

# Mixed eigenvalues of $p$ -Laplacian

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**Abstract** The mixed principal eigenvalue of  $p$ -Laplacian (equivalently, the optimal constant of weighted Hardy inequality in  $L^p$  space) is studied in this paper. Several variational formulas for the eigenvalue are presented. As applications of the formulas, a criterion for the positivity of the eigenvalue is obtained. Furthermore, an approximating procedure and some explicit estimates are presented case by case. An example is included to illustrate the power of the results of the paper.

**Keywords**  $p$ -Laplacian, Hardy inequality in  $L^p$  space, mixed boundaries, explicit estimates, eigenvalue, approximating procedure

**MSC** 60J60, 34L15

## 1 Introduction

As a natural extension of Laplacian from linear to nonlinear,  $p$ -Laplacian plays a typical role in mathematics, especially in nonlinear analysis. Refer to [1,10] for recent progresses on this subject. Motivated by the study on stability speed, we come to this topic, see [2,3] and references therein. The present paper is a continuation of [5] in which the estimates of the mixed principal eigenvalue for discrete  $p$ -Laplacian were carefully studied. This paper deals with the same problem but for continuous  $p$ -Laplacian, its principal eigenvalue is equivalent to the optimal constant in the weighted Hardy inequality. Even though the discrete case is often harder than the continuous one, the latter has its own difficulty. For instance, the existence of the eigenfunction is rather hard in the nonlinear context, but it is not a problem in the discrete situation. Similar to the case of  $p = 2$  ([3,4]), there are four types of boundaries: Neumann (denoted by code ‘N’) or Dirichlet (denoted by code ‘D’) boundary at the left- or right-endpoint of the half line  $[0, D]$ . In [7], Jin and Mao studied a class of weighted Hardy inequality and presented two variational formulas in the DN-case. Here,

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we study ND-case carefully and add some results to [7]. The DD- and NN-cases will be handled elsewhere. Comparing with our previous study, here the general weights are allowed.

The paper is organized as follows. In the next section, restricted in the ND-case, we introduce the main results: variational formulas and the basic estimates for the optimal constant (cf. [8,9]). As an application, we improve the basic estimates step by step through an approximating procedure. To illustrate the power of the results, an example is included. The sketched proofs of the results in Section 2 are presented in Section 3. For another mixed case: DN-case studied in [7], some complementary are presented in Section 4.

## 2 ND-case

Let  $\mu$  and  $\nu$  be two positive Borel measures on  $[0, D]$ ,  $D \leq \infty$  (replace  $[0, D]$  by  $[0, \infty)$  if  $D = \infty$ ),  $d\mu = u(x)dx$ , and  $d\nu = v(x)dx$ . Next, let

$$L_p f = (v|f'|^{p-2}f')', \quad p > 1.$$

Then the eigenvalue problem with ND-boundary conditions reads

$$\begin{cases} \text{Eigenequation:} & L_p g(x) = -\lambda u(x)|g|^{p-2}g(x); \\ \text{ND-boundaries:} & g'(0) = 0, \quad g(D) = 0 \quad \text{if } D < \infty. \end{cases} \quad (1)$$

If  $(\lambda, g)$  is a solution to the eigenvalue problem above,  $g \neq 0$ , then we call  $\lambda$  an 'eigenvalue' and  $g$  is an 'eigenfunction' of  $\lambda$ . When  $p = 2$ , the operator  $L_p$  defined above returns to the diffusion operator defined in [4]:  $u^{-1}(vf)'$ , where  $u(x)dx$  is the invariant measure of the diffusion process and  $v$  is a Borel measurable function related to its recurrence criterion. For  $\alpha \leq \beta$ , define

$$\mathcal{C}[\alpha, \beta] = \{f: f \text{ is continuous on } [\alpha, \beta]\},$$

$$\mathcal{C}^k(\alpha, \beta) = \{f: f \text{ has continuous derivatives of order } k \text{ on } (\alpha, \beta)\}, \quad k \geq 1,$$

and

$$\mu_{\alpha, \beta}(f) = \int_{\alpha}^{\beta} f d\mu, \quad D_p^{\alpha, \beta}(f) = \int_{\alpha}^{\beta} |f'|^p d\nu.$$

Similarly, one may define  $\mathcal{C}(\alpha, \beta)$ . In this section, we study the first eigenvalue (the minimal one), denoted by  $\lambda_p$ , described by the following classical variational formula:

$$\lambda_p = \inf\{D_p(f): f \in \mathcal{C}_K[0, D], \mu(|f|^p) = 1, f(D) = 0 \text{ if } D < \infty\}, \quad (2)$$

where

$$\mu(f) = \mu_{0, D}(f), \quad D_p(f) = D_p^{0, D}(f),$$

and

$$\mathcal{C}_K[\alpha, \beta] = \{f \in \mathcal{C}[\alpha, \beta]: v^{p^*-1} f' \in \mathcal{C}(\alpha, \beta) \text{ and } f \text{ has compact support}\},$$

with  $p^*$  the conjugate number of  $p$  (i.e.,  $p^{-1} + p^{*-1} = 1$ ). When  $p = 2$ , it reduces to the linear case studied in [4]. Thus, the aim of the paper is extending the results in linear case ( $p = 2$ ) to nonlinear one. Set

$$\mathcal{A}[\alpha, \beta] = \{f: f \text{ is absolutely continuous on } [\alpha, \beta]\}.$$

