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RESEARCH ARTICLE

Central limit theorems for ergodic continuous-time Markov chains with applications to single birth processes

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Abstract We obtain sufficient criteria for central limit theorems (CLTs) for ergodic continuous-time Markov chains (CTMCs). We apply the results to establish CLTs for continuous-time single birth processes. Moreover, we present an explicit expression of the time average variance constant for a single birth process whenever a CLT exists. Several examples are given to illustrate these results.

Keywords Markov process, single birth processes, central limit theorem (CLT), ergodicity
MSC 60J27, 60F05

1 Introduction

Let $\{X_t,\, t \in \mathbb{R}_+\}$ be a continuous-time Markov chain (CTMC) on a countable state space $\mathbb{E}$ with the Q-matrix $Q = (q_{ij})$ and the transition function $P^t(i,j)$. Throughout this paper, we assume that Q is irreducible, totally stable, and regular, which implies that Q is conservative and the Q-process is unique. We further assume that X_t is positive recurrent with the unique invariant distribution π. Then we have

$$\|P^t(i,\cdot) - \pi\| := \sum_{j\in\mathbb{E}} |P^t(i,j) - \pi_j| \to 0, \quad t \to +\infty,$$

for any $i \in \mathbb{E}$. Define

$$\tau_i = \inf\{t > 0\colon X_t = i\}$$

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to be the first hitting time on the state i. Let J_1 be the first jump time of the process X_t, and define

$$\delta_i = \inf\{t > J_1 \colon X_t = i\}$$

to be the first return time on the state i. Define

$$\pi(g) = \sum_{i\in\mathbb{E}} \pi_i g_i$$

for a function g on $\mathbb{E}$.

When $\pi(|f|) < +\infty$, by [19, Proposition 4.2 (ii)], we know that the sample mean $S(t)$ is well defined by

$$S(t) = \frac{1}{t}\int_0^t f_{X_s}\mathrm{d}s, \quad t \geqslant 0.$$

Moreover, it is known that the strong law of large numbers holds, i.e., $S(t) \to \pi(f)$ with probability 1 as $t \to +\infty$. We say that a CLT holds if there exists a scaling constant $0 \leqslant \sigma^2(f) < +\infty$ such that

$$t^{1/2}[S(t) - \pi(f)] \Rrightarrow N(0, \sigma^2(f)), \quad t \to +\infty,$$

for any initial distribution, where $N(0,1)$ denotes the standard normal distribution, and '$\Rrightarrow$' means convergence in distribution. The (deterministic) constant $\sigma^2(f)$ is called the time average variance constant.

It is a fundamental issue to find conditions on X_t and f, under which a CLT holds. Let

$$\overline{f} := f - \pi(f).$$

From [8], we immediately know that if $\pi(|f|) < +\infty$, then a CLT holds if and only if

$$E_i\Big[\Big(\int_0^{\delta_i} \overline{f}_{X_s}\mathrm{d}s\Big)^2\Big] < +\infty$$

for any $i \in \mathbb{E}$. Obviously, this condition holds whenever

$$E_i\Big[\Big(\int_0^{\delta_i} |\overline{f}_{X_s}|\mathrm{d}s\Big)^2\Big] < +\infty. \tag{1.1}$$

It is hard to check the above two conditions directly to verify a CLT for Markov models, since both expressions involve the function f and the first return time δ_i simultaneously. It is well known that the moments of the first return time are closely related with ergodicity (e.g., [6]). We will establish three easily checked criteria in Theorem 3.1 for (1.1) in terms of the conditions on the function f and ergodicity separately, in which, the first one is new and the other two are analogous to those for discrete-time Markov chains (DTMCs) (see [13,20]).

This current research is also motivated by investigating CLTs for a continuous-time single birth process (see [6]) on the state space $\mathbb{E} = \mathbb{Z}_+$, which is a CTMC with the following Q-matrix:

$$Q = \begin{pmatrix} q_{00} & q_{01} & 0 & 0 & \cdots \\ q_{10} & q_{11} & q_{12} & 0 & \cdots \\ q_{20} & q_{21} & q_{22} & q_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The single birth processes, also called continuous-time Markov processes skip-free to the right ([1]), are a class of important Markov processes, which cover many interesting Markov processes, such as the birth-death processes, the population processes, and the level-dependent queues. It has been shown by [18] that the single birth processes can be used to investigate generalized Markov branching processes and multidimensional CTMCs. The single birth processes have been applied to modeling real problems in queues, biology, and so on. However, to model a real-world phenomenon, we have to judge whether this model is appropriate or not. That is to say, we need to test this model through the sample path, which raises a problem of statistical hypothesis testing.

