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Probabilistic Meanings of Numerical Characteristics for Single Birth Processes *

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Abstract: We consider probabilistic meanings for some numerical characteristics of single birth processes. Some probabilities of events, such as extinction probability, returning probability, are represented in terms of these numerical characteristics. Two examples are also presented to illustrate the value of the results.

Keywords: single birth (upwardly skip-free) processes; extinction probability; birth-death processes

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§1. Introduction and Main Results

Wang and Yang[1] present probabilistic meanings for a lot of numerical characteristics of birth-death processes, such as returning probability, extinction probability. This paper is devoted to considering the corresponding problems for the single birth processes described as follows.

On a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, consider a continuous-time, homogeneous and irreducible Markov chain $\{X(t) : t \geqslant 0\}$ with transition probability matrix $P(t) = (p_{ij}(t))$ and state space $\mathbf{Z}_+ = \{0, 1, 2, \ldots\}$. We call $\{X(t) : t \geqslant 0\}$ a single birth process if its

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density matrix $Q = (q_{ij} : i, j \in \mathbf{Z}_+)$ has the following form

$$Q = \begin{pmatrix} -q_0 & q_{01} & 0 & 0 & 0 \\ q_{10} & -q_1 & q_{12} & 0 & 0 \\ q_{20} & q_{21} & -q_2 & q_{23} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{1}$$

where $q_i := -q_{ii} \geqslant \sum_{j \neq i} q_{ij}$, $q_{i,i+1} > 0$, $q_{i,i+j} = 0$ for $i \in \mathbf{Z}_+$ and $j \geqslant 2$. The matrix in (1) is called a single birth Q-matrix deduced by

$$p_{ij}(t) = \begin{cases} q_{ij}t + o(t), & \text{if } j < i \text{ or } j = i+1; \\ 1 - q_i t + o(t), & \text{if } j = i \end{cases}$$

as $t \to 0$. Throughout the rest of the paper, we consider only totally stable and conservative single birth Q-matrix: $q_i = -q_{ii} = \sum_{j \neq i} q_{ij} < \infty$ for $i \in \mathbf{Z}_+$. Especially, if $q_{ij} = 0$ for $0 \leqslant i \leqslant j-2$ and $j \geqslant 2$, then (1) is just a birth death Q-matrix.

Some notations are necessary before moving on. Define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ for $0 \leqslant k < n$ $(k, n \in \mathbf{Z}_+)$ and

$$\begin{aligned} m_0 &= \frac{1}{q_{01}}, & m_n &= \frac{1}{q_{n,n+1}}\Big(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k\Big), \quad n \geqslant 1, \\ d_0 &= 0, & d_n &= \frac{1}{q_{n,n+1}}\Big(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\Big), \quad n \geqslant 1, \\ F_n^{(n)} &= 1, & F_n^{(i)} &= \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leqslant i < n. \end{aligned}$$

Then the numerical characteristics defined below play important roles in studying single birth processes:

$$R = \sum_{n=0}^{\infty} m_n, \quad Z_m = \sum_{n=m}^{\infty} F_n^{(m)}, \quad d = \sup_{i>0} \Big[\sum_{n=0}^{i-1} d_n \Big/ \sum_{n=0}^{i-1} F_n^{(0)}\Big], \quad S = \sup_{k \geqslant 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n).$$

To explain what the numerical characters might mean in probability, we introduce some stoping times. Denote the first leaping time and the n-th jumping time by η and η_n respectively, i.e.,

$$\eta_n = \inf\{t > \eta_{n-1} : X(t) \neq X(\eta_{n-1})\}, \quad n \geqslant 1; \qquad \eta = \lim_{n \to \infty} \eta_n,$$

where $\eta_0 \equiv 0$. The first hitting time and the first returning time of the state i are defined respectively as follows

$$\tau_i = \inf\{t > 0 : X(t) = i\}, \qquad \sigma_i = \inf\{t \geqslant \eta_1 : X(t) = i\}.$$