As will be proved soon (see Lemmas 3.3 and 3.4), we can rewrite  $\lambda_p$  as

$$\tilde{\lambda}_{*,p} := \inf\{D_p(f): \mu(|f|^p) = 1, f \in \mathcal{A}[0, D], f(D) = 0\}. \quad (3)$$

By making inner product with  $g$  on both sides of eigenequation (1) with respect to the Lebesgue measure over  $(\alpha, \beta)$ , we obtain

$$\lambda \mu_{\alpha,\beta}(|g|^p) = D_p^{\alpha,\beta}(g) - (v|g'|^{p-2} g' g)|_{\alpha}^{\beta}.$$

Moreover, since  $g'(0) = 0$ , we have

$$\lambda \mu(|g|^p) = D_p(g) - (v|g'|^{p-2} g' g)(D),$$

where, throughout this paper,  $f(D) := \lim_{x \rightarrow D} f(x)$  provided  $D = \infty$ . Hence, with

$$\mathcal{D}(D_p) = \{f: f \in \mathcal{A}[0, D], D_p(f) < \infty\},$$

$A := \lambda_p^{-1}$  is the optimal constant of the following *weighted Hardy inequality*:

$$\begin{aligned} \text{Hardy inequality:} & \quad \mu(|f|^p) \leq AD_p(f), \quad f \in \mathcal{D}(D_p); \\ \text{Boundary condition:} & \quad f(D) = 0. \end{aligned}$$

Note that the boundary condition ' $f'(0) = 0$ ' is unnecessary in the inequality.

Throughout this paper, we concentrate on  $p \in (1, \infty)$  since the degenerated cases that either  $p = 1$  or  $\infty$  are often easier to handle (cf. [11; Lemmas 5.4, 5.6]).

## 2.1 Main notation and results

For  $p > 1$ , let  $p^*$  be its conjugate number. Define

$$\hat{v}(x) = v^{1-p^*}(x), \quad \hat{v}(dx) = \hat{v}(x)dx.$$

We use the following hypothesis throughout the paper:

$u, \hat{v}$  are locally integrable with respect to the Lebesgue measure on  $[0, D]$ , without mentioned time by time.

Our main operators are defined as follows.

$$I(f)(x) = -\frac{1}{(vf'|f'|^{p-2})(x)} \int_0^x f^{p-1} d\mu \quad (\text{single integral form}),$$

$$II(f)(x) = \frac{1}{f^{p-1}(x)} \left[ \int_{(x,D) \cap \text{supp}(f)} \hat{v}(s) \left( \int_0^s f^{p-1} d\mu \right)^{p^*-1} ds \right]^{p-1}$$

(double integral form),

$$R(h)(x) = u(x)^{-1} [-|h|^{p-2}(v'h + (p-1)(h^2 + h')v)](x) \quad (\text{differential form}).$$

These operators have domains, respectively, as follows:

$$\begin{aligned} \mathcal{F}_I &= \{f \in \mathcal{C}[0, D] : v^{p^*-1}f' \in \mathcal{C}(0, D), f|_{(0,D)} > 0, f'|_{(0,D)} < 0\}, \\ \mathcal{F}_{II} &= \{f : f \in \mathcal{C}[0, D], f|_{(0,D)} > 0\}, \\ \mathcal{H} &= \{h : h \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], h(0) = 0, h|_{(0,D)} < 0 \text{ if } \hat{v}(0, D) < \infty, \\ &\quad \text{and } h|_{(0,D)} \leq 0 \text{ if } \hat{v}(0, D) = \infty\}, \end{aligned}$$

where  $\nu(\alpha, \beta) = \int_\alpha^\beta d\nu$  for a measure  $\nu$ . To avoid the non-integrability problem, some modifications of these sets are needed for studying the upper estimates.

$$\begin{aligned} \widetilde{\mathcal{F}}_I &= \{f \in \mathcal{C}[x_0, x_1] : v^{p^*-1}f' \in \mathcal{C}(x_0, x_1), f'|_{(x_0, x_1)} < 0 \text{ for some} \\ &\quad x_0, x_1 \in (0, D) \text{ with } x_0 < x_1, \text{ and } f = f(\cdot \vee x_0)\mathbb{1}_{[0, x_1]}\}, \\ \widetilde{\mathcal{F}}_{II} &= \{f : f = f\mathbb{1}_{[0, x_0]} \text{ for some } x_0 \in (0, D) \text{ and } f \in \mathcal{C}[0, x_0]\}, \\ \widetilde{\mathcal{H}} &= \{h : \exists x_0 \in (0, D) \text{ such that } h \in \mathcal{C}[0, x_0] \cap \mathcal{C}^1(0, x_0), h|_{(0, x_0)} < 0, \\ &\quad h|_{[x_0, D]} = 0, h(0) = 0, \text{ and } \sup_{(0, x_0)} (v'h + (p-1)(h^2 + h')v) < 0\}, \end{aligned}$$

