q_i}\cdot x_j+\frac{1}{q_i},\ \ i\neq i_0.$$

From [5, Theorem 2.13] (Localization Theorem) and single death property, it follows directly that $(\mathbb{E}_i\tau_{i_0};\ i\geqslant i_0)$ is the minimal non-negative solution to the following equation:

$$x_{i_0}=0,\quad x_i=\sum_{j\neq i,j>i_0}\frac{q_{ij}}{q_i}\cdot x_j+\frac{1}{q_i},\ \ i>i_0. \tag{2.10}$$

Define

$$y_i=\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}}{q_{\ell,\ell-1}},\quad i\geqslant i_0.$$

It is not difficult to check that $(y_i;\ i\geqslant i_0)$ is a non-negative solution to Equation (2.10). Hence, $\mathbb{E}_i\tau_{i_0}\leqslant y_i$ for all $i\geqslant i_0$. So the assertion is proven. □

Proof of Proposition 2.5 On the one hand, for all $k>i-1\geqslant 0$, from strong Markov property and single death property, it follows that

$$\begin{aligned}\mathbb{E}_k\tau_{i-1}&=\mathbb{E}_k(\tau_{k-1}+\tau_{i-1}-\tau_{k-1})\\&=\mathbb{E}_k\tau_{k-1}+\mathbb{E}_k(\mathbb{E}_k(\tau_{i-1}-\tau_{k-1}\mid\mathscr{F}_{\tau_{k-1}}))\\&=\mathbb{E}_k\tau_{k-1}+\mathbb{E}_k(\mathbb{E}_{X_{\tau_{k-1}}}\tau_{i-1})\\&=\mathbb{E}_k\tau_{k-1}+\mathbb{E}_{k-1}\tau_{i-1}\\&=\cdots\\&=\sum_{\ell=i}^{k}\mathbb{E}_\ell\tau_{\ell-1}.\end{aligned}$$

Denote $\mathbb{E}_i\tau_{i-1}$ $(i\geqslant 1)$ by m_i. By strong Markov property and the equality above, one gets that

$$\begin{aligned}m_i&=\mathbb{E}_i\eta_1+\mathbb{E}_i((\tau_{i-1}-\eta_1)\mathbb{1}_{\{X_{\eta_1}=i-1\}})+\sum_{k\geqslant i+1}\mathbb{E}_i((\tau_{i-1}-\eta_1)\mathbb{1}_{\{X_{\eta_1}=k\}})\\&=\frac{1}{q_i}+\mathbb{E}_i(\mathbb{E}_i((\tau_{i-1}-\eta_1)\mathbb{1}_{\{X_{\eta_1}=i-1\}}\mid\mathscr{F}_{\eta_1}))\\&\quad+\sum_{k\geqslant i+1}\mathbb{E}_i(\mathbb{E}_i((\tau_{i-1}-\eta_1)\mathbb{1}_{\{X_{\eta_1}=k\}}\mid\mathscr{F}_{\eta_1}))\\&=\frac{1}{q_i}+\mathbb{E}_i(\mathbb{1}_{\{X_{\eta_1}=i-1\}}\mathbb{E}_{X_{\eta_1}}\tau_{i-1})+\sum_{k\geqslant i+1}\mathbb{E}_i(\mathbb{1}_{\{X_{\eta_1}=k\}}\mathbb{E}_{X_{\eta_1}}\tau_{i-1})\end{aligned}$$

$$= \frac{1}{q_i} + \sum_{k\geqslant i+1} \frac{q_{ik}}{q_i} \sum_{\ell=i+1}^{k} \mathbb{E}_\ell \tau_{\ell-1} + \sum_{k\geqslant i+1} \frac{q_{ik}}{q_i} \mathbb{E}_i \tau_{i-1}$$
$$= \frac{q_i^{(i+1)}}{q_i} \cdot m_i + \sum_{\ell\geqslant i+1} \frac{q_i^{(\ell)}}{q_i} \cdot m_\ell + \frac{1}{q_i}.$$

Hence, $(m_i;\ i \geqslant 1)$ is a solution of Equation (2.5). From Lemma 2.6, it follows immediately that $m_i \geqslant h_i$ for all $i \geqslant 1$.

On the other hand, fix i_0 arbitrarily. By Lemma 2.7, we obtain that

$$\mathbb{E}_i \tau_{i_0} \leqslant \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}, \quad i \geqslant i_0.$$

In particular, it holds that

$$m_{i_0+1} = \mathbb{E}_{i_0+1} \tau_{i_0} \leqslant h_{i_0+1}.$$

From the arbitrariness of i_0, it follows that $m_n \leqslant h_n$ for all $n \geqslant 1$.

Summing up the arguments above, we know that $m_n = h_n$ for all $n \geqslant 1$. The proof of the assertion is finished. □

Remark 2.8 Under the assumption of Proposition 2.5, by Lemma 2.6, it is obtained that $(m_n;\ n \geqslant 1)$ is the minimal non-negative solution of (2.5), which satisfies

$$m_n = \frac{1}{q_{n,n-1}} \left(1 + \sum_{k\geqslant n+1} q_n^{(k)} m_k \right), \quad n \geqslant 1. \tag{2.11}$$

Moreover, for all $i \geqslant k$, it holds that

$$\mathbb{E}_i \tau_k = \sum_{k+1\leqslant j\leqslant i} m_j = \sum_{k+1\leqslant j\leqslant i} \sum_{\ell\geqslant j} \frac{G_j^{(\ell)}}{q_{\ell,\ell-1}}.$$

Now, we come to give the proof of Theorem 1.1 in details.

Proof of Theorem 1.1 By Remark 2.8, we know that for all $i \geqslant 1$,

$$\mathbb{E}_i \sigma_0 = \mathbb{E}_i \tau_0 = \sum_{1\leqslant j\leqslant i} m_j = \sum_{1\leqslant j\leqslant i} \sum_{\ell\geqslant j} \frac{G_j^{(\ell)}}{q_{\ell,\ell-1}}.$$

Furthermore, we have

$$\mathbb{E}_0 \sigma_0 = \frac{1}{q_0} + \sum_{j\geqslant 1} \frac{q_{0j}}{q_0} \mathbb{E}_j \tau_0 = \frac{1}{q_0} + \sum_{j\geqslant 1} \frac{q_{0j}}{q_0} \sum_{1\leqslant k\leqslant j} m_k = \frac{1}{q_0} + \frac{1}{q_0} \sum_{k\geqslant 1} q_0^{(k)} m_k.$$

Finally, note that

$$\mathbb{E}_0 \sigma_0 = \frac{1+D}{q_0}, \quad D \leqslant q_0 S.$$

From [5, Theorem 4.44 (1)], it follows immediately that the single death process is ergodic if and only if $\mathbb{E}_0\sigma_0 < +\infty$, which is now equivalent to $D < +\infty$. By [5, Theorem 4.44 (3)] or [9], the process is strongly ergodic if and only if $\sup_{k\geqslant 0}\mathbb{E}_i\sigma_0 < +\infty$, equivalently, $S < +\infty$. As mentioned in the proof of the cited book, for ergodicity, the uniqueness assumption is enough instead of the recurrence one. The proof is now finished. □

Now, we illustrate our results by two examples.