To attack this problem, we apply Theorem 3.1 and the known ergodicity results to obtain sufficient criteria for a CLT for a single birth process. Moreover, we get the explicit expression of the variance constant, which is extremely important since it is a key parameter for determining the confidence interval for statistical hypothesis testing. In the literature, the variance constant has been investigated for continuous-time birth-death process on a finite state space (see, e.g., [21]), the continuous-time queues driven by a Markovian marked point process ([3]), the discrete-time waiting time of the M/G/1 queue [7] and the PH/PH/1 queue [4], the discrete-time birth-death process ([14]) and single birth process ([12]), the continuous-time matrix-analytical models ([15]), and so on.

2 Preliminaries on ergodicity

In this section, we review the definitions and known criteria of several types of ergodicities. We also state the known ergodicity results for continuous-time single birth processes.

A positive recurrent CTMC X_t is said to be strongly ergodic if

$$\sup_{i\in\mathbb{E}} \|P^t(i,\cdot) - \pi\| \to 0, \quad t \to +\infty.$$

From [6], we know that X_t is strongly ergodic if and only if

$$\sup_{k\in\mathbb{E}} E_k[\tau_i] < +\infty$$

for some $i \in \mathbb{E}$. A positive recurrent CTMC X_t is said to be exponentially ergodic if

$$e^{\gamma t}\|P^t(i,\cdot) - \pi\| \to 0, \quad t \to +\infty,$$

for some $\gamma > 0$ and for any $i \in \mathbb{E}$. From [6] again, we know that the chain X_t is exponentially ergodic if and only if

$$E_i[e^{r\delta_i}] < +\infty$$

for some $r > 0$ and some $i \in \mathbb{E}$. For a positive real number ℓ such that $\ell \geqslant 1$, a positive recurrent CTMC X_t is said to be ℓ-ergodic (see, [11,17]) if

$$t^{\ell-1}\|P^t(i,\cdot) - \pi\| \to 0, \quad t \to +\infty,$$

for any $i \in \mathbb{E}$. It follows from [16] that the chain X_t is ℓ-ergodic if and only if

$$E_i[\delta_i^{\ell}] < +\infty$$

for some $i \in \mathbb{E}$. The equivalent criteria of the three types of ergodicity can be characterized in terms of different drift functions on the intensity matrix Q (see, e.g., [6,16]).

The sufficient and necessary criteria for strong ergodicity, exponential ergodicity, and ℓ-ergodicity have been established for single-birth processes by [22], [18], and [24], respectively. The invariant distribution is obtained by [23]. We do not present those results here in order to avoid introducing too many notations.

We note that

$$\text{strong ergodicity} \Longrightarrow \text{exponential ergodicity} \Longrightarrow \ell\text{-ergodicity}$$

for any $\ell \in \mathbb{R}_+$ and $\ell \geqslant 1$. Generally speaking, it is easiest to investigate strong ergodicity for a CTMC among the three types of ergodicities. Strong ergodicity is a restrictive condition for DTMCs, since a DTMC on a infinitely countable state space fails to be strongly ergodic whenever its transition matrix $P = (P_{ij})$ is a Feller transition one (see [10, Section 2] for details), i.e.,

$$\lim_{i \to +\infty} P_{ij} = 0$$

for any fixed $j \in \mathbb{E}$. However, it is rather different for CTMCs with unbounded Q-matrices, for which, strong ergodicity may hold under feasible conditions.

3 CLTs for general CTMCs

In the following, we will present sufficient criteria of a CLT for polynomially, exponentially, or strongly ergodic CTMCs. The proofs of these results are postponed to Section 5.