Though these numerical characteristics may seem complex, they do have explicit probabilistic meanings and make a positive contribution towards understanding the process clearly. Let $\mathsf{P}_i(A) = \mathsf{P}(A \,|\, X_0 = i)$, i.e., the condition probability given $\{X_0 = i\}$ and $\mathsf{E}_i \mathbf{1}_A = \mathsf{P}_i(A)$ for some measurable set A. Then Zhang [2] proved that $m_n = \mathsf{E}_n \tau_{n+1}$, $R = \mathsf{E}_0 \eta$ and pointed out that

$$\mathsf{P}_0(\sigma_0 < \eta) = 1 - \frac{1}{Z_0}.$$

So R is the mean time of the first hitting ∞ of the single birth process with starting from 0 and $\mathsf{P}_0(\sigma_0 < \eta) = 1$ once $1/Z_0 = \infty$. In [3], we see that $d = \mathsf{E}_1 \tau_0$, $\mathsf{E}_0 \sigma_0 = 1/q_{01} + d$ and

$$\mathsf{E}_i \tau_0 = \sum_{n=0}^{i-1} (F_n^{(0)} d - d_n), \qquad i \geqslant 1.$$

It is easy to see that $S = \sup\limits_{i \geqslant 0} \mathsf{E}_i \tau_0$.

Based on the above results, the following explicit criteria for several classical problems can be understood clearly (cf. [2, 4–7]).

The process is unique if and only if $R = \infty$. Assume that the Q-matrix is irreducible and regular. Then the process is recurrent if and only if $Z_0 = \infty$. For the regular case, the process is ergodic if and only if $d < \infty$, and the process is strongly ergodic if and only if $S < \infty$.

Now we still need to study the probabilistic meanings of Z_m and $Z_{m,n}$ defined as

$$Z_{m,n} = \sum_{i=m}^{n-1} F_i^{(m)}, \qquad n > m \geqslant 0$$

with the convention that $Z_{m,n} = \sum\limits_{i=m}^{n-1} F_i^{(m)} = 0$ if $m \geqslant n$. It will be seen later that these quantities are related to $\mathsf{P}_k(\tau_m < \tau_n)$, which is the probability of arriving at m along the trajectory before reaching n with starting from k.

Before presenting our main results, we mention that if the single birth process is ergodic, then the stationary distribution (π_i) can be described as (cf. [8])

$$\pi_k = \frac{1}{q_{k,k+1} c_k}, \qquad c_k = \sup_{i > k} \Big[\sum_{n=k}^{i-1} m_n \Big/ \sum_{n=k}^{i-1} F_n^{(k)} \Big], \qquad k \geqslant 0. \tag{2}$$

Moreover,

$$c_k \sum_{n=k}^{i-1} F_n^{(k)} = \mathsf{E}_k \tau_i + \mathsf{E}_i \tau_k, \qquad 0 \leqslant k < i. \tag{3}$$

It is easy to see that c_k is the mean commute time between k and $k+1$. Now, we present our main results as follows.

Theorem 1 Suppose that $m < n$. Then $\mathsf{P}_k(\tau_n < \tau_m) + \mathsf{P}_k(\tau_m < \tau_n) = 1$, and

(i) for $0 \leqslant k \leqslant n$,

$$\mathsf{P}_k(\tau_n < \tau_m) = \frac{Z_{m,k}}{Z_{m,n}}, \qquad \mathsf{P}_k(\tau_m < \tau_n) = 1 - \frac{Z_{m,k}}{Z_{m,n}};$$

(ii) for $k > n$,

$$\mathsf{P}_k(\tau_m < \tau_n) = Z_{n,k}\mathsf{P}_{n+1}(\tau_m < \tau_n) + \frac{Z_{n,k}F_n^{(m)}}{Z_{m,n}} - \frac{Z_{m,k}}{Z_{m,n}} + 1.$$

Moreover, if the process is ergodic, then

$$\mathsf{P}_{n+1}(\tau_m < \tau_n) = \frac{1}{Z_{m,n}}\Big(\frac{c_n}{c_m} + \frac{1}{q_{n,n+1}}\sum_{j=m+1}^{n-1} q_{nj}Z_{m,j}\Big) - \frac{q_n^{(n-1)}}{q_{n,n+1}}.$$

It is easy to see that $\mathsf{P}_k(\tau_m < \tau_n) = 1$ and $\mathsf{P}_k(\tau_n < \tau_m) = 0$ for $0 \leqslant k \leqslant m$, $\mathsf{P}_n(\tau_m < \tau_n) = 0$ and $\mathsf{P}_n(\tau_n < \tau_m) = 1$.