Example 2.9 Assume that the birth-death Q-matrix (a_i, b_i) is totally stable and conservative. Define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i},\ i \geqslant 1; \quad \mu[i, +\infty) = \sum_{k\geqslant i}\mu_k,\ i \geqslant 0.$$

Then

$$G_n^{(i)} = \frac{\mu_i a_i}{\mu_n a_n}, \quad 1 \leqslant n \leqslant i.$$

Suppose that the corresponding birth-death chain is recurrent. Then

$$m_n = \frac{\mu[n, +\infty)}{\mu_n a_n}, \quad n \geqslant 1,$$

and

$$\mathbb{E}_i\tau_{i_0} = \sum_{i_0+1\leqslant k\leqslant i}\frac{\mu[k, +\infty)}{\mu_k a_k} = \sum_{i_0\leqslant k\leqslant i-1}\frac{\mu[k+1, +\infty)}{\mu_k b_k}, \quad i > i_0.$$

Assume that the chain is unique. Then the chain is ergodic if and only if $D = \mu[1, +\infty) < +\infty$, equivalently, $\mu := \mu[0, +\infty) < +\infty$; the chain is strongly ergodic if and only if

$$S = \sum_{n\geqslant 1}\frac{\mu[n, +\infty)}{\mu_n a_n} = \sum_{n\geqslant 0}\frac{\mu[n+1, +\infty)}{\mu_n b_n} < +\infty.$$

These results are well known. Refer to [5].

Example 2.10 Give a constant $b > 2$. Define a totally stable, conservative, and irreducible single death Q-matrix $Q = (q_{ij})$ as follows:

$$q_{ij} = \frac{b-1}{b^{j-i+2}},\ j \geqslant i+1; \quad q_{i,i-1} = \frac{b-1}{b},\ q_i = -q_{ii} = \frac{b^2-b+1}{b^2},\ i \geqslant 1;$$

$$q_{0j} = \frac{b-1}{b^{j+1}},\ j \geqslant 1; \quad q_0 = -q_{00} = \frac{1}{b}.$$

In Section 5, we know that the corresponding is recurrent. Then

$$q_n^{(k)} = \frac{1}{b^{k-n+1}},\ 1 \leqslant n < k; \quad q_0^{(k)} = \frac{1}{b^k},\ k \geqslant 1,$$

and

$$G_n^{(i)} = \frac{1}{b(b-1)^{i-n}}, \ 1 \leqslant n < i; \quad m_n = \frac{b-1}{b-2}, \ n \geqslant 1.$$

$$\mathbb{E}_i \tau_{i_0} = (i - i_0) \frac{b-1}{b-2}, \quad i > i_0.$$

Therefore, by $D = 1/(b-2)$ and $S = +\infty$, we know that the process is ergodic but not strongly ergodic.

At last, we present a result which is directly obtained by [12] and Proposition 2.5 and omit the detailed proof here.

Theorem 2.11 *Assume that the single death Q-matrix $Q = (q_{ij})$ is regular such that the state 0 is an absorbing state and the absorption occurs almost surely. Furthermore, assume that $Q = (q_{ij})$ is irreducible on $\mathbb{N} := \{1, 2, \ldots\}$. If*

$$\sum_{n \geqslant 1} \sum_{k \geqslant n} \frac{G_n^{(k)}}{q_{k,k-1}} < +\infty,$$

then there exists a unique quasi-stationary distribution ρ for the single death process. Moreover, for any probability measure μ on $\mathbb{N}$, we have

$$\|\mathbb{P}_\mu(X(t) \in \cdot | t < \tau_0) - \rho\|_{TV} \leqslant 2(1-\gamma)^{[t]}, \quad t \geqslant 0,$$

for some positive constant γ independent of μ.

3 An extended class of branching processes

For the extended class of branching processes defined in (1.1), it is easy to check that

$$q_i^{(k)} = r_i \sum_{j=k}^{+\infty} p_{j-i+1} = r_i \sum_{j=k-i+1}^{+\infty} p_j = r_i f_{k-i+1}, \quad k > i \geqslant 1,$$

and

$$G_n^{(k)} = \frac{1}{p_0} \sum_{\ell=n+1}^{k} f_{\ell-n+1} G_\ell^{(k)}, \quad 1 \leqslant n < k, \tag{3.1}$$

in particular,

$$G_{k-1}^{(k)} = \frac{f_2}{p_0}, \quad k \geqslant 2.$$

Define

$$M_1 = \sum_{k=1}^{+\infty} k p_k, \quad \Gamma = \sum_{k \geqslant 1} k p_{k+1}.$$

Note that

$$M_1 = \sum_{k\geqslant 1} f_k, \quad \Gamma = \sum_{k\geqslant 2} f_k = M_1 - f_1 = M_1 + p_0 - 1.$$

From the definition of convolution, it follows that

$$\boldsymbol{a} * \boldsymbol{b} = \boldsymbol{b} * \boldsymbol{a}, \quad (\boldsymbol{a} * \boldsymbol{b}) * \mathbf{c} = \boldsymbol{a} * (\boldsymbol{b} * \mathbf{c}).$$

When $a_n \leqslant b_n$ $(n \geqslant 2)$, we denote $\boldsymbol{a} \leqslant \boldsymbol{b}$. If $\boldsymbol{a} \leqslant \boldsymbol{b}$, then $\boldsymbol{a} * \mathbf{c} \leqslant \boldsymbol{b} * \mathbf{c}$.

Before proving Theorem 1.2, we present one important lemma as follows.

Lemma 3.1 *Give a nonnegative vector* $\boldsymbol{g} = (g_n;\, n \geqslant 2)$. *Then*

$$\sum_{1\leqslant n\leqslant k-1} g_{n+1} G_n^{(k)} = \sum_{1\leqslant n\leqslant k-1} \frac{(\boldsymbol{f}^{*n} * \boldsymbol{g})_{k-n+1}}{p_0^n}. \tag{3.2}$$

Proof In fact, by $G_k^{(k)} = 1$ and (3.1), it is derived that

$$\begin{aligned}\sum_{1\leqslant n\leqslant k-1} g_{n+1} G_n^{(k)} &= \frac{1}{p_0} \sum_{1\leqslant n\leqslant k-1}\ \sum_{n+1\leqslant \ell\leqslant k} g_{n+1} f_{\ell-n+1} G_\ell^{(k)} \\ &= \frac{1}{p_0} \sum_{2\leqslant \ell\leqslant k}\ \sum_{1\leqslant n\leqslant \ell-1} g_{n+1} f_{\ell-n+1} G_\ell^{(k)} \\ &= \frac{1}{p_0} \sum_{2\leqslant \ell\leqslant k}\ \sum_{2\leqslant m\leqslant \ell} f_{\ell+2-m} g_m G_\ell^{(k)} \\ &= \frac{1}{p_0} \sum_{2\leqslant \ell\leqslant k} (\boldsymbol{f} * \boldsymbol{g})_\ell G_\ell^{(k)}.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\sum_{1\leqslant n\leqslant k-1} g_{n+1} G_n^{(k)} &= \frac{(\boldsymbol{f}^{*1} * \boldsymbol{g})_k}{p_0} \mathbb{1}_{\{k\geqslant 2\}} + \frac{1}{p_0} \sum_{2\leqslant n\leqslant k-1} (\boldsymbol{f} * \boldsymbol{g})_n G_n^{(k)} \\ &= \frac{(\boldsymbol{f}^{*1} * \boldsymbol{g})_k}{p_0} \mathbb{1}_{\{k\geqslant 2\}} + \frac{1}{p_0^2} \sum_{2\leqslant n\leqslant k-1} (\boldsymbol{f} * \boldsymbol{g})_n \sum_{n+1\leqslant \ell\leqslant k} f_{\ell-n+1} G_\ell^{(k)} \\ &= \frac{(\boldsymbol{f}^{*1} * \boldsymbol{g})_k}{p_0} \mathbb{1}_{\{k\geqslant 2\}} + \frac{1}{p_0^2} \sum_{3\leqslant \ell\leqslant k}\ \sum_{2\leqslant n\leqslant \ell-1} f_{\ell+1-n} (\boldsymbol{f} * \boldsymbol{g})_n G_\ell^{(k)} \\ &= \frac{(\boldsymbol{f}^{*1} * \boldsymbol{g})_k}{p_0} \mathbb{1}_{\{k\geqslant 2\}} + \frac{1}{p_0^2} \sum_{3\leqslant \ell\leqslant k} (\boldsymbol{f}^{*2} * \boldsymbol{g})_{\ell-1} G_\ell^{(k)} \\ &= \frac{(\boldsymbol{f}^{*1} * \boldsymbol{g})_k}{p_0} \mathbb{1}_{\{k\geqslant 2\}} + \frac{(\boldsymbol{f}^{*2} * \boldsymbol{g})_{k-1}}{p_0^2} \mathbb{1}_{\{k\geqslant 3\}} \\ &\quad + \frac{1}{p_0^2} \sum_{3\leqslant n\leqslant k-1} (\boldsymbol{f}^{*2} * \boldsymbol{g})_{n-1} G_n^{(k)}\end{aligned}$$