Theorem 3.1 *Let X_t be a positive recurrent CTMC with the invariant distribution π. Let f be a function on $\mathbb{E}$ such that $\pi(|f|) < +\infty$ and i_0 be any fixed state in $\mathbb{E}$. Then any one of the following conditions is sufficient for a CLT to hold*:

(i) *X_t is ℓ-ergodic and $\pi(|f|^{2+\eta}) < +\infty$ for some $\eta > 0$ and $\ell > 2 + \frac{4}{\eta}$*;

(ii) *X_t is exponentially ergodic and $\pi(|f|^{2+\eta}) < +\infty$ for some $\eta > 0$*;

(iii) *X_t is strongly ergodic and $\pi(|f|^2) < +\infty$.*

Moreover, if any of the above condition holds, then the variance constant is given by

$$\sigma^2(f) = 2\pi(\overline{f}\hat{f}) = 2\sum_{i\in\mathbb{E}} \pi_i \overline{f}_i \hat{f}_i, \tag{3.1}$$

where the function $\hat{f}$, given by

$$\hat{f}_k = E_k\bigg[\int_0^{\delta_{i_0}} \overline{f}_{X_s} \mathrm{d}s\bigg], \quad k \in \mathbb{E},$$

is a solution of the Poisson equation $Qx = -\overline{f}$. Note that $\hat{f}_{i_0} = 0$.

Remark 3.2 (1) The discrete-time analogue of (ii) and (iii) can be found in the survey papers [13] or [20]. However, the arguments in the proof are new.

(2) As we note in Section 2, strong ergodicity rarely holds for discrete-time Markov chains, which is however the easiest checked ergodicity condition for CTMCs. Hence, assertion (iii) has a wider application potential than the corresponding discrete-time result.

(3) Result (i) is new, which is useful for polynomially but not exponentially ergodic Markov processes. The proof of (i) is modified from the proof of [20, Lemma 34], by extending it to the continuous-time case and weakening the geometric ergodicity condition there with the polynomial ergodicity condition.

We now give an example to illustrate Theorem 3.1 under various ergodic situations.

Example 3.3 Let X_t be a birth-death process with birth coefficients b_i and death coefficients a_i given by $b_0 = 1$, and $b_i = a_i = i^\gamma$, $i \geqslant 1$. From [5,17], we know that X_t is ℓ-ergodic if and only if $\gamma \in (1, 2)$ and $\gamma > 2 - \frac{1}{\ell}$; X_t is exponentially ergodic if and only if $\gamma = 2$; and X_t is strongly ergodic if and only if $\gamma > 2$. Now, take $f_i = i^\alpha$, where α is a positive constant to be determined. Let η be an arbitrarily given positive constant. It is easy to obtain the following results.

(1) Let ℓ, γ, and α be such that

$$\ell > 2 + \frac{4}{\eta}, \quad 2 - \frac{1}{\ell} < \gamma < 2, \quad \alpha < \frac{\gamma - 1}{2 + \eta}.$$

Then Theorem 3.1 (i) holds.

(2) Let γ and α be such that $\gamma = 2$ and $\alpha < 1/(2+\eta)$. Then Theorem 3.1 (ii) holds.

(3) Let γ and α be such that $\gamma > 2$ and $\alpha < (\gamma-1)/2$. Then Theorem 3.1 (iii) holds.

4 Application to single birth processes

In this section, we will apply Theorem 3.1 to single birth processes. Define

$$q_n^{(k)} = \sum_{j=0}^{k} q_{nj}, \quad 0 \leqslant k < n,$$
$$F_i^{(i)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad n > i \geqslant 0,$$

Theorem 4.1 *Let X_t be a positive recurrent single birth process with the invariant distribution π. Let f be a function on $\mathbb{E}$ such that $\pi(|f|) < +\infty$. Suppose that any one of the conditions in Theorem* 3.1 (i)–(iii) *is satisfied. Then a CLT holds and the variance constant is given by*

$$\sigma^2(f) = 2\sum_{n=0}^{+\infty} \bigg(\sum_{k=0}^{n} \frac{F_n^{(k)}\overline{f}_k}{q_{k,k+1}}\bigg) \sum_{k=0}^{n} \pi_k \overline{f}_k. \tag{4.1}$$

Remark 4.2 (i) [12, Theorem 3.1] derives the explicit expression of the variance constant for discrete-time single birth processes. Note that there is a difference between both expressions, specifically, one more item, $\sum_{n\in\mathbb{E}} \pi_n \overline{f}_n^2$, appears in the discrete-time expression.

(ii) The proof of Theorem 4.1 is a little different from the one of [12], which can be modified to give a shorter and more complete proof of [12, Theorem 2.1].