In Theorem 2.1 below, for each  $f \in \mathcal{F}_I$ ,  $\inf_{x \in (0, D)} I(f)(x)^{-1}$  produces a lower bound of  $\lambda_p$ . So the part having ‘sup inf’ in each of the formulas is used for the lower estimates of  $\lambda_p$ . Dually, the part having ‘inf sup’ is used for the upper estimates. These formulas deduce the basic estimates in Theorem 2.3 and the approximating procedure in Theorem 2.4.

**Theorem 2.1** (Variational formulas) *For  $p > 1$ , we have*

(i) *single integral forms:*

$$\inf_{f \in \widetilde{\mathcal{F}}_I} \sup_{x \in (0, D)} I(f)(x)^{-1} = \lambda_p = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1};$$

(ii) *double integral forms:*

$$\inf_{f \in \widetilde{\mathcal{F}}_{II}} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} = \lambda_p = \sup_{f \in \mathcal{F}_{II}} \inf_{x \in (0, D)} II(f)(x)^{-1}.$$

Moreover, if  $u$  and  $v'$  are continuous, then we have additionally

(iii) *differential forms:*

$$\inf_{h \in \tilde{\mathcal{H}}} \sup_{x \in (0,D)} R(h)(x) = \lambda_p = \sup_{h \in \mathcal{H}} \inf_{x \in (0,D)} R(h)(x).$$

Furthermore, the supremum on the right-hand side of the above three formulas can be attained.

The following proposition adds some additional sets of functions for operators  $I$  and  $II$ . It then provides alternative descriptions of the lower and upper estimates of  $\lambda_p$ .

**Proposition 2.2** *For  $p > 1$ , we have*

$$\begin{aligned} \lambda_p &= \sup_{f \in \mathcal{F}_I} \inf_{x \in (0,D)} II(f)(x)^{-1}; \\ \lambda_p &= \inf_{f \in \tilde{\mathcal{F}}_{II} \cup \tilde{\mathcal{F}}'_{II}} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} \\ &= \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} \\ &= \inf_{f \in \tilde{\mathcal{F}}'_I} \sup_{x \in (0,D)} I(f)(x)^{-1}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{F}}'_{II} &= \{f : f \in \mathcal{C}[0, D] \text{ and } fII(f) \in L^p(\mu)\}, \\ \tilde{\mathcal{F}}'_I &= \{f : \exists x_0 \in (0, D), f = f\mathbb{1}_{[0,x_0]} \in \mathcal{C}[0, x_0], \\ &\quad f'|_{(0,x_0)} < 0, \text{ and } v^{p^*-1}f' \in \mathcal{C}(0, x_0)\}. \end{aligned}$$

Define  $k(p) = pp^{*p-1}$  for  $p > 1$  and

$$\sigma_p = \sup_{x \in (0,D)} \mu(0, x)\hat{v}(x, D)^{p-1}.$$

As applications of the variational formulas in Theorem 2.1 (i), we have the following basic estimates known in [11].

**Theorem 2.3** (Criterion and basic estimates) *For  $p > 1$ , the eigenvalue  $\lambda_p > 0$  if and only if  $\sigma_p < \infty$ . Moreover, the following basic estimates hold:*

$$(k(p)\sigma_p)^{-1} \leq \lambda_p \leq \sigma_p^{-1},$$

In particular, we have  $\lambda_p = 0$  if  $\hat{v}(0, D) = \infty$  and  $\lambda_p > 0$  if

$$\int_0^D \mu(0, s)^{p^*-1}\hat{v}(s)ds < \infty.$$

The approximating procedure below is an application of variational formulas in Theorem 2.1 (ii). The main idea is an iteration, its first step

produces Corollary 2.5 below. Noticing that  $\lambda_p$  is trivial once  $\sigma_p = \infty$ , we may assume that  $\sigma_p < \infty$  for further study on the estimates of  $\lambda_p$ .