$$= \cdots$$
$$= \sum_{1\leqslant n\leqslant k-1} \frac{(\boldsymbol{f}^{*n} * \boldsymbol{g})_{k-n+1}}{p_0^n}.$$

Hence, the proof of (3.2) is finished. □

In the following, we come to verify Theorem 1.2.

Proof of Theorem 1.2 Define $\mathbf{1} = (1, 1, \ldots)$ and $\boldsymbol{q} = (q_0^{(n-1)}; n \geqslant 2)$. Replacing $\boldsymbol{g}$ in (3.2) by $\mathbf{1}$ and $\boldsymbol{q}$, respectively, it follows that

$$\sum_{1\leqslant n\leqslant k-1} G_n^{(k)} = \sum_{1\leqslant n\leqslant k-1} \frac{(\boldsymbol{f}^{*n} * \mathbf{1})_{k-n+1}}{p_0^n}$$

and

$$\sum_{1\leqslant n\leqslant k-1} q_0^{(n)} G_n^{(k)} = \sum_{1\leqslant n\leqslant k-1} \frac{(\boldsymbol{f}^{*n} * \boldsymbol{q})_{k-n+1}}{p_0^n}.$$

Furthermore, we get

$$\begin{aligned} D &= \sum_{k\geqslant 1} q_0^{(k)} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} \\ &= \sum_{\ell\geqslant 1} \frac{1}{r_\ell p_0} \sum_{1\leqslant k\leqslant \ell} q_0^{(k)} G_k^{(\ell)} \\ &= \sum_{\ell\geqslant 1} \frac{1}{r_\ell p_0} \left(q_0^{(\ell)} + \sum_{1\leqslant k\leqslant \ell-1} \frac{(\boldsymbol{f}^{*k} * \boldsymbol{q})_{\ell-k+1}}{p_0^k} \right) \end{aligned}$$

and

$$\begin{aligned} S &= \sum_{k\geqslant 1} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} \\ &= \sum_{\ell\geqslant 1} \frac{1}{r_\ell p_0} \sum_{1\leqslant k\leqslant \ell} G_k^{(\ell)} \\ &= \sum_{\ell\geqslant 1} \frac{1}{r_\ell p_0} \left(1 + \sum_{1\leqslant k\leqslant \ell-1} \frac{(\boldsymbol{f}^{*k} * \mathbf{1})_{\ell-k+1}}{p_0^k} \right). \end{aligned}$$

By the argument above and Theorem 1.1, we know that the assertions hold immediately. □

Come back to Example 2.10. Fix a positive constant a such that

$$a < 1 - \frac{1}{b^2 - b + 1}.$$

Then Example 2.10 is the special case of (1.1):

$$r_i = \frac{b-1}{ab}, \quad p_0 = a, \quad p_1 = 1 - a - \frac{a}{b^2 - b}, \quad p_j = \frac{a}{b^j}, \ j \geqslant 2.$$

Furthermore,

$$f_n = \frac{a}{(b-1)b^{n-1}},\ n \geqslant 2; \quad q_0^{(\ell)} = \frac{1}{b^\ell},\ \ell \geqslant 1.$$

Now, let us verify

$$(\boldsymbol{f}^{*k} * \boldsymbol{q})_{\ell-k+1} = \frac{a^k}{(b-1)^k b^\ell}\,\mathrm{C}_{\ell-1}^k, \quad 1 \leqslant k \leqslant \ell - 1. \tag{3.3}$$

At first, when $k = 1$, it holds that

$$(\boldsymbol{f} * \boldsymbol{q})_\ell = \sum_{n=2}^{\ell} f_{\ell+2-n} q_0^{(n-1)} = \sum_{n=2}^{\ell} \frac{a}{(b-1)b^{\ell+1-n}} \cdot \frac{1}{b^{n-1}} = \frac{a}{(b-1)b^\ell}\,\mathrm{C}_{\ell-1}^1.$$

So (3.3) holds for $k = 1$. Assume that (3.3) holds until $k = s$. For $k = s + 1$, by

$$\sum_{n=0}^{k} \mathrm{C}_{m+n}^m = \mathrm{C}_{m+k+1}^{m+1},$$

we obtain

$$\begin{aligned}
(\boldsymbol{f}^{*(s+1)} * \boldsymbol{q})_{\ell-s} &= \sum_{n=2}^{\ell-s} f_{\ell-s+2-n} (\boldsymbol{f}^{*s} * \boldsymbol{q})_n \\
&= \sum_{n=2}^{\ell-s} \frac{a}{(b-1)b^{\ell-s+1-n}} \cdot \frac{a^s}{(b-1)^s b^{n+s-1}}\,\mathrm{C}_{n+s-2}^s \\
&= \frac{a^{s+1}}{(b-1)^{s+1} b^\ell} \sum_{n=2}^{\ell-s} \mathrm{C}_{n+s-2}^k \\
&= \frac{a^{s+1}}{(b-1)^{s+1} b^\ell} \sum_{n=0}^{\ell-s-2} \mathrm{C}_{s+n}^s \\
&= \frac{a^{s+1}}{(b-1)^{s+1} b^\ell}\,\mathrm{C}_{\ell-1}^{s+1}.
\end{aligned}$$

Hence, (3.3) holds for $k = s + 1$. By induction, (3.3) holds for all $1 \leqslant k \leqslant \ell - 1$. Then

$$\begin{aligned}
\sum_{1 \leqslant k \leqslant \ell-1} \frac{(\boldsymbol{f}^{*k} * \boldsymbol{q})_{\ell-k+1}}{p_0^k} &= \sum_{1 \leqslant k \leqslant \ell-1} \frac{1}{(b-1)^k b^\ell}\,\mathrm{C}_{\ell-1}^k \\
&= \frac{1}{b^\ell}\left(\left(1 + \frac{1}{b-1}\right)^{\ell-1} - 1\right) \\
&= \frac{1}{b(b-1)^{\ell-1}} - \frac{1}{b^\ell}.
\end{aligned}$$

Furthermore, we see that

$$\sum_{\ell\geqslant 1}\frac{1}{r_\ell}\bigg(q_0^{(\ell)}+\sum_{1\leqslant k\leqslant \ell-1}\frac{(\boldsymbol{f}^{*k}*\boldsymbol{q})_{\ell-k+1}}{p_0^k}\bigg)=\frac{a}{b-2}<+\infty.$$