As for $\mathsf{P}_k(\sigma_m < \tau_n)$, it is obvious that $\mathsf{P}_k(\sigma_m < \tau_n) = \mathsf{P}_k(\tau_m < \tau_n)$ for $k \neq m$ and $\mathsf{P}_k(\sigma_n < \tau_m) = \mathsf{P}_k(\tau_n < \tau_m)$ for $k \neq n$. Moreover, we have the following theorem.

Theorem 2 Suppose that $m < n$.

(i) Suppose the single birth process is ergodic. Then

$$\mathsf{P}_n(\tau_m < \sigma_n) = \frac{q_{n,n+1}c_n}{q_n c_m Z_{m,n}}, \qquad \mathsf{P}_n(\sigma_n < \tau_m) = 1 - \frac{q_{n,n+1}c_n}{q_n c_m Z_{m,n}}.$$

(ii)

$$\mathsf{P}_m(\sigma_m < \tau_n) = 1 - \frac{q_{m,m+1}}{q_m Z_{m,n}}, \qquad \mathsf{P}_m(\tau_n < \sigma_m) = \frac{q_{m,m+1}}{q_m Z_{m,n}},$$

and $\mathsf{P}_m(\tau_m < \sigma_m) = 1$, $\mathsf{P}_m(\sigma_m < \tau_m) = 0$.

$\mathsf{P}_k(\sigma_m < \eta)$ is the probability of reaching m along the trajectory through finitely many jumps with starting from k. In particular, $\mathsf{P}_m(\sigma_m < \eta)$ is the probability, starting from m, of returning to m along the trajectory through finitely many jumps after leaving m, which is called a returning probability.

Corollary 3 For $\mathsf{P}_k(\sigma_m < \eta)$, we have

$$\mathsf{P}_k(\sigma_m < \eta) = \begin{cases} 1 & \text{if } k < m; \\ 1 - \dfrac{q_{m,m+1}}{q_m Z_m} & \text{if } k = m; \\ 1 - \dfrac{Z_{m,k}}{Z_m} & \text{if } k > m, \end{cases}$$

where we use the convention that $1/\infty = 0$.

In practical applications, $\mathsf{P}_k(\sigma_0 < \eta)$ is called an extinction probability, i.e., the probability that there exist k individuals initially but (through finitely many steps of transition) they finally die out (namely reach the state 0). About extinction probability, one may also refer to [9; Chapter 9] for the case $m = 0$ in Corollary 3.

§2. Proofs of the Main Results

Proof of Theorem 1 It is easy to see that $\mathsf{P}_k(\tau_m < \tau_n) = 1$ and $\mathsf{P}_k(\tau_n < \tau_m) = 0$ for $0 \leqslant k \leqslant m$. To prove the remainders, denote $\mathsf{P}_k(\tau_m < \tau_n)$ by p_k. By the strong Markov property of the process, for $m < k \neq n$, we have

$$p_k = \frac{q_{k,k+1}}{q_k} p_{k+1} + \sum_{j=0}^{k-1} \frac{q_{kj}}{q_k} p_j.$$

Then by the conservative property of Q-matrix and $p_k = 1$ for $0 \leqslant k \leqslant m$, it follows from the above equality that

$$q_{k,k+1}(p_k - p_{k+1}) = \sum_{i=m}^{k-1} q_k^{(i)}(p_i - p_{i+1}), \qquad m < k \neq n. \tag{4}$$

Denote $p_i - p_{i+1}$ by v_i for $i \geqslant 0$. So we have the difference equation

$$v_k = \frac{1}{q_{k,k+1}} \sum_{i=m}^{k-1} q_k^{(i)} v_i, \qquad m < k < n$$

with the boundary conditions $p_m = 1$ and $p_n = 0$. By the induction, it is seen that

$$v_i = v_m \cdot F_i^{(m)}, \qquad m \leqslant i < n. \tag{5}$$

By definitions of v_i and $Z_{m,n}$, it is derived that

$$1 = p_m - p_n = \sum_{i=m}^{n-1} v_i = v_m \sum_{i=m}^{n-1} F_i^{(m)} = v_m \cdot Z_{m,n}.$$

So $v_m = 1/Z_{m,n}$ and $v_i = F_i^{(m)}/Z_{m,n} (m \leqslant i < n)$. Therefore, it follows from $p_n = 0$ that

$$p_k = p_k - p_n = \sum_{i=k}^{n-1} v_i = \Big(\sum_{i=k}^{n-1} F_i^{(m)}\Big)\Big/Z_{m,n} = \Big(\sum_{i=m}^{n-1} F_i^{(m)} - \sum_{i=m}^{k-1} F_i^{(m)}\Big)\Big/Z_{m,n} = 1 - \frac{Z_{m,k}}{Z_{m,n}}$$

for $m < k < n$. By the similar argument, one can prove the second part of the assertion (i). Of course, it is followed immediately from the property of $\mathsf{P}_k(\tau_m < \tau_n) + \mathsf{P}_k(\tau_n < \tau_m) = 1$ too.