Remark 4.3 When X_t is a birth-death process under the condition of Theorem 4.1, we obtain the variance constant

$$\sigma^2(f) = 2\sum_{n=0}^{+\infty} \frac{1}{b_n \mu_n} \bigg(\sum_{k=0}^{n} \mu_k \overline{f}_k\bigg)^2,$$

where

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geqslant 1.$$

Example 4.4 Let $q_{n,n+1} = 1$ for all $n \geqslant 0$, $q_{10} = 1$, $q_{n,n-2} = 1$ for all $n \geqslant 2$, and $q_{ij} = 0$ for other $i \neq j$. The single birth process is exponentially ergodic but not strongly ergodic (refer to [18]). By computing, we know that $\{F_n^{(k)}\}_{n\geqslant k}$ are Fibonacci numbers for every k:

$$F_{k+n}^{(k)} = \frac{1}{\sqrt{5}} (A^{n+1} - (-B)^{n+1}), \quad n, k \geqslant 0,$$

where

$$A = \frac{\sqrt{5}+1}{2}, \quad B = \frac{\sqrt{5}-1}{2}.$$

The stationary distribution is

$$\pi_n = (1-B)B^n, \quad \forall\, n \geqslant 0,$$

which is obtained in [23]. Let $f_i = i$. Then we have

$$\pi(f) = \pi(|f|) = A, \quad \pi(|f|^3) = 6A^4 + A.$$

So by Theorem 4.1, (4.1) holds. Now,

$$\sum_{k=0}^{n} \frac{F_n^{(k)} \overline{f}_k}{q_{k,k+1}} = -n + (-B)^{n+3} + A - 3, \quad \sum_{k=0}^{n} \pi_k \overline{f}_k = -(n+1)B^{n+1}, \quad n \geqslant 0.$$

Hence, from (4.1), one gets

$$\sigma^2(f) = \frac{44\sqrt{5}+98}{5}.$$

Example 4.4 is the so-called uniform catastrophes in population models.

Example 4.5 Let $q_{n,n+1} = n+1$ for all $n \geqslant 0$, $q_{nj} = 1$ for all $0 \leqslant j < n$, and $q_{ij} = 0$ for other $i \neq j$. The single birth process is strongly ergodic (see [25]). By computing, we know that

$$F_k^{(k)} = 1, \quad F_n^{(k)} = 2^{n-k-1}\frac{k+1}{n+1}, \quad n > k \geqslant 0.$$

The stationary distribution is

$$\pi_n = 2^{-n-1}, \quad \forall\, n \geqslant 0.$$

Let $f_i = i$. Then we have

$$\pi(f) = \pi(|f|) = 1, \quad \pi(|f|^2) = 3.$$

So by Theorem 4.1, (4.1) holds. Now, we have

$$\sum_{k=0}^{n} \frac{F_n^{(k)} \overline{f}_k}{q_{k,k+1}} = -\frac{1}{n+1}, \quad \sum_{k=0}^{n} \pi_k \overline{f}_k = -(n+1)2^{-(n+1)}, \quad n \geqslant 0.$$

Hence, we derive that

$$\sigma^2(f) = 2.$$

5 Proof of Theorem 3.1

Note that assertion (ii) follows from (i) directly by observing that if $E_i[\mathrm{e}^{r\delta_i}] < +\infty$ for some $r > 0$, then $E_i[\delta_i^\ell] < +\infty$ for any $\ell > 0$. Hence, we only need to prove (i), (iii), and the last part of Theorem 3.1.

We prove Theorem 3.1 (i) and (iii) by verifying condition (1.1). To proceed our arguments, we need the following lemma, which presents a functional Cauchy-Schwarz inequality for CTMCs.

Lemma 5.1 *Let V be a nonnegative function. If*

$$\int_0^{+\infty} V(t, X_t)\mathrm{d}t, \quad \int_0^{+\infty} \sqrt{E_x[V^2(t, X_t)]}\,\mathrm{d}t$$

exist, then we have

$$E_x\bigg[\bigg(\int_0^{+\infty} V(t, X_t)\mathrm{d}t\bigg)^2\bigg] \leqslant \bigg(\int_0^{+\infty} \sqrt{E_x[V^2(t, X_t)]}\,\mathrm{d}t\bigg)^2.$$