Note that

$$(\boldsymbol{f}*\mathbf{1})_\ell=\frac{a}{(b-1)^2}\Big(1-\frac{1}{b^{\ell-1}}\Big).$$

From the equality above, it follows that

$$\begin{aligned}\sum_{\ell\geqslant 1}\frac{1}{r_\ell}\bigg(1+\sum_{1\leqslant k\leqslant \ell-1}\frac{(\boldsymbol{f}^{*k}*\mathbf{1})_{\ell-k+1}}{p_0^k}\bigg)&\geqslant\sum_{\ell\geqslant 2}\frac{1}{r_\ell}\cdot\frac{(\boldsymbol{f}*\mathbf{1})_\ell}{p_0}\\&=\frac{ab}{(b-1)^3}\sum_{\ell\geqslant 2}\Big(1-\frac{1}{b^{\ell-1}}\Big)\\&=+\infty.\end{aligned}$$

So the process in Example 2.10 is ergodic but not strong ergodic. This is the case of $b>2$. Note that

$$M_1=\frac{a}{(b-1)^2}+1-a.$$

So $M_1\leqslant 1$ if and only if $b\geqslant 2$. Hence, it is easy to know that

the process is unique when $b>2$ by [7, Theorem 1.2 (i)];

the process is unique and null recurrent when $b=2$ by [7, Theorems 1.2 (i), 1.3 (i)];

the process is unique and transient when $1<b<2$ by [7, Theorems 1.2 (ii), 1.3 (i)].

4 Exponential ergodicity

In this section, we consider the exponential ergodicity of single death processes. Note that

$$\frac{G_n^{(m)}}{G_1^{(m)}}=\frac{1}{q_{n,n-1}}\sum_{k\geqslant n+1}q_n^{(k)}\frac{G_k^{(m)}}{G_1^{(m)}}\mathbb{1}_{[1,m]}(k).$$

By Fatou's Lemma, we know that

$$\begin{aligned}\varliminf_{m\to+\infty}\frac{G_n^{(m)}}{G_1^{(m)}}&\geqslant\frac{1}{q_{n,n-1}}\sum_{k\geqslant n+1}q_n^{(k)}\varliminf_{m\to+\infty}\frac{G_k^{(m)}}{G_1^{(m)}}\mathbb{1}_{[1,m]}(k)\\&=\frac{1}{q_{n,n-1}}\sum_{k\geqslant n+1}q_n^{(k)}\varliminf_{m\to+\infty}\frac{G_k^{(m)}}{G_1^{(m)}}.\end{aligned}$$

Define

$$G_n = \varliminf_{m\to+\infty} \frac{G_n^{(m)}}{G_1^{(m)}}, \quad n \geqslant 1. \tag{4.1}$$

Then

$$G_n \geqslant \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{+\infty} q_n^{(k)} G_k, \quad n \geqslant 1. \tag{4.2}$$

Now, we present a sufficient condition for the exponential ergodicity of single death processes as follows.

Theorem 4.1 *Let the single death Q-matrix be regular and irreducible. Assume*

$$\sum_{k\geqslant 1} q_0^{(k)} G_k < +\infty. \tag{4.3}$$

If

$$q := \inf_{n\geqslant 0} q_n > 0, \quad M := \sup_{n\geqslant 1} \left(\sum_{k=1}^{n} G_k \sum_{j=n}^{+\infty} \frac{1}{q_{j,j-1} G_j} \right) < +\infty,$$

then the process is exponentially ergodic.

Proof In view of [5, Theorem 4.45 (2)], the condition $q > 0$ is indeed necessary. From [5, Theorem 4.45 (2)], the process is exponentially ergodic if and only if, for some λ with $0 < \lambda < q_i$ for all $i \in \mathbb{Z}_+$, the system of inequalities

$$\begin{cases} y_i \geqslant 1, & i \geqslant 0, \\ \sum_j q_{ij} y_j \leqslant -\lambda y_i, & i \geqslant 1, \\ \sum_{j\geqslant 1} q_{0j} y_j < +\infty, \end{cases} \tag{4.4}$$

has a finite solution (y_i). We need to construct a solution (g_i) to (4.4) for a fixed λ with $0 < \lambda < q$. First, define an operator

$$\mathit{II}_i(f) = \frac{1}{f_i} \sum_{j=1}^{i} G_j \sum_{k=j}^{+\infty} \frac{f_k}{q_{k,k-1} G_k}, \quad i \geqslant 1.$$

Next, define

$$\varphi_i = \sum_{j=1}^{i} G_j, \quad i \geqslant 1.$$

Then φ is increasing in i and $\varphi_1 = G_1$. Let $f = cq_{10}\sqrt{\varphi}/\sqrt{G_1}$ for some $c > 1$. Then f is increasing and $f_1 = cq_{10}$. Finally, define $g = f\mathit{II}(f)$. Then g is increasing and

$$g_1 = G_1 \sum_{k=1}^{+\infty} \frac{f_k}{q_{k,k-1} G_k} \geqslant \frac{f_1}{q_{10}} = c > 1.$$

By [3, Lemma 3.6], it follows that

$$\begin{aligned} g_i &= \frac{cq_{10}}{\sqrt{G_1}} \sum_{j=1}^{i} G_j \sum_{k=j}^{+\infty} \frac{\sqrt{\varphi_k}}{q_{k,k-1}G_k} \\ &\leqslant \frac{2Mcq_{10}}{\sqrt{G_1}} \sum_{j=1}^{i} G_j \varphi_j^{-1/2} \\ &\leqslant \frac{2Mcq_{10}}{G_1} \sum_{j=1}^{i} G_j \\ &< +\infty, \quad i \geqslant 1. \end{aligned}$$

Then, by the argument above and (4.3), it is obtained that

$$\sum_{j\geqslant 1} q_{0j} g_j \leqslant \frac{2Mcq_{10}}{G_1} \sum_{j\geqslant 1} q_{0j} \sum_{1\leqslant k\leqslant j} G_k = \frac{2Mcq_{10}}{G_1} \sum_{k\geqslant 1} q_0^{(k)} G_k < +\infty.$$

Let $g_0 = 1$. Then $1 \leqslant g_i < +\infty$ for all $i \geqslant 0$. We now determine λ in terms of (4.4). When $i = 1$, we get

$$\lambda g_1 \leqslant q_{10}(g_1 - g_0) - \sum_{\ell=2}^{+\infty} q_1^{(\ell)} G_\ell \sum_{k=\ell}^{+\infty} \frac{f_k}{q_{k,k-1}G_k}.$$

By (4.2) and $g_0 = 1$, it suffices that

$$\begin{aligned} \lambda g_1 &\leqslant q_{10} G_1 \sum_{k=1}^{+\infty} \frac{f_k}{q_{k,k-1}G_k} - q_{10} - q_{10} G_1 \sum_{k=2}^{+\infty} \frac{f_k}{q_{k,k-1}G_k} \\ &= f_1 - q_{10} \\ &= (c-1)q_{10} \\ &= \frac{c-1}{c} \cdot f_1. \end{aligned}$$

We need

$$\lambda \leqslant \frac{c-1}{cII_1(f)}. \tag{4.5}$$

When $i \geqslant 2$, we should have

$$\lambda g_i \leqslant q_{i,i-1} G_i \sum_{k=i}^{+\infty} \frac{f_k}{q_{k,k-1}G_k} - \sum_{\ell=i+1}^{+\infty} q_i^{(\ell)} G_\ell \sum_{k=\ell}^{+\infty} \frac{f_k}{q_{k,k-1}G_k}.$$