To prove the assertion (ii), we will discuss firstly the relation between p_{n+1} and p_k with $k \geqslant n+1$. By (4) and (5), it is seen that

$$v_k = \frac{1}{q_{k,k+1}}\Big(\sum_{i=m}^{n-1} q_k^{(i)} F_i^{(m)} v_m + \sum_{i=n}^{k-1} q_k^{(i)} v_i\Big), \qquad k > n.$$

Then

$$F_k^{(m)} v_m - v_k = \frac{1}{q_{k,k+1}} \sum_{i=n}^{k-1} q_k^{(i)} (F_i^{(m)} v_m - v_i), \qquad k > n.$$

Define $u_i = F_i^{(m)} v_m - v_i$ $(i \geqslant n)$. Thus, one obtains that

$$u_k = \frac{1}{q_{k,k+1}} \sum_{i=n}^{k-1} q_k^{(i)} u_i, \qquad k > n.$$

By the equalities above and the induction, it follows that $u_i = F_i^{(n)} u_n$ $(i \geqslant n)$. Hence, one deduces that

$$\begin{aligned} v_i &= F_i^{(m)} v_m - u_i = F_i^{(m)} v_m - F_i^{(n)} u_n = F_i^{(m)} v_m - F_i^{(n)} (F_n^{(m)} v_m - v_n) \\ &= (F_i^{(m)} - F_i^{(n)} F_n^{(m)}) v_m + F_i^{(n)} v_n, \qquad i \geqslant n. \end{aligned}$$

Note that $v_n = p_n - p_{n+1} = -p_{n+1}$. Furthermore, it is obtained that for $k > n$,

$$\begin{aligned} p_k &= p_k - p_n = -\sum_{i=n}^{k-1} v_i = \sum_{i=n}^{k-1} \big((F_i^{(n)} F_n^{(m)} - F_i^{(m)}) v_m + F_i^{(n)} p_{n+1}\big) \\ &= \Big(Z_{n,k} F_n^{(m)} - \sum_{i=n}^{k-1} F_i^{(m)}\Big) / Z_{m,n} + Z_{n,k} p_{n+1} \\ &= Z_{n,k} p_{n+1} + \frac{Z_{n,k} F_n^{(m)}}{Z_{m,n}} - \frac{Z_{m,k}}{Z_{m,n}} + 1. \end{aligned}$$

By the similar argument or the property $\mathsf{P}_k(\tau_m < \tau_n) + \mathsf{P}_k(\tau_n < \tau_m) = 1$, one can prove the second part of the assertion (ii).

Now it remains to show the assertion on the expression of p_{n+1}. Using the strong Markov property with Theorem 1, we have

$$\begin{aligned} \mathsf{P}_n(\tau_m < \sigma_n) &= \frac{q_{n,n+1}}{q_n} p_{n+1} + \sum_{j=0}^{n-1} \frac{q_{nj}}{q_n} p_j \\ &= \frac{q_{n,n+1}}{q_n} p_{n+1} + \frac{q_n^{(m)}}{q_n} + \sum_{j=m+1}^{n-1} \frac{q_{nj}}{q_n}\Big(1 - \frac{Z_{m,j}}{Z_{m,n}}\Big) \\ &= \frac{q_{n,n+1}}{q_n} p_{n+1} + \frac{q_n^{(n-1)}}{q_n} - \sum_{j=m+1}^{n-1} \frac{q_{nj}}{q_n} \cdot \frac{Z_{m,j}}{Z_{m,n}}. \end{aligned}$$

Combining the above equality with the assertion (i) in Theorem 2, which needs only some simple calculations, the required assertion holds immediately. □

Before proving Theorem 2, we introduce the following result (refer to [10]).