**Theorem 2.4** (Approximating procedure) *Assume that  $\sigma_p < \infty$ .*

(i) *Let*

$$f_1 = \hat{\nu}(\cdot, D)^{1/p^*}, \quad f_{n+1} = f_n H(f_n)^{p^*-1}, \quad \delta_n = \sup_{x \in (0, D)} H(f_n)(x), \quad n \geq 1.$$

*Then  $\delta_n$  is decreasing and*

$$\lambda_p \geq \delta_n^{-1} \geq (k(p)\sigma_p)^{-1}.$$

(ii) *For fixed  $x_0, x_1 \in (0, D)$  with  $x_0 < x_1$ , define*

$$f_1^{x_0, x_1} = \hat{\nu}(\cdot \vee x_0, x_1) \mathbb{1}_{[0, x_1]}, \quad f_n^{x_0, x_1} = f_{n-1}^{x_0, x_1} H(f_{n-1}^{x_0, x_1})^{p^*-1} \mathbb{1}_{[0, x_1]},$$

*and*

$$\delta'_n = \sup_{x_0, x_1: x_0 < x_1} \inf_{x < x_1} H(f_n^{x_0, x_1})(x), \quad n \geq 1.$$

*Then  $\delta'_n$  is increasing and*

$$\sigma_p^{-1} \geq \delta'_n{}^{-1} \geq \lambda_p.$$

*Next, define*

$$\bar{\delta}_n = \sup_{x_0 < x_1} \frac{\|f_n^{x_0, x_1}\|_p^p}{D_p(f_n^{x_0, x_1})}, \quad n \geq 1.$$

*Then  $\bar{\delta}_n^{-1} \geq \lambda_p$  and  $\bar{\delta}_{n+1} \geq \delta'_n$  for  $n \geq 1$ .*

The following Corollary 2.5 can be obtained directly from Theorem 2.4. It provides us some improved and explicit estimates of the eigenvalue (see Example 2.6 below).

**Corollary 2.5** (Improved estimates) *Assume that  $\sigma_p < \infty$ . Then*

$$\sigma_p^{-1} \geq \delta'_1{}^{-1} \geq \lambda_p \geq \delta_1^{-1} \geq (k(p)\sigma_p)^{-1},$$

*where*

$$\delta_1 = \sup_{x \in (0, D)} \left[ \frac{1}{\hat{\nu}(x, D)^{1/p^*}} \int_x^D \hat{\nu}(s) \left( \int_0^s \hat{\nu}(t, D)^{(p-1)/p^*} \mu(dt) \right)^{p^*-1} ds \right]^{p-1},$$

$$\delta'_1 = \sup_{x \in (0, D)} \frac{1}{\hat{\nu}(x, D)^{p-1}} \left[ \int_x^D \hat{\nu}(s) \left( \int_0^s \hat{\nu}(t \vee x, D)^{p-1} \mu(dt) \right)^{p^*-1} ds \right]^{p-1}.$$

*Moreover,*

$$\bar{\delta}_1 = \sup_{x \in (0, D)} \left[ \mu(0, x) \hat{\nu}(x, D)^{p-1} + \frac{1}{\hat{\nu}(x, D)} \int_x^D \hat{\nu}(t, D)^p \mu(dt) \right] \in [\sigma_p, p\sigma_p],$$

$\bar{\delta}_1 \leq \delta'_1$  for  $1 < p \leq 2$  and  $\bar{\delta}_1 \geq \delta'_1$  for  $p \geq 2$ .

When  $p = 2$ , the assertion that  $\bar{\delta}_1 = \delta'_1$  was proved in [4; Theorem 3]. To illustrate the results above, we present an example as follows.

## 2.2 Example

**Example 2.6** Let  $d\mu = d\nu = dx$  on  $(0, 1)$ . In the ND-case, the eigenvalue  $\lambda_p$  satisfies

$$\lambda_p^{1/p} = \frac{\pi(p-1)^{1/p}}{p} \sin^{-1} \frac{\pi}{p}. \quad (4)$$

For the basic estimates, we have

$$\sigma_p^{1/p} = \left(\frac{1}{p}\right)^{1/p} \left(\frac{1}{p^*}\right)^{1/p^*}.$$

Furthermore, we have

$$\bar{\delta}_1^{1/p} = p^{\frac{1}{p}-2} (p^2 - 1)^{1-\frac{1}{p}},$$

$$\delta_1^{1/p} = \frac{1}{(p + \frac{1}{p} - 1)^{1/p}} \left\{ \sup_{x \in (0,1)} \frac{1}{(1-x)^{1/p^*}} \int_0^{1-x} (1 - z^{p+\frac{1}{p}-1})^{p^*-1} dz \right\}^{1/p^*}.$$

The exact value  $\lambda_p^{1/p}$  and its basic estimates are shown in Fig. 1. Then, the improved upper bound  $\delta_1^{1/p}$  and lower one  $\bar{\delta}_1^{1/p}$  are added to Fig. 1, as shown in Fig. 2. It is quite surprising and unexpected that both of  $\delta_1^{1/p}$  and  $\bar{\delta}_1^{1/p}$  are almost overlapped with the exact value  $\lambda_p^{1/p}$  except in a small neighborhood of  $p = 2$ , where  $\delta_1^{1/p}$  is a little bigger and  $\bar{\delta}_1^{1/p}$  is a little smaller than  $\lambda_p^{1/p}$ . Here,  $\delta_1^{\prime 1/p}$  is ignored since it improves  $\bar{\delta}_1^{1/p}$  only a little bit for  $p \in (1, 2)$ .