Proof Fix any $b > 0$. Divide the interval $[0, b]$ into $n + 1$ parts:

$$[s_0, s_1],\ [s_1, s_2],\ \ldots,\ [s_{n-1}, s_n],$$

where $s_0 = 0$ and $s_n = b$. Let

$$\Delta s_i = s_i - s_{i-1},\ 1 \leqslant i \leqslant n, \quad d = \max_{1\leqslant i\leqslant n} \Delta s_i.$$

We simply write $V_i = V(t_i, X_{t_i})$. Since both

$$\int_0^{+\infty} V(t, X_t)\mathrm{d}t, \quad \int_0^{+\infty} \sqrt{E_x[V^2(t, X_t)]}\,\mathrm{d}t$$

exist, we have

$$\begin{aligned} E_x\bigg[\bigg(\int_0^{b} V(t, X_t)\mathrm{d}t\bigg)^2\bigg] &= E_x\bigg[\bigg(\lim_{d\to 0} \sum_{i=0}^{n} V_i \Delta s_i\bigg)^2\bigg] \\ &\leqslant \liminf_{d\to 0} E_x\bigg[\bigg(\sum_{i=0}^{n} V_i \Delta s_i\bigg)^2\bigg] \\ &\leqslant \liminf_{d\to 0} \bigg(\sum_{i=0}^{n} \sqrt{E_x[V_i^2]}\bigg)^2 \\ &= \bigg(\int_0^{b} \sqrt{E_x[V^2(t, X_t)]}\,\mathrm{d}t\bigg)^2, \end{aligned} \tag{5.1}$$

where the first inequality follows from the Fatou lemma, and the second inequality follows from the Cauchy-Schwarz inequality, i.e.,

$$E\bigg[\bigg(\sum_{n=0}^{+\infty} X_n\bigg)^2\bigg] \leqslant \bigg(\sum_{n=0}^{+\infty} \sqrt{E[X_n^2]}\bigg)^2.$$

We obtain the assertion by letting $b \to +\infty$ in (5.1). □

5.1 Proof of Theorem 3.1 (i)

Let

$$V(t, X_t) = I_{t<\delta_i} |\overline{f}_{X_t}|.$$

By Lemma 5.1, we have

$$E_i\bigg[\bigg(\int_0^{\delta_i} |\overline{f}_{X_s}| \mathrm{d}s\bigg)^2\bigg] \leqslant \bigg(\int_0^{+\infty} \sqrt{E_i[I_{t<\delta_i} |\overline{f}_{X_t}|^2]}\, \mathrm{d}t\bigg)^2. \tag{5.2}$$

Let

$$p = 1 + \frac{2}{\eta}, \quad q = 1 + \frac{\eta}{2}.$$

It follows from Hölder inequality that

$$E_i[I_{t<\delta_i} |\overline{f}_{X_t}|^2] \leqslant (E_i[I_{t<\delta_i}])^{1/p} (E_i[|\overline{f}_{X_t}|^{2q}])^{1/q}. \tag{5.3}$$

Since $\pi(|f|^{2+\eta}) < +\infty$, we have

$$(E_i[|\overline{f}_{X_t}|^{2q}])^{1/q} \leqslant \Big(\frac{1}{\pi_i} E_\pi[|\overline{f}_{X_t}|^{2q}]\Big)^{1/q} = \Big(\frac{1}{\pi_i} \pi(|\overline{f}|^{2q})\Big)^{1/q} < +\infty. \tag{5.4}$$

It follows from the Markov inequality that

$$E_i[I_{t<\delta_i}] = P_i[t < \delta_i] \leqslant \frac{E_i[\delta_i^\ell]}{t^\ell}. \tag{5.5}$$

From (5.2)–(5.5), we obtain (1.1). □

5.2 Proof of Theorem 3.1 (iii)

To bound (1.1), we let the process start at the state i. Recall that J_1 is the first jump time. Define

$$T_0 = 0, \quad T_k := \inf\{t \geqslant T_{k-1} + J_1 \colon X_t = i\}, \quad i \geqslant 1.$$

Then T_k denotes the k-th return time on i. Obviously, X_t is a undelayed regenerative process with T_i, $i \geqslant 0$, as the regenerative time points, i.e.,

$$\{X_t,\, T_i \leqslant t < T_{i+1},\, i \geqslant 0\}$$

are independently identically distributed random variables. We divide the proof of this assertion into three steps.