Proposition 4 Given an ergodic Markov chain $\{X(t) : t \geqslant 0\}$ with the stationary distribution (π_i). Then for $j \neq i$, we have

$$\mathsf{P}_i(\tau_j < \sigma_i) = \frac{1}{q_i \pi_i (\mathsf{E}_i \tau_j + \mathsf{E}_j \tau_i)}.$$

Proof of Theorem 2 The assertion (i) follows directly from (2), (3) and Proposition 4 by some simple calculations. By the strong Markov property and the assertion below Theorem 1, it turns out that

$$\begin{aligned}
\mathsf{P}_m(\tau_n < \sigma_m) &= \sum_{j=0}^{m-1} \frac{q_{mj}}{q_m} \mathsf{P}_j(\tau_n < \tau_m) + \frac{q_{m,m+1}}{q_m} \mathsf{P}_{m+1}(\tau_n < \tau_m) \\
&= \frac{q_{m,m+1}}{q_m} \mathsf{P}_{m+1}(\tau_n < \tau_m) \\
&= \begin{cases} \dfrac{q_{m,m+1}}{q_m} = \dfrac{q_{m,m+1}}{q_m Z_{m,m+1}} & \text{if } n = m+1 \\[2ex] \dfrac{q_{m,m+1} Z_{m,m+1}}{q_m Z_{m,n}} = \dfrac{q_{m,m+1}}{q_m Z_{m,n}} & \text{if } n > m+1 \end{cases} \\
&= \frac{q_{m,m+1}}{q_m Z_{m,n}}.
\end{aligned}$$

The remainders of the assertion (ii) are easily obtained. □

Remark 5 By induction, it is not difficult to obtain that $F_i^{(n)} F_n^{(m)} \leqslant F_i^{(m)}$, $i \geqslant n \geqslant m$. Further, we get the following inequality: $Z_{m,n} \leqslant Z_{m,k} - Z_{n,k} F_n^{(m)}$, $m < n < k$. Hence, it follows that $Z_n F_n^{(m)} \leqslant Z_m - Z_{m,n}$, $m < n$. In particular, we see that $Z_n F_n^{(0)} \leqslant Z_0 - Z_{0,n}$ for all $n > 0$.

Proof of Corollary 3 Note that $\tau_n \uparrow \eta$ as $n \to \infty$ almost surely with respect to P_k. Hence $\mathsf{P}_k(\tau_m < \tau_n) \uparrow \mathsf{P}_k(\sigma_m < \eta)$ as $n \to \infty$ for $k \neq m$. Combining these facts with the assertions proved above, one gets easily the first and the third parts of the assertion. By the strong Markov property and argument above, it is seen that

$$\begin{aligned}
\mathsf{P}_m(\sigma_m < \eta) &= \sum_{j=0}^{m-1} \frac{q_{mj}}{q_m} \mathsf{P}_j(\sigma_m < \eta) + \frac{q_{m,m+1}}{q_m} \mathsf{P}_{m+1}(\sigma_m < \eta) \\
&= \frac{q_m^{(m-1)}}{q_m} + \frac{q_{m,m+1}}{q_m}\Big(1 - \frac{Z_{m,m+1}}{Z_m}\Big) \\
&= 1 - \frac{q_{m,m+1}}{q_m Z_m}.
\end{aligned}$$

In the last equality, we use the fact that $Z_{m,m+1} = F_m^{(m)} = 1$. □

§3. Examples

The first example is about the birth-death process which is a special class of single birth processes.

Example 6 For birth-death processes with birth rate a_i and death rate b_i at i, denoted by (a_i, b_i). We have these important quantities with simple forms, as follows

$$m_n = \mu[0,n]\nu_n, \qquad d_n = \mu[1,n]\nu_n, \qquad F_n^{(m)} = \frac{\nu_n}{\nu_m}, \qquad n \geqslant m \geqslant 0,$$

where $\mu[i,k] = \sum_{j=i}^{k} \mu_j$ with $\{\mu_i\}$ is the invariant measure having the following form

$$\mu_0 = 1, \qquad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i} \quad (i \geqslant 1);$$

and ν_i is another measure related to the recurrence of the process with $\nu_i = 1/\mu_i b_i$ $(i \geqslant 0)$. In the following, we always let $\nu[i,k]$ denote the term $\sum_{j=i}^{k} \nu_j$ and $\nu[i,\infty) := \sum_{j=i}^{\infty} \nu_j$ for some measure.