## 3 Proofs of main results

Some preparations for the proofs are collected in Subsection 3.1. They may not be used completely in the proofs but are helpful to understand the idea in this paper and may be useful in other cases. The proofs of the main results are presented in Subsection 3.2. For simplicity, we let  $\uparrow$  (resp.  $\uparrow\uparrow$ ,  $\downarrow$ ,  $\downarrow\downarrow$ ) denote increasing (resp. strictly increasing, decreasing, strictly decreasing) throughout this paper.

### 3.1 Preparations

The next lemma is taken from [1; Theorem 1.1, p.170] (see [13] for its original idea). Combining with the following Remark 3.2, Lemmas 3.3 and 3.4, it guarantees the existence of the solution  $(\lambda_p, g)$  to the eigenvalue problem.

**Lemma 3.1** (Existence and uniqueness) (i) *Suppose that  $u$  and  $v$  are locally integrable on  $[0, D] \subseteq \mathbb{R}$  (or  $[0, D] \subseteq \mathbb{R}$  provided  $D = \infty$ ) and  $v > 0$ .*

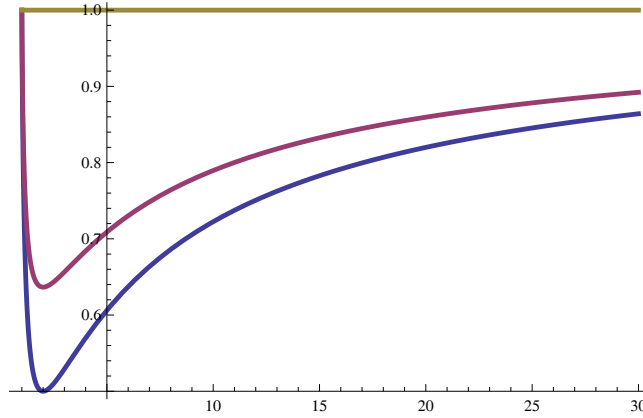


Fig. 1 Middle curve is exact value of  $\lambda_p^{1/p}$ . Top straight line and bottom curve are basic estimates of  $\lambda_p^{1/p}$ .

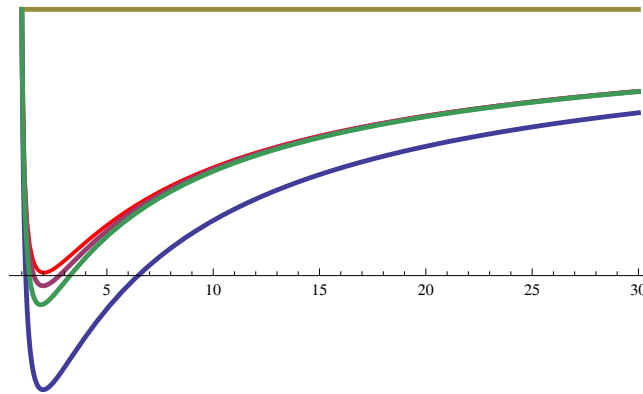


Fig. 2 Improved bounds  $\delta_1^{1/p}$  and  $\bar{\delta}_1^{1/p}$  are added to Figure 1.

Given constants  $A$  and  $B$ , for each fixed  $\lambda$ , there is uniquely a solution  $g$  such that  $g(0) = A$ ,  $g'(0) = B$ , and eigenequation (1) holds almost everywhere. Moreover,  $v^{p^*-1}g'$  is absolutely continuous.

(ii) Suppose additionally that  $u$  and  $v$  are continuous. Then  $g \in \mathcal{C}^2[0, D]$  and the eigenequation holds everywhere on  $[0, D]$ .

If eigenequation (1) holds (almost) everywhere for  $(\lambda_p, g)$ , then  $g$  is called an (a.e.) eigenfunction of  $\lambda_p$ .

**Remark 3.2** (i) One may also refer to [6; Lemma 2.1] for the existence of solution to eigenvalue problem with ND boundary conditions provided  $D < \infty$ . When  $D = \infty$ , the Dirichlet boundary at  $D$  means  $g(D) = 0$ , which is proved by Proposition 3.7 below.

(ii) By [11; Theorems 4.1, 4.7], we see that the eigenequation in (1) has solutions if and only if the following equation has solutions:

$$(|g'|^{p-2}g')'(x) = -\lambda\tilde{u}(x)|g|^{p-2}g(x),$$



where  $\tilde{u}$  is related to  $v$  and  $u$  in the eigenequation. Hence, the weight function  $v$  in the eigenquation is not a sensitive or key quantity to the existence of solution to the eigenequation and can be seen as a constant.