(a) Since

$$E_x[\tau_i] = \int_0^{+\infty} t \mathrm{d}P_x[\tau_i \leqslant t] \geqslant s P_x[\tau_i > s], \quad \forall\, s > 0,$$

we know that if

$$\sup_{x \in \mathbb{E}} E_x[\tau_i] < +\infty,$$

then there exists constants $h < +\infty$ and $\beta < 1$ such that

$$\sup_{x \in \mathbb{E}} P_x[\tau_i \geqslant h] < \beta.$$

(b) Now, we can use the time length h given by (a) to divide this interval $[J_1, \delta_i]$. Let

$$t_n = J_1 + nh, \quad n \geqslant 0.$$

Let us define a Bernoulli random sequence $\{Z_n,\, n \geqslant 0\}$ as follows:

$$Z_0 = 0, \quad Z_n = \begin{cases} 1, & X_t = i \text{ for some } t \in [t_{n-1}, t_n], \\ 0, & \text{otherelse.} \end{cases}$$

Let

$$N = \inf\{n \geqslant 1\colon Z_n = 1\}$$

be the first hitting time of the state 1. Then we have

$$P[N = n] = \prod_{m=1}^{n-1} P[Z_m = 0] P[Z_n = 1] \leqslant \beta^{n-1} \cdot 1 = \beta^{n-1}, \quad n \geqslant 1,$$

where we make the convention that

$$\prod_{m=1}^{0} P[Z_m = 0] = 1.$$

Thus, we have

$$J_1 + (N-1)h = t_{N-1} \leqslant \delta_i \leqslant t_N = J_1 + Nh. \tag{5.6}$$

(c) Now, we use (5.6) to bound $E_i[(\int_0^{\delta_i} |\overline{f}_{X_s}| \mathrm{d}s)^2]$.

$$\begin{aligned} E_i\Big[\Big(\int_0^{\delta_i} |\overline{f}_{X_s}| \mathrm{d}s\Big)^2\Big] &= E_i\Big[\Big(\int_0^{t_0} |\overline{f}_{X_t}| \mathrm{d}t + \int_{t_0}^{\delta_i} |\overline{f}_{X_t}| \mathrm{d}t\Big)^2\Big] \\ &\leqslant E_i\Big[\Big(\int_0^{t_0} |\overline{f}_{X_t}| \mathrm{d}t\Big)^2\Big] + E_i\Big[\Big(\int_{t_0}^{\delta_i} |\overline{f}_{X_t}| \mathrm{d}t\Big)^2\Big] \\ &\quad + 2E_i\Big[\int_0^{t_0} |\overline{f}_{X_t}| \mathrm{d}t\Big] E_i\Big[\int_{t_0}^{\delta_i} |\overline{f}_{X_t}| \mathrm{d}t\Big] \\ &\leqslant \Big(\frac{1}{\lambda} + \frac{1}{\lambda^2}\Big) |\overline{f}|^2(i) + \frac{2|\overline{f}(i)|}{\lambda} \frac{\pi(|\overline{f}|)}{\pi_i} + D, \end{aligned} \tag{5.7}$$

where

$$D = E_i\Big[\Big(\int_{t_0}^{\delta_i} |\overline{f}_{X_t}| \mathrm{d}t\Big)^2\Big].$$

Now, we focus on bounding on D. Let

$$S_k = \int_{t_{k-1}}^{t_k} |\overline{f}_{X_t}|\mathrm{d}t, \quad k \geqslant 1.$$

Then we have

$$D \leqslant E_i\bigg[\bigg(\int_{t_0}^{t_N} |\overline{f}_{X_t}|\mathrm{d}t\bigg)^2\bigg] = E_i\bigg[\sum_{k=1}^{N} S_k^2\bigg] + 2E_i\bigg[\sum_{k=1}^{N} S_k \sum_{m=k+1}^{N} S_m\bigg].$$

Conditioning on J_1 and applying Lemma 5.1 gives that for any $k \geqslant 1$,

$$E_\pi[S_k^2] = \int_0^{+\infty} E_\pi[S_k^2|J_1 = s] \cdot \mathrm{d}P[J_1 \leqslant s] \leqslant h\sqrt{\pi(f^2)},$$

which implies

$$E_i[S_k^2] \leqslant \frac{1}{\pi_i} E_\pi[S_k^2] \leqslant \frac{1}{\pi_i} h\sqrt{\pi(f^2)} < +\infty.$$