For this, it suffices that

$$\lambda g_i \leqslant q_{i,i-1} G_i \sum_{k=i}^{+\infty} \frac{f_k}{q_{k,k-1}G_k} - q_{i,i-1} G_i \sum_{k=i+1}^{+\infty} \frac{f_k}{q_{k,k-1}G_k} = f_i.$$

In other words, for (4.4), we need only

$$\lambda \leqslant \frac{f_i}{g_i} = \frac{1}{II_i(f)}, \quad i \geqslant 2,$$

and (4.5). Then we can take any λ satisfying

$$0 < \lambda < \frac{c-1}{cII_1(f)} \wedge \inf_{i\geqslant 2} \frac{1}{II_i(f)} \wedge q, \tag{4.6}$$

provided the right-hand side of (4.6) is positive or, equivalently,

$$\sup_{i\geqslant 1} II_i(f) < +\infty.$$

To prove the last property, define another operator

$$I_i(f) = \frac{G_i}{f_i - f_{i-1}} \sum_{k=i}^{+\infty} \frac{f_k}{q_{k,k-1} G_k}, \quad i \geqslant 1,$$

where $f_0 := 0$. By the proportion property, we get

$$\sup_{i\geqslant 1} II_i(f) \leqslant \sup_{i\geqslant 1} I_i(f).$$

By [3, Lemma 3.6] and the condition that $M < +\infty$, it follows that

$$I_i(f) = \frac{G_i}{\sqrt{\varphi_i} - \sqrt{\varphi_{i-1}}} \sum_{k=i}^{+\infty} \frac{\sqrt{\varphi_k}}{q_{k,k-1} G_k} \leqslant \frac{2MG_i}{(\sqrt{\varphi_i} - \sqrt{\varphi_{i-1}})\sqrt{\varphi_i}} \leqslant 4M$$

for all $i \geqslant 2$ and

$$I_1(f) = \frac{G_1}{\sqrt{\varphi_1}} \sum_{k=1}^{+\infty} \frac{\sqrt{\varphi_k}}{q_{k,k-1} G_k} \leqslant \frac{2MG_1}{\varphi_1} = 2M.$$

Therefore,

$$\sup_{i\geqslant 1} II_i(f) \leqslant 4M < +\infty,$$

as required. We have thus constructed a solution (g_i) to (4.4) with $1 \leqslant g_i < +\infty$ for all $i \geqslant 0$ and $\sum_{j\geqslant 1} q_{0j} g_j < +\infty$. This implies the exponential ergodicity of the single death process. The proof is finished. □

For the birth-death chain (a_i, b_i),

$$\frac{G_n^{(m)}}{G_1^{(m)}} = \frac{\mu_1 a_1}{\mu_n a_n}.$$

Thus,

$$G_n = \lim_{m\to+\infty} \frac{G_n^{(m)}}{G_1^{(m)}} = \frac{\mu_1 a_1}{\mu_n a_n}, \quad n \geqslant 1,$$

and

$$\sum_{k\geqslant 1} q_0^{(k)} G_k = b_0 < +\infty.$$

The equality in (4.2) holds. Now,

$$M = \sup_{n\geqslant 1} \left(\sum_{k=1}^{n} \frac{1}{\mu_k a_k} \sum_{j=n}^{+\infty} \mu_j \right) = \sup_{n\geqslant 1} \left(\sum_{k=0}^{n-1} \frac{1}{\mu_k b_k} \sum_{j=n+1}^{+\infty} \mu_j \right).$$

As we have known (refer to [3]), the birth-death chain is exponentially ergodic if and only if $M < +\infty$.

Consider Example 2.10 with $b > 2$. We see that

$$\frac{G_n^{(m)}}{G_1^{(m)}} = (b-1)^{n-1}.$$

Then

$$G_n = \lim_{m\to+\infty} \frac{G_n^{(m)}}{G_1^{(m)}} = (b-1)^{n-1}, \quad n \geqslant 1,$$

and

$$\sum_{k\geqslant 1} q_0^{(k)} G_k = 1 < +\infty.$$

The equality in (4.2) holds. Note that $q = 1/b > 0$ and

$$M = \frac{b(b-1)}{(b-2)^2} < +\infty.$$

Hence, by Theorem 4.1, the process is exponentially ergodic.

Given a positive constant c arbitrarily, define a birth-death chain

$$b_0 = \frac{c}{G_1}, \quad b_i = \frac{q_{i,i-1} G_i}{G_{i+1}}, \; a_i = q_{i,i-1}, \; i \geqslant 1.$$

Here, we take $c \geqslant q_0 G_1$. Then

$$b_0 \geqslant q_0 = q_0^{(1)}.$$

By (4.2), it is seen that

$$b_i \geqslant q_i^{(i+1)}, \quad \forall\, i \geqslant 0.$$

Denote the first hitting time of the state 0 for the birth-death chain by $\overline{\tau}_0$. Then the following assertion holds.

Proposition 4.2 *Let the single death Q-matrix be regular and irreducible. Assume that the birth-death Q-matrix defined above is regular. Then*

$$\mathbb{E}_i\tau_0^n \leqslant \mathbb{E}_i\overline{\tau}_0^n, \quad n \geqslant 0,\, i \geqslant 1.$$

Proof Define

$$m_i^{(n)} = \mathbb{E}_i\tau_0^n, \quad \overline{m}_i^{(n)} = \mathbb{E}_i\overline{\tau}_0^n, \quad n \geqslant 0,\, i \geqslant 1.$$

Obviously,

$$m_i^{(0)} = \overline{m}_i^{(0)} = 1, \quad i \geqslant 1.$$

By [8, Chapter 9] or [14], we know that

$$\overline{m}_i^{(n)} = n\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=j+1}^{+\infty}\mu_k\overline{m}_k^{(n-1)} = n\sum_{j=0}^{i-1}G_{j+1}\sum_{k=j+1}^{+\infty}\frac{\overline{m}_k^{(n-1)}}{q_{k,k-1}G_k}, \quad i \geqslant 1,\, n \geqslant 1. \tag{4.7}$$

By [10] or [5, Proposition 4.56], $(m_i^{(n)})$ is the minimal nonnegative solution to the equation

$$x_0^{(n)} = 0, \quad x_i^{(n)} = \sum_{k\neq i}\frac{q_{ik}}{q_i}\,x_k^{(n)} + \frac{n}{q_i}\,m_i^{(n-1)},\ i \geqslant 1. \tag{4.8}$$

Then, by (4.7), it is not difficult to check that $(\overline{m}_i^{(n)})$ satisfies

$$\overline{x}_0^{(n)} = 0, \quad \overline{x}_i^{(n)} \geqslant \sum_{k\neq i}\frac{q_{ik}}{q_i}\,\overline{x}_k^{(n)} + \frac{n}{q_i}\,\overline{m}_i^{(n-1)},\ i \geqslant 1. \tag{4.9}$$

By (4.8), (4.9), and Comparison Theorem, we see that $m_i^{(n)} \leqslant \overline{m}_i^{(n)}$ for all $i \geqslant 1$ and $n \geqslant 0$. The proof is finished. □

Furthermore, we can get that

$$\mathbb{E}_i\mathrm{e}^{\lambda\tau_0} \leqslant \mathbb{E}_i\mathrm{e}^{\lambda\overline{\tau}_0}, \quad i \geqslant 1,\, \lambda > 0,$$

by Tayor's expansion. Hence, based on [5, Theorems 4.44, 4.55] and the argument above, some sufficient conditions on several ergodicity are obtained immediately as follows.