Define  $\mathcal{A}_K[0, D] = \{f : f \in \mathcal{A}[0, D], f \text{ has compact support}\}$  and

$$\lambda_{*,p} = \inf\{D_p(f) : f \in \mathcal{A}_K[0, D], \|f\|_p = 1, \text{ and } f(D) = 0 \text{ if } D < \infty\}, \quad (5)$$

$$\tilde{\lambda}_p = \inf\{D_p(f) : v^{p^*-1}f' \in \mathcal{C}(0, D), f \in \mathcal{C}[0, D], \|f\|_p = 1, f(D) = 0\}, \quad (6)$$

where  $\|\cdot\|_p$  means the norm in  $L^p(\mu)$  space. The following quantities are also useful for us. Set  $\alpha \in (0, D)$  and define

$$\lambda_{*,p}^{(0,\alpha)} = \inf\{D_p(f) : \mu(|f|^p) = 1, f \in \mathcal{A}[0, \alpha] \text{ and } f|_{[\alpha, D]} = 0\},$$

$$\lambda_p^{(0,\alpha)} = \inf\{D_p(f) : f \in \mathcal{C}[0, \alpha], v^{p^*-1}f' \in \mathcal{C}(0, \alpha), \mu(|f|^p) = 1, f|_{[\alpha, D]} = 0\}.$$

The following three lemmas describe in a refined way the first eigenvalue and lead to, step by step, the conclusion that

$$\tilde{\lambda}_p = \lambda_p = \lambda_{*,p} = \tilde{\lambda}_{*,p}.$$

**Lemma 3.3** *We have  $\lambda_p = \lambda_{*,p}$ .*

*Proof* It is obvious that  $\lambda_p \geq \lambda_{*,p}$ . Next, let  $g$  be the a.e. eigenfunction of  $\lambda_{*,p}$ . Then  $g \in \mathcal{C}[0, D]$  and  $v^{p^*-1}g' \in \mathcal{C}(0, D)$  by Lemma 3.1. Since  $L_p g = -\lambda_{*,p}|g|^{p-2}g$ , by the arguments after formula (3), we have

$$-(vg|g'|^{p-2}g')|_0^D + D_p(g) = \lambda_{*,p}\|g\|_p^p.$$

Since  $g'(0) = 0$  and  $(gg')(D) \leq 0$ , we have  $\lambda_{*,p} \geq D_p(g)/\|g\|_p^p$ . Because  $g \in \mathcal{C}_K[0, D]$ , it is clear that  $D_p(g)/\|g\|_p^p \geq \lambda_p$ . We have thus obtained that

$$\lambda_p \leq \lambda_{*,p} \leq \lambda_p,$$

and so  $\lambda_p = \lambda_{*,p}$ . There is a small gap in the proof above since in the case of  $D = \infty$ , the a.e. eigenfunction  $g$  may not belong to  $L^p(\mu)$  and we have not yet proved that  $(gg')(D) \leq 0$ . However, one may avoid this by a standard approximating procedure, using  $[0, \alpha_n]$  instead of  $[0, D)$  with  $\alpha_n \uparrow D$  provided  $D = \infty$ :

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_p^{(0,\alpha_n)} &= \lim_{n \rightarrow \infty} \inf\{D_p(f) : \mu(|f|^p) = 1, f \in \mathcal{C}[0, \alpha_n], \\ &\quad v^{p^*-1}f' \in \mathcal{C}(0, \alpha_n), f|_{[\alpha_n, D]} = 0\} \\ &= \lambda_p. \end{aligned}$$

Similarly,  $\lambda_{*,p}^{(0,\alpha_n)} \rightarrow \lambda_{*,p}$  as  $n \rightarrow \infty$ . □

**Lemma 3.4** *For  $\tilde{\lambda}_{*,p}$  defined in (3), we have  $\tilde{\lambda}_{*,p} = \lambda_{*,p}$ . Furthermore,*

$$\tilde{\lambda}_p = \lambda_p = \lambda_{*,p} = \tilde{\lambda}_{*,p}.$$

*Proof* On one hand, by definition, if  $\beta_{n+1} \geq \beta_n$ , then  $\lambda_{*,p}^{(0,\beta_n)} \geq \lambda_{*,p}^{(0,\beta_{n+1})}$ . We have thus obtained

$$\lim_{n \rightarrow \infty} \lambda_{*,p}^{(0,\beta_n)} \geq \lambda_{*,p}^{(0,D)} = \tilde{\lambda}_{*,p}.$$

On the other hand, by definition of  $\tilde{\lambda}_{*,p}$ , for any fixed  $\varepsilon > 0$ , there exists  $f$  satisfying

$$\|f\|_p = 1, \quad f(D) = 0, \quad D_p(f) \leq \tilde{\lambda}_{*,p} + \varepsilon.$$

Let  $\beta_n \uparrow D$  and  $f_n = (f - f(\beta_n))\mathbb{1}_{[0,\beta_n]}$ . Then  $D_p(f_n) \uparrow D_p(f)$  as  $n \uparrow \infty$ . Choose subsequence  $\{n_m\}_{m \geq 1}$  if necessary such that

$$\overline{\lim}_{n \rightarrow \infty} \frac{D_p(f_n)}{\|f_n\|_p^p} = \lim_{m \rightarrow \infty} \frac{D_p(f_{n_m})}{\|f_{n_m}\|_p^p}.$$