For the process we have the following results, which can also refer to [1].

Corollary 7 Suppose that $m < n$. For birth-death processes, we have

(i) $\mathsf{P}_k(\tau_m < \tau_n) = 1$ and $\mathsf{P}_k(\tau_n < \tau_m) = 0$ for all $0 \leqslant k \leqslant m$; $\mathsf{P}_k(\tau_m < \tau_n) = 0$ and $\mathsf{P}_k(\tau_n < \tau_m) = 1$ for all $k \geqslant n$;

(ii) For $m < k < n$,

$$\mathsf{P}_k(\tau_n < \tau_m) = \frac{\nu_i[m,k-1]}{\nu_i[m,n-1]}, \qquad \mathsf{P}_k(\tau_m < \tau_n) = \frac{\nu_i[k,n-1]}{\nu_i[m,n-1]};$$

(iii) $\mathsf{P}_m(\tau_m < \sigma_m) = 1$, $\mathsf{P}_m(\sigma_m < \tau_m) = 0$ and

$$\mathsf{P}_m(\tau_n < \sigma_m) = \frac{1}{(a_m + b_m)\mu_m \nu[m,n-1]} = 1 - \mathsf{P}_m(\sigma_m < \tau_n);$$

(iv)

$$\mathsf{P}_n(\tau_m < \sigma_n) = \frac{1}{(a_n + b_n)\mu_n \nu[m,n-1]} = 1 - \mathsf{P}_n(\sigma_n < \tau_m);$$

(v)

$$\mathsf{P}_k(\sigma_m < \eta) = \begin{cases} 1 & \text{if } k < m \\ 1 - \dfrac{1}{\mu_m(a_m + b_m)\nu[m,\infty)} & \text{if } k = m \\ 1 - \dfrac{\nu[m,k-1]}{\nu[m,\infty)} & \text{if } k > m \end{cases}$$

in convention that $1/\infty = 0$.

Proof By Theorems 1, 2 and Corollary 3, all the assertions are derived directly except the assertion (iv), which is proven as follows. By the strong Markov property and the assertion (i) as well as (ii), we have

$$\begin{aligned}\mathsf{P}_n(\tau_m<\sigma_n)&=\frac{b_n}{a_n+b_n}\mathsf{P}_{n+1}(\tau_m<\tau_n)+\frac{a_n}{a_n+b_n}\mathsf{P}_{n-1}(\tau_m<\tau_n)\\&=\frac{a_n}{a_n+b_n}\mathsf{P}_{n-1}(\tau_m<\tau_n)\\&=\begin{cases}\dfrac{a_n}{a_n+b_n} & \text{if } m=n-1\\ a_n\nu_{n-1}\Big/\Big[(a_n+b_n)\sum\limits_{i=m}^{n-1}\nu_i\Big] & \text{if } m<n-1\end{cases}\\&=1\Big/\Big[(a_n+b_n)\mu_n\sum_{i=m}^{n-1}\nu_i\Big].\end{aligned}$$

The proof is finished. □

Especially, the extinction probability

$$\mathsf{P}_k(\sigma_0<\eta)=1-\sum_{n=0}^{k-1}\nu_n\Big/\sum_{n=0}^{\infty}\nu_n,\qquad k\geqslant 1.$$

The following example is an extension of the one in [8] or [11].

Example 8 Let $q_{n,n+1}=1$ for all $n\geqslant 0$, $q_{10}=b$, $q_{n,n-1}=b-a$, $q_{n,n-2}=a$ for all $n\geqslant 2$ and $q_{ij}=0$ for other $i\neq j$, where a and b are constants satisfying $b\geqslant a>0$.