By conditioning on N, we have

$$E_i\bigg[\sum_{k=1}^{N} S_k^2\bigg] = E_i\bigg[E\bigg[\sum_{k=1}^{N} S_k^2 \,\Big|\, N\bigg]\bigg] \leqslant \sum_{n=1}^{+\infty} \frac{nh\sqrt{\pi(f^2)}}{\pi_i}\beta^{n-1} < +\infty.$$

Similarly, we have

$$\begin{aligned}
E_i\bigg[\sum_{k=1}^{N} S_k \sum_{m=k+1}^{N} S_m\bigg] &= 2\sum_{n=1}^{+\infty} E_i\bigg[\sum_{k=1}^{n} S_k \sum_{m=k+1}^{n} S_m\bigg] P[N=n] \\
&\leqslant 2\sum_{n=1}^{+\infty}\bigg(\sum_{k=1}^{n}\sum_{m=k+1}^{n} \sqrt{E_i[S_k^2]}\sqrt{E_i[S_m^2]}\bigg)\beta^{n-1} \\
&\leqslant 2\sum_{n=1}^{+\infty} \frac{n^2-n}{\pi_i} h\sqrt{\pi(f^2)}\,\beta^{n-1} \\
&< +\infty, \qquad (5.8)
\end{aligned}$$

where we use the Hölder inequality with $p = q = 2$ in the second inequality. The assertion follows immediately from (5.7) and (5.8). □

5.3 Proof of last part of Theorem 3.1

It follows from [8] and the similar arguments in the proof of [3, Lemma 3.1], we have

$$\sigma^2(f) = \frac{2}{E_{i_0}[\delta_{i_0}]} E_{i_0}\bigg[\int_0^{\delta_{i_0}} \overline{f}_{X_t}\hat{f}_{X_t}\mathrm{d}t\bigg] = 2\pi(\overline{f}\hat{f}).$$

[9, Theorem 9.5.2] shows that the sequence $\{x_k,\, k \in \mathbb{E}\}$, given by

$$x_k = E_k\bigg[\int_0^{\delta_{i_0}} g_{X_t}\mathrm{d}t\bigg],$$

is a solution of the following equations:

$$\sum_{k\neq i_0} q_{ik}x_k = -g_i, \quad i \in \mathbb{E}, \tag{5.9}$$

where g is a nonnegative function. [2, Theorem VI.1.2] shows that

$$E_{i_0}\left[\int_0^{\delta_{i_0}} f_{X_t}\mathrm{d}t\right] = \pi(f)E_{i_0}[\delta_{i_0}],$$

which implies that $x_{i_0} = 0$. For a real number a, define

$$a^+ = \max\{a, 0\}, \quad a^- = \max\{-a, 0\}.$$

Obviously,

$$a = a^+ - a^-.$$

By considering

$$x_k^+ = E_k\left[\int_0^{\delta_{i_0}} (\overline{f}_{X_t})^+\mathrm{d}t\right], \quad x_k^- = E_k\left[\int_0^{\delta_{i_0}} (\overline{f}_{X_t})^-\mathrm{d}t\right],$$

we know that the sequence $\{x_k,\ k \in \mathbb{E}\}$ satisfies the Poisson equation $Qx = -\overline{f}$. The proof of the theorem is finished. □

6 Proof of Theorem 4.1

The fist part of Theorem 4.1 follows from Theorem 3.1 directly. We only need to prove the expression (4.1) of the variance constant.

Lemma 6.1 *Give a function f on $\mathbb{E}$. The sequence of functions $\{h_n,\ n \in \mathbb{Z}_+\}$, defined by*

$$h_i = \frac{f_i}{q_{i,i+1}}, \quad h_n = \frac{1}{q_{n,n+1}}\left[f_n + \sum_{k=i}^{n-1} q_n^{(k)}h_k\right], \quad n \geqslant i+1, \tag{6.1}$$

has the the following equivalent presentation:

$$h_n = \sum_{k=i}^{n} \frac{F_n^{(k)}f_k}{q_{k,k+1}}, \quad n \geqslant i. \tag{6.2}$$

Proof We use the induction. For $n = i$,

$$h_i = \frac{f_i}{q_{i,i+1}} = \frac{F_i^{(i)}f_i}{q_{i,i+1}} = \sum_{k=i}^{i} \frac{F_i^{(k)}f_k}{q_{k,k+1}}.$$