By Fatou's lemma and the fact that  $f(D) = 0$ , we have

$$\underline{\lim}_{m \rightarrow \infty} \|f_{n_m}\|_p^p \geq \left\| \underline{\lim}_{m \rightarrow \infty} f_{n_m} \right\|_p^p = \|f\|_p^p = 1.$$

Therefore, we obtain

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} \lambda_{*,p}^{(0,\beta_n)} &\leq \overline{\lim}_{n \rightarrow \infty} \frac{D_p(f_n)}{\|f_n\|_p^p} \\ &= \lim_{m \rightarrow \infty} \frac{D_p(f_{n_m})}{\|f_{n_m}\|_p^p} \\ &\leq \frac{\lim_{m \rightarrow \infty} D_p(f_{n_m})}{\underline{\lim}_{m \rightarrow \infty} \|f_{n_m}\|_p^p} \\ &\leq D_p(f) \\ &\leq \tilde{\lambda}_{*,p} + \varepsilon. \end{aligned}$$

Since  $\lim_{n \rightarrow \infty} \lambda_{*,p}^{(0,\beta_n)} = \lambda_{*,p}$ , we get  $\tilde{\lambda}_{*,p} = \lambda_{*,p}$ . Moreover,

$$\tilde{\lambda}_p \geq \tilde{\lambda}_{*,p} = \lambda_{*,p} = \lambda_p \geq \tilde{\lambda}_p,$$

and the required assertion holds. □

The following lemma, which serves for Lemma 3.6, presents us that  $\{\lambda_{*,p}^{(0,\alpha)}\}$  is strictly decreasing with respect to  $\alpha$ .

**Lemma 3.5** For  $\alpha, \beta \in (0, D)$  with  $\alpha < \beta$ , we have  $\lambda_{*,p}^{(0,\alpha)} > \lambda_{*,p}^{(0,\beta)}$ . Furthermore,  $\lambda_{*,p}^{(0,\beta_n)} \downarrow \lambda_{*,p}$  as  $\beta_n \uparrow D$ .

*Proof* Let  $g (\neq 0)$  be an a.e. eigenfunction of  $\lambda_{*,p}^{(0,\alpha)}$ . Then  $g'(0) = 0$ ,  $g(\alpha) = 0$ , and  $L_p g = -\lambda_{*,p}^{(0,\alpha)} |g|^{p-2} g$  on  $(0, \alpha)$ . Moreover,

$$\lambda_{*,p}^{(0,\alpha)} = \frac{D_p^{0,\alpha}(g)}{\|g\|_{L^p(0,\alpha;\mu)}^p}, \quad D_p^{\alpha,\beta}(f) = \int_{\alpha}^{\beta} |f'|^p d\nu$$

(see arguments after formula (3)). By the proof of Lemma 3.3, the proof of the first assertion will be done once we choose a function  $\tilde{g} \in \mathcal{A}[0, \beta]$  such that  $\tilde{g}'(0) = 0$ ,  $\tilde{g}(\beta) = 0$ , and

$$\frac{D_p^{0,\alpha}(g)}{\|g\|_{L^p(0,\alpha;\mu)}^p} > \frac{D_p^{0,\beta}(\tilde{g})}{\|\tilde{g}\|_{L^p(0,\beta;\mu)}^p} \quad (\geq \lambda_{*,p}^{(0,\beta)}). \tag{7}$$

To do so, without loss of generality, assume that  $g|_{(0,\alpha)} > 0$  (see [12; Lemma 2.4]). Then the required assertion follows for

$$\tilde{g}(x) = (g + \varepsilon)\mathbb{1}_{[0,\alpha)}(x) + \frac{\varepsilon(\beta - x)}{\beta - \alpha} \mathbb{1}_{[\alpha,\beta]}(x), \quad x \in [0, \beta],$$

once  $\varepsilon$  is sufficiently small. Actually, by simple calculation, we have

$$D_p^{0,\beta}(\tilde{g}) = D_p^{0,\alpha}(g) + \frac{\varepsilon^p}{(\beta - \alpha)^p} \nu(\alpha, \beta),$$

$$\|\tilde{g}\|_{L^p(0,\beta;\mu)}^p = \|g\|_{L^p(0,\alpha;\mu)}^p + \int_0^\alpha (|g + \varepsilon|^p - |g|^p) d\mu + \int_\alpha^\beta \frac{\varepsilon^p(\beta - x)^p}{(\beta - \alpha)^p} \mu(dx).$$