Theorem 4.3 *Let the single death Q-matrix be regular and irreducible. Assume $\sum_{k\geqslant 1} G_k = +\infty$. If the birth-death chain defined above is ergodic (resp. exponentially ergodic, strongly ergodic), then so is the single death process, equivalently,*

if

$$\sum_{i\geqslant 1}\frac{1}{q_{i,i-1}G_i} < +\infty,$$

then the single death process is ergodic;

if $M < +\infty$, *then the single death process is exponentially ergodic*;

if

$$\sum_{n\geqslant 1} G_n \sum_{k\geqslant n} \frac{1}{q_{k,k-1}G_k} < +\infty,$$

then the single death process is strongly ergodic.

The condition of '$\sum_{k\geqslant 1} G_k = +\infty$' guarantees that the single death process and the birth-death chain are recurrent simultaneously. Refer to Theorem 5.1 below. So the birth-death Q-matrix is regular.

5 Recurrence and return probability

In this section, we consider the recurrence of single death processes. By Proposition 2.4, we know that

$$\frac{G_k^{(m)}}{G_1^{(m)}} \leqslant \frac{1}{G_1^{(k)}}, \quad 1 \leqslant k \leqslant m.$$

Define

$$\widetilde{G}_n = \overline{\lim_{m\to+\infty}} \frac{G_n^{(m)}}{G_1^{(m)}}, \quad n \geqslant 1.$$

Assume the following condition holds:

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} < +\infty, \quad n \geqslant 1. \tag{5.1}$$

Then, from Fatou's Lemma and (5.1), it follows that

$$\begin{aligned}\overline{\lim_{m\to+\infty}} \frac{G_n^{(m)}}{G_1^{(m)}} &\leqslant \frac{1}{q_{n,n-1}} \sum_{k\geqslant n+1} q_n^{(k)} \overline{\lim_{m\to+\infty}} \frac{G_k^{(m)}}{G_1^{(m)}} \mathbb{1}_{[1,m]}(k) \\ &= \frac{1}{q_{n,n-1}} \sum_{k\geqslant n+1} q_n^{(k)} \overline{\lim_{m\to+\infty}} \frac{G_k^{(m)}}{G_1^{(m)}},\end{aligned}$$

i.e.,

$$\widetilde{G}_n \leqslant \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{+\infty} q_n^{(k)} \widetilde{G}_k, \quad n \geqslant 1. \tag{5.2}$$

Now, we present the criterion on the recurrence of single death processes as follows.

Theorem 5.1 *Let the single death Q-matrix be regular and irreducible. If G_n defined in* (4.1) *is finite for all $n \geqslant 1$ and*

$$\sum_{n\geqslant 1} G_n = +\infty,$$

then the process is recurrent. Conversely, assume that condition (5.1) *holds. If the process is recurrent, then*

$$\sum_{n\geqslant 1} \widetilde{G}_n = +\infty.$$

Proof First, we prove that the process is recurrent provided that $\sum_{n\geqslant 1} G_n = +\infty$. By [5, Theorems 4.34, 4.24], the process is recurrent if and only if the inequality

$$\sum_{j\neq i} \frac{q_{ij}}{q_i} y_j \leqslant y_i, \quad i \geqslant 1, \tag{5.3}$$

has a finite solution (y_i) so that

$$\lim_{i\to+\infty} y_i = +\infty.$$

Take

$$g_0 = 0, \quad g_i = \sum_{1\leqslant n\leqslant i} G_n, \ i \geqslant 1.$$

It follows that (g_i) is a finite solution of inequality (5.3) satisfying $\lim_{i\to+\infty} g_i = +\infty$ from (4.2) immediately. So the process is recurrent.

Second, under assumption (5.1), when the process is recurrent, we consider the case that

$$\widetilde{G} := \sum_{n\geqslant 1} \widetilde{G}_n < +\infty.$$

Take

$$x_i = \frac{1}{\widetilde{G}} \sum_{2\leqslant n\leqslant i} \widetilde{G}_n, \ i \geqslant 2, \quad x_1 = \frac{q_{10}\widetilde{G}_1}{q_1\widetilde{G}}, \quad x_0 = \frac{1}{q_0} \sum_{k\geqslant 1} q_{0k} x_k.$$

Obviously, $0 \leqslant x_i \leqslant 1$ for all $i \geqslant 0$ and

$$\sum_{k\neq 0} \frac{q_{0k}}{q_0} x_k = x_0.$$

By (5.2), it can be derived that

$$\sum_{k\neq 0,1} \frac{q_{1k}}{q_1} x_k = \frac{1}{q_1\widetilde{G}} \sum_{k\geqslant 2} q_{1k} \sum_{2\leqslant n\leqslant k} \widetilde{G}_n = \frac{1}{q_1\widetilde{G}} \sum_{n\geqslant 2} q_1^{(n)} \widetilde{G}_n \geqslant \frac{q_{10}\widetilde{G}_1}{q_1\widetilde{G}} = x_1.$$

Moreover, by (5.2), we see that

$$\begin{aligned}\sum_{k\neq 0,2}\frac{q_{2k}}{q_2}\,x_k &= \frac{q_{21}}{q_2}x_1+\frac{1}{q_2\widetilde{G}}\sum_{k\geqslant 3}q_{2k}\sum_{2\leqslant n\leqslant k}\widetilde{G}_n\\ &= \frac{q_{21}}{q_2}\,x_1+\frac{1}{q_2\widetilde{G}}\sum_{k\geqslant 3}q_{2k}\Big(\widetilde{G}_2+\sum_{3\leqslant n\leqslant k}\widetilde{G}_n\Big)\\ &= \frac{q_{21}}{q_2}\,x_1+\frac{q_2^{(3)}\widetilde{G}_2}{q_2\widetilde{G}}+\frac{1}{q_2\widetilde{G}}\sum_{n\geqslant 3}q_2^{(n)}\widetilde{G}_n\\ &\geqslant \frac{q_{21}}{q_2}\,x_1+\frac{q_2^{(3)}\widetilde{G}_2}{q_2G}+\frac{q_{21}\widetilde{G}_2}{q_2\widetilde{G}}\\ &\geqslant \frac{\widetilde{G}_2}{\widetilde{G}}\\ &= x_2,\end{aligned}$$

and for all $i\geqslant 3$,

$$\begin{aligned}\sum_{k\neq 0,i}\frac{q_{ik}}{q_i}\,x_k &= \frac{q_{i,i-1}}{q_i\widetilde{G}}\sum_{2\leqslant n\leqslant i-1}\widetilde{G}_n+\frac{1}{q_i\widetilde{G}}\sum_{k\geqslant i+1}q_{ik}\sum_{2\leqslant n\leqslant k}\widetilde{G}_n\\ &= \frac{q_{i,i-1}}{q_i\widetilde{G}}\sum_{2\leqslant n\leqslant i-1}\widetilde{G}_n+\frac{1}{q_i\widetilde{G}}\sum_{k\geqslant i+1}q_{ik}\Big(\sum_{2\leqslant n\leqslant i}\widetilde{G}_n+\sum_{i+1\leqslant n\leqslant k}\widetilde{G}_n\Big)\\ &= \frac{q_{i,i-1}}{q_i\widetilde{G}}\sum_{2\leqslant n\leqslant i-1}\widetilde{G}_n+\frac{q_i^{(i+1)}}{q_i\widetilde{G}}\sum_{2\leqslant n\leqslant i}\widetilde{G}_n+\frac{1}{q_i\widetilde{G}}\sum_{n\geqslant i+1}q_i^{(n)}\widetilde{G}_n\\ &\geqslant \frac{1}{\widetilde{G}}\sum_{2\leqslant n\leqslant i-1}\widetilde{G}_n+\frac{q_i^{(i+1)}\widetilde{G}_i}{q_i\widetilde{G}}+\frac{q_{i,i-1}\widetilde{G}_i}{q_i\widetilde{G}}\\ &= \frac{1}{\widetilde{G}}\sum_{2\leqslant n\leqslant i}\widetilde{G}_n\\ &= x_i.\end{aligned}$$