Assume that (6.2) holds for all $n \leqslant m$. When $n = m+1$, from (6.1), we see that

$$\begin{aligned} h_{m+1} &= \frac{f_{m+1}}{q_{m+1,m+2}} + \frac{1}{q_{m+1,m+2}} \sum_{k=i}^{m} q_{m+1}^{(k)} \sum_{\ell=i}^{k} \frac{F_k^{(\ell)} f_\ell}{q_{\ell,\ell+1}} \\ &= \frac{f_{m+1}}{q_{m+1,m+2}} + \sum_{\ell=i}^{m} \left(\frac{1}{q_{m+1,m+2}} \sum_{k=\ell}^{m} q_{m+1}^{(k)} F_k^{(\ell)} \right) \frac{f_\ell}{q_{\ell,\ell+1}} \\ &= \frac{f_{m+1}}{q_{m+1,m+2}} + \sum_{\ell=i}^{m} \frac{F_{m+1}^{(\ell)} f_\ell}{q_{\ell,\ell+1}} \\ &= \sum_{\ell=i}^{m+1} \frac{F_{m+1}^{(\ell)} f_\ell}{q_{\ell,\ell+1}}. \end{aligned}$$

Hence, (6.2) holds for $n = m+1$. By induction, we know that (6.2) hold for all $n \geqslant i$. □

Lemma 6.2 *For a single birth Q-matrix $Q = (q_{ij})$, a finite solution $x = (x_i,\ i \in \mathbb{E})$ of Poisson's equation $Qx = -g$ with $x_j = 0$ is given by*

$$x_i = \begin{cases} \displaystyle\sum_{n=i}^{j-1} \left(\sum_{k=0}^{n} \frac{F_n^{(k)} g_k}{q_{k,k+1}} \right), & i < j, \\ 0, & i = j, \\ \displaystyle -\sum_{n=j}^{i-1} \left(\sum_{k=0}^{n} \frac{F_n^{(k)} g_k}{q_{k,k+1}} \right), & i > j. \end{cases} \tag{6.3}$$

Proof From Poisson's equation $Qx = -g$, one easily sees that

$$x_0 - x_1 = \frac{g_0}{q_{01}}, \quad x_n - x_{n+1} = \frac{1}{q_{n,n+1}} \left[g_n + \sum_{k=0}^{n-1} q_n^{(k)} (x_k - x_{k+1}) \right], \quad n \geqslant 1.$$

So, by Lemma 6.1, we know that

$$x_n - x_{n+1} = \sum_{k=0}^{n} \frac{F_n^{(k)} g_k}{q_{k,k+1}}, \quad n \geqslant 0.$$

Note that

$$x_i = -\sum_{n=0}^{i-1} (x_n - x_{n+1}) + x_0 = -\sum_{n=0}^{i-1} \left(\sum_{k=0}^{n} \frac{F_n^{(k)} g_k}{q_{k,k+1}} \right) + x_0, \quad i \geqslant 1.$$

Since $x_j = 0$, we know that

$$x_0 = \sum_{n=0}^{j-1} \left(\sum_{k=0}^{n} \frac{F_n^{(k)} g_k}{q_{k,k+1}} \right).$$

Then we obtain the assertion. □

Proof of Theorem 4.1 It is known that the sequence $\widetilde{x} = (\widetilde{x}_i,\, i \in \mathbb{E})$, defined by

$$\widetilde{x}_i = \mathrm{E}_i\bigg[\int_0^{\delta_j} \overline{f}_{X_t}\mathrm{d}t\bigg],$$

is a solution of Poisson's equation with $\widetilde{x}_j = 0$. Due to the special structure of the single birth processes, we know that Poisson's equation has a unique finite solution. Hence, $\widetilde{x} = x$, where x is given by (6.3). By (3.1), we obtain

$$\sigma^2(f) = -2\sum_{n=1}^{+\infty}\pi_n\overline{f}_n\sum_{k=0}^{n-1}\bigg(\sum_{m=0}^{k}\frac{F_k^{(m)}\overline{f}_m}{q_{m,m+1}}\bigg). \tag{6.4}$$

The assertion (4.1) follows by exchanging the order of summation in the right-hand side of (6.4) and using the fact that

$$\sum_{k=0}^{+\infty}\pi_k\overline{f}(k) = 0.$$ □

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