By computing, we know that $\{F_n^{(k)}\}_{n\geqslant k}$ are generalized Fibonacci numbers for every k.

$$F_{k+n}^{(k)}=\frac{p^{n+1}-q^{n+1}}{p-q},\qquad n\geqslant 0,\ k\geqslant 0,$$

where $p=(b+\sqrt{b^2+4a})/2$ and $q=(b-\sqrt{b^2+4a})/2$. Note that $p>b$ and $-1<q<0$. Now

$$Z_{m,n}=\begin{cases}\dfrac{1}{p-q}\Big(\dfrac{p^{n-m+1}-p}{p-1}-\dfrac{q^{n-m+1}-q}{q-1}\Big) & \text{if } p\neq 1;\\ \dfrac{n-m}{1-q}+\dfrac{q^{n-m+1}-q}{(1-q)^2} & \text{if } p=1,\end{cases}$$

$$m_n=\begin{cases}\dfrac{1}{p-q}\Big(\dfrac{p^{n+2}-p}{p-1}-\dfrac{q^{n+2}-q}{q-1}\Big) & \text{if } p\neq 1;\\ \dfrac{n+1}{1-q}+\dfrac{q^{n+2}-q}{(1-q)^2} & \text{if } p=1,\end{cases}$$

and

$$d_n=\begin{cases}\dfrac{1}{p-q}\Big(\dfrac{p^{n+1}-p}{p-1}-\dfrac{q^{n+1}-q}{q-1}\Big) & \text{if } p\neq 1;\\ \dfrac{n}{1-q}+\dfrac{q^{n+1}-q}{(1-q)^2} & \text{if } p=1.\end{cases}$$

Hence, it turns out that $R = \sum_{n=0}^{\infty} m_n = \infty$, i.e., the process is always unique for all $b \geqslant a > 0$. Moreover, we get that

$$Z_m = \begin{cases} \infty & \text{if } p \geqslant 1; \\ \dfrac{1}{p-q}\Big(\dfrac{p}{1-p} + \dfrac{q}{q-1}\Big) & \text{if } p < 1. \end{cases}$$

Thus, when $p \geqslant 1$ (equivalently, $a + b \geqslant 1$), we have $Z_0 = \infty$, the process is recurrent and $\mathsf{P}_k(\sigma_m < \eta) = 1$ for all $k \geqslant 0$. When $p < 1$ (equivalently, $a + b < 1$), we have $Z_0 < \infty$ and the process is transient,

$$\mathsf{P}_k(\sigma_m < \eta) = \begin{cases} 1 & \text{if } k < m; \\ a + b & \text{if } k = m = 0; \\ 1 - \dfrac{(1-p)(1-q)}{1+b} & \text{if } k = m > 0; \\ 1 - \dfrac{p(1-q)(1-p^{k-m}) - (1-p)q(1-q^{k-m})}{p-q} & \text{if } k > m. \end{cases}$$

Moreover, for $p \neq 1$, we get that

$$\mathsf{P}_k(\tau_n < \tau_m) = \frac{pq(p^{k-m} - q^{k-m}) - p^{k-m+1} + q^{k-m+1}) + p - q}{pq(p^{n-m} - q^{n-m}) - p^{n-m+1} + q^{n-m+1} + p - q}, \qquad m < k < n;$$
$$\mathsf{P}_0(\tau_n < \sigma_0) = \frac{(p-q)(p-1)(q-1)}{pq(p^n - q^n) - p^{n+1} + q^{n+1} + p - q}, \qquad 0 < n;$$
$$\mathsf{P}_m(\tau_n < \sigma_m) = \frac{(p-q)(p-1)(q-1)}{(1+b)(pq(p^{n-m} - q^{n-m}) - p^{n-m+1} + q^{n-m+1} + p - q)}, \qquad 1 \leqslant m < n,$$

for $p = 1$, one obtains that

$$\mathsf{P}_k(\tau_n < \tau_m) = \frac{k - m - (k-m+1)q + q^{k-m+1}}{n - m - (n-m+1)q + q^{n-m+1}}, \qquad m < k < n;$$
$$\mathsf{P}_0(\tau_n < \sigma_0) = \frac{(1-q)^2}{n - m - (n-m+1)q + q^{n-m+1}}, \qquad 0 < n;$$
$$\mathsf{P}_m(\tau_n < \sigma_m) = \frac{(1-q)^2}{(1+b)(n - m - (n-m+1)q + q^{n-m+1})}, \qquad 1 \leqslant m < n.$$

By the way, when $p = 1$, the process is null recurrent because $d = \infty$.

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单生过程数字特征的概率含义

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摘　要: 本文考虑了单生过程的一些数字特征, 借助这些数字特征刻画了某些事件(如灭绝时间、回返时等)发生的概率. 最后, 利用本文结果计算了两个例子的相关数字特征.

关键词: 单生过程; 灭绝概率; 生灭过程

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