Thus, (x_i) is a non-trivial solution of the inequality

$$x_i\leqslant\sum_{k\neq 0,i}\frac{q_{ik}}{q_i}\,x_k,\quad 0\leqslant x_i\leqslant 1. \tag{5.4}$$

By the comparison lemma, we know that the equation

$$x_i=\sum_{k\neq 0,i}\frac{q_{ik}}{q_i}\,x_k,\quad 0\leqslant x_i\leqslant 1,$$

has a non-trivial solution if and only if inequality (5.4) also has a non-trivial solution. Thus, from [5, Lemma 4.51], it follows that the process is transient immediately. It is a contradiction. So $\widetilde{G}=+\infty$ holds. The proof is finished. □

For the birth-death chain (a_i, b_i), we have

$$\sum_{k\geqslant n+1}\frac{q_n^{(k)}}{G_1^{(k)}}=\frac{b_n}{G_1^{(n+1)}}=\frac{b_0}{\mu_n}<+\infty,\quad n\geqslant 1,$$

and

$$G_n=\widetilde{G}_n=\frac{\mu_1 a_1}{\mu_n a_n},\quad n\geqslant 1.$$

So assumption (5.1) holds and

$$\sum_{n\geqslant 1}G_n=\sum_{n\geqslant 1}\widetilde{G}_n=\sum_{n\geqslant 1}\frac{\mu_1 a_1}{\mu_n a_n}=\sum_{n\geqslant 0}\frac{b_0}{\mu_n b_n}.$$

Hence, the birth-death chain is recurrent if and only if

$$\sum_{n\geqslant 0}\frac{1}{\mu_n b_n}=+\infty,$$

which is well known before.

Consider Example 2.10 with $b>1$. Then

$$\sum_{k\geqslant n+1}\frac{q_n^{(k)}}{G_1^{(k)}}=\sum_{k\geqslant n+1}\frac{(b-1)^{k-1}}{b^{k-n}}=(b-1)^n<+\infty,\quad n\geqslant 1,$$

and

$$G_n=\widetilde{G}_n=(b-1)^{n-1},\quad n\geqslant 1.$$

So assumption (5.1) holds and

$$\sum_{n\geqslant 1}G_n=\sum_{n\geqslant 1}\widetilde{G}_n=\sum_{n\geqslant 1}(b-1)^{n-1},$$

which is infinite when $b\geqslant 2$ and finite when $1<b<2$. So the process is recurrent if and only if $b\geqslant 2$. By the way, from [7, Therem 1.3], it follows that the unique process is recurrent if and only if $b\geqslant 2$.

In the following, let us consider the return probability for single death processes. Define

$$G_0=\frac{1}{q_0}\sum_{n\geqslant 1}q_0^{(n)}G_n,\quad G=\sum_{n\geqslant 1}G_n,\quad \widetilde{G}_0=\frac{1}{q_0}\sum_{n\geqslant 1}q_0^{(n)}\widetilde{G}_n,\quad \widetilde{G}=\sum_{n\geqslant 1}\widetilde{G}_n.$$

It is obvious that

$$G_0\leqslant G,\quad \widetilde{G}_0\leqslant\widetilde{G}.$$

Then our result is presented as follows.

Theorem 5.2 *Let the single death Q-matrix be regular and irreducible. Assume that* (5.1) *holds. Then*

$$\mathbb{P}_0(\sigma_0 < +\infty) \leqslant 1 - \frac{\widetilde{G}_0}{\widetilde{G}}, \quad \mathbb{P}_n(\sigma_0 < +\infty) \leqslant 1 - \frac{1}{\widetilde{G}} \sum_{1 \leqslant k \leqslant n} \widetilde{G}_k, \ n \geqslant 1.$$

In addition, assume

$$\varliminf_{m \to +\infty} \frac{1}{G_1^{(m)}} \sum_{1 \leqslant j \leqslant k} G_j^{(m)} = \sum_{1 \leqslant j \leqslant k} G_j, \quad k \geqslant 1. \tag{5.5}$$

Then

$$\mathbb{P}_0(\sigma_0 < +\infty) \geqslant 1 - \frac{G_0}{G}, \quad \mathbb{P}_n(\sigma_0 < +\infty) \geqslant 1 - \frac{1}{G} \sum_{1 \leqslant k \leqslant n} G_k, \ n \geqslant 1.$$

Proof First, take

$$x_0 = 1 - \frac{\widetilde{G}_0}{\widetilde{G}}, \quad x_k = 1 - \frac{1}{\widetilde{G}} \sum_{1 \leqslant n \leqslant k} \widetilde{G}_n, \ k \geqslant 1.$$

Then, by (5.2), we see that

$$\sum_{k \neq 0} \frac{q_{0k}}{q_0} x_k = \frac{1}{q_0} \Big(q_0^{(1)} - \frac{1}{\widetilde{G}} \sum_{n \geqslant 1} q_0^{(n)} \widetilde{G}_n \Big) = 1 - \frac{\widetilde{G}_0}{\widetilde{G}} = x_0,$$

and

$$\begin{aligned}
\sum_{k \neq 0,1} \frac{q_{1k}}{q_1} x_k + \frac{q_{10}}{q_1} &= \frac{1}{q_1} \Big(q_1^{(2)} \Big(1 - \frac{\widetilde{G}_1}{\widetilde{G}}\Big) - \frac{1}{\widetilde{G}} \sum_{n \geqslant 2} q_1^{(n)} \widetilde{G}_n \Big) + \frac{q_{10}}{q_1} \\
&\leqslant \frac{1}{q_1} \Big(q_1^{(2)} \Big(1 - \frac{\widetilde{G}_1}{\widetilde{G}}\Big) - \frac{q_{10}\widetilde{G}_1}{\widetilde{G}} \Big) + \frac{q_{10}}{q_1} \\
&= \frac{1}{q_1} \Big(q_1 \Big(1 - \frac{\widetilde{G}_1}{\widetilde{G}}\Big) - q_{10} \Big) + \frac{q_{10}}{q_1} \\
&= 1 - \frac{\widetilde{G}_1}{\widetilde{G}} \\
&= x_1.
\end{aligned}$$

For $i \geqslant 2$, by $q_{i0} = 0$, we have

$$\begin{aligned}
&\sum_{k \neq 0,i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i} \\
&\quad = \frac{1}{q_i} \sum_{k \neq i} q_{ik} \Big(1 - \frac{1}{\widetilde{G}} \sum_{1 \leqslant n \leqslant k} \widetilde{G}_n \Big)
\end{aligned}$$

$$= \frac{1}{q_i}\left(q_{i,i-1}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i-1}\widetilde{G}_n\right)+\sum_{k\geqslant i+1}q_{ik}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant k}\widetilde{G}_n\right)\right)$$

$$= \frac{1}{q_i}\left(q_{i,i-1}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i-1}\widetilde{G}_n\right)\right.$$

$$\left.+\,q_i^{(i+1)}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i}\widetilde{G}_n\right)-\frac{1}{\widetilde{\widetilde{G}}}\sum_{n\geqslant i+1}q_i^{(n)}\widetilde{G}_n\right);$$

and furthermore, by (5.2), we get that

$$\sum_{k\neq 0,i}\frac{q_{ik}}{q_i}\,x_k+\frac{q_{i0}}{q_i}$$

$$\leqslant \frac{1}{q_i}\left(q_{i,i-1}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i-1}\widetilde{G}_n\right)+q_i^{(i+1)}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i}\widetilde{G}_n\right)-\frac{q_{i,i-1}\widetilde{G}_i}{\widetilde{\widetilde{G}}}\right)$$

$$= \frac{1}{q_i}\left(q_{i,i-1}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i}\widetilde{G}_n\right)+q_i^{(i+1)}\left(1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i}\widetilde{G}_n\right)\right)$$

$$= 1-\frac{1}{\widetilde{\widetilde{G}}}\sum_{1\leqslant n\leqslant i}\widetilde{G}_n$$

$$= x_i.$$

Thus, (x_i) satisfies the inequality

$$x_i \geqslant \sum_{k\neq 0,i}\frac{q_{ik}}{q_i}\,x_k+\frac{q_{i0}}{q_i}\,(1-\delta_{i0}), \quad i\geqslant 0.$$

It is well known that $(\mathbb{P}_i(\sigma_0<+\infty))$ is the minimal nonnegative solution to the equation

$$x_i = \sum_{k\neq 0,i}\frac{q_{ik}}{q_i}\,x_k+\frac{q_{i0}}{q_i}\,(1-\delta_{i0}), \quad i\geqslant 0.$$

Thus, from Comparison Theorem and the argument above, it follows that

$$\mathbb{P}_i(\sigma_0<+\infty)\leqslant x_i, \quad i\geqslant 0.$$

Second, under assumption (5.5), from [16, Theorem 1], it follows that

$$\begin{aligned}\mathbb{P}_i(\sigma_0<+\infty) &\geqslant \overline{\lim_{m\to+\infty}}\,\mathbb{P}_i(\sigma_0<\tau_m)\\ &= 1-\underline{\lim}_{m\to+\infty}\frac{\sum_{1\leqslant j\leqslant i}G_j^{(m)}}{\sum_{1\leqslant j\leqslant m}G_j^{(m)}}\\ &\geqslant 1-\frac{\overline{\lim}_{m\to+\infty}\sum_{1\leqslant j\leqslant i}G_j^{(m)}/G_1^{(m)}}{\underline{\lim}_{m\to+\infty}\sum_{1\leqslant j\leqslant m}G_j^{(m)}/G_1^{(m)}}\end{aligned}$$

$$\begin{aligned}&\geqslant 1-\frac{\sum_{1\leqslant j\leqslant i}G_j}{\sum_{j\geqslant 1}G_j}\\&=1-\frac{1}{G}\sum_{1\leqslant j\leqslant i}G_j,\quad i\geqslant 1.\end{aligned}$$

So we obtain that

$$\begin{aligned}\mathbb{P}_0(\sigma_0<+\infty)&=\frac{1}{q_0}\sum_{i\geqslant 1}q_{0i}\mathbb{P}_i(\sigma_0<+\infty)\\&\geqslant\frac{1}{q_0}\sum_{i\geqslant 1}q_{0i}\bigg(1-\frac{1}{G}\sum_{1\leqslant j\leqslant i}G_j\bigg)\\&=\frac{1}{q_0}\bigg(q_0^{(1)}-\frac{1}{G}\sum_{j\geqslant 1}q_0^{(j)}G_j\bigg)\\&=1-\frac{G_0}{G}.\end{aligned}$$

The assertion is proven. □

Remark 5.3 If there exist the limits $\lim_{m\to+\infty}G_n^{(m)}/G_1^{(m)}$ for all $n\geqslant 1$ and assumption (5.1) holds, then $G_n=\widetilde{G}_n\ (n\geqslant 0)$ and (5.5) holds. Hence,

$$\mathbb{P}_0(\sigma_0<+\infty)=1-\frac{G_0}{G},\quad \mathbb{P}_n(\sigma_0<+\infty)=1-\frac{1}{G}\sum_{1\leqslant k\leqslant n}G_k,\ n\geqslant 1.$$

Thus, $\mathbb{P}_i(\sigma_0<+\infty)=1$ for all $i\geqslant 1$ if and only if $\mathbb{P}_0(\sigma_0<+\infty)=1$, equivalently, if and only if $G=+\infty$ ($G_0<+\infty$). By the way, from the dominated convergence theorem, it follows that equality (4.2) or (5.2) hold. See Examples 2.9 and 2.10.

Acknowledgements The author acknowledges the constructive discussion with Professor Mu-Fa Chen, Professor Yong-Hua Mao, Yue-Shuang Li, and Yan-Yan Yan. This work was supported by the National Natural Science Foundation of China (Grant Nos. 11571043, 11771047, 11871008).

References

1. Chen A Y, Pollett P, Zhang H J, Cairns B. Uniqueness criteria for continuous-time Markov chains with general transition structure. Adv Appl Probab, 2005, 37(4): 1056–1074
2. Chen M F. Single birth processes. Chin Ann Math Ser B, 1999, 20: 77–82
3. Chen M F. Explicit bounds of the first eigenvalue. Sci China Ser A, 2000, 43(10): 1051–1059
4. Chen M F. Explicit criteria for several types of ergodicity. Chinese J Appl Probab Statist, 2001, 17(2): 1–8

5. Chen M F. From Markov Chains to Non-Equilibrium Particle Systems. 2nd ed. Singapore: World Scientific, 2004
6. Chen M F, Zhang Y H. Unified representation of formulas for single birth processes. Front Math China, 2014, 9(4): 761–796
7. Chen R R. An extended class of time-continuous branching processes. J Appl Probab, 1997, 34(1): 14–23
8. Hou Z T, Guo Q F. Homogeneous Denumerable Markov Processes. Beijing: Science Press, 1978 (in Chinese); English translation, Beijing: Science Press and Springer, 1988
9. Isaacson D, Arnold B. Strong ergodicity for continuous-time Markov chains. J Appl Probab, 1978, 15, 699–706
10. Mao Y H. Ergodic degrees for continuous-time Markov chains. Sci China Math, 2004, 47(2): 161–174
11. Mao Y H, Zhang Y H. Exponential ergodicity for single-birth processes. J Appl Probab, 2004, 41: 1022–1032
12. Martínez S, Martín J S, Villemonais D. Existence and uniqueness of a quasi-stationary distribution for Markov processes with fast return from infinity. J Appl Probab, 2014, 51(3): 756–768
13. Wang L D, Zhang Y H. Criteria for zero-exit (-entrance) of single-birth (-death) Q-matrices. Acta Math Sinica (Chin Ser), 2014, 57(4): 681–692 (in Chinese)
14. Wang Z K, Yang X Q. Birth and Death Processes and Markov Chains. Science Press: Beijing, 1992
15. Yan S J, Chen M F. Multidimensional Q-processes. Chin Ann Math Ser B, 1986, 7: 90–110
16. Yan Y Y, Zhang Y H. Probabilistic meanings for some numerical characteristics of single death processes. J Beijing Normal Univ (Natur Sci), 2018 (to appear) (in Chinese)
17. Zhang J K. On the generalized birth and death processes (I). Acta Math Sci, 1984, 4: 241–259
18. Zhang Y H. Strong ergodicity for single-birth processes. J Appl Probab, 2001, 38(1): 270–277