Since

$$\lambda_{*,p}^{(0,\alpha)} = \frac{D_p^{0,\alpha}(g)}{\|g\|_{L^p(0,\alpha;\mu)}^p},$$

inequality (7) holds if and only if

$$\frac{\varepsilon^p \nu(\alpha, \beta)}{(\beta - \alpha)^p} < \left( \int_0^\alpha (|g + \varepsilon|^p - |g|^p) d\mu + \frac{\varepsilon^p}{(\beta - \alpha)^p} \int_\alpha^\beta (\beta - x)^p \mu(dx) \right) \lambda_{*,p}^{(0,\alpha)}.$$

It suffices to show that

$$\frac{\varepsilon^{p-1}}{(\beta - \alpha)^p} \nu(\alpha, \beta) < \lambda_{*,p}^{(0,\alpha)} \left( \int_0^\alpha \frac{|g(x) + \varepsilon|^p - |g(x)|^p}{\varepsilon} \mu(dx) \right).$$

By letting  $\varepsilon \rightarrow 0$ , the right-hand side is equal to

$$\lambda_{*,p}^{(0,\alpha)} \int_0^\alpha p g^{p-1} d\mu,$$

which is positive. So the required inequality is obvious for sufficiently small  $\varepsilon$  and the first assertion holds. The second assertion was proved at the end of the proofs of Lemma 3.4.  $\square$

The following lemma is about the eigenfunction of  $\lambda_p$ , which is the basis of the test functions used for the corresponding operators.

**Lemma 3.6** *Let  $g$  be the first eigenfunction of eigenvalue problem (1). Then both  $g$  and  $g'$  do not change sign. Moreover, if  $g > 0$ , then  $g' < 0$ .*

*Proof* If there exists  $\alpha \in (0, D)$  such that  $g(\alpha) = 0$ , then  $\lambda_{*,p}^{(0,\alpha)} \leq \lambda_{*,p}$  by the minimum property of  $\lambda_{*,p}^{(0,\alpha)}$ . However, by Lemma 3.5, we get  $\lambda_{*,p}^{(0,\alpha)} \downarrow \lambda_{*,p}$  as  $\alpha \uparrow \uparrow D$ . This is a contradiction. So  $g$  does not change its sign. Next, consider  $g'$ . By [12; Lemma 2.3], if there exists  $x \in (0, D)$  such that  $g'(x) = 0$ , then  $\exists x_0 \in (0, x)$  such that  $g(x_0) = 0$ , which is impossible by the strictly decreasing property of  $\lambda_{*,p}^{(0,\alpha)}$  with respect to  $\alpha$ . So the assertion holds.  $\square$

Before moving on, we introduce a general equation, non-linear ‘Poisson equation’ as follows:

$$L_p g(x) = -u(x)|f|^{p-2}f(x), \quad x \in (0, D). \quad (8)$$

Integration by parts yields that for  $x, y \in (0, D)$  with  $x < y$ ,

$$v(x)|g'|^{p-2}g'(x) - v(y)|g'|^{p-2}g'(y) = \int_x^y |f|^{p-2}f d\mu. \quad (9)$$

By replacing  $f$  with  $\lambda^{p^*-1}g$ , it is not hard to understand where the operator  $I$  comes from. Moreover, if  $g$  is positive and decreasing,  $g'(0) = 0$ , then

$$g(y) - g(D) = \int_y^D \hat{v}(x) \left( \int_0^x |f|^{p-2}f d\mu \right)^{p^*-1} dx, \quad y \in (0, D). \quad (10)$$

Replacing  $f$  with  $\lambda^{p^*-1}g$ , it is easy to see where the operator  $II$  comes from, provided  $g(D) = 0$  (which is affirmative by Proposition 3.7 below). Finally, assume that  $(\lambda_p, g)$  is a solution to (1). Then  $\lambda_p = -L_p g / (|g|^{p-2}g)$ . Hence, by letting  $h = g'/g$ , we deduce the operator  $R$  from the eigenequation.

### 3.2 Proof of main results

*Proofs of Theorem 2.1 and Proposition 2.2* We adopt the circle arguments below to prove the lower estimates:

$$\begin{aligned} \lambda_p &\geq \tilde{\lambda}_p \\ &\geq \sup_{f \in \mathcal{F}_{II}} \inf_{x \in (0, D)} II(f)(x)^{-1} \\ &= \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} II(f)(x)^{-1} \\ &= \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1} \\ &\geq \sup_{h \in \mathcal{H}} \inf_{x \in (0, D)} R(h)(x) \\ &\geq \lambda_p. \end{aligned}$$

**Step 1** Prove that

$$\lambda_p \geq \tilde{\lambda}_p \geq \sup_{f \in \mathcal{F}_{II}} \inf_{x \in (0, D)} II(f)(x)^{-1}.$$

It suffices to show the second inequality. For each fixed  $h > 0$  and  $g \in \mathcal{C}[0, D]$  with  $\|g\|_p = 1$ ,  $g(D) = 0$ , and  $v^{p^*-1}g' \in \mathcal{C}(0, D)$ , we have

$$\begin{aligned} \int_0^D |g^p| d\mu &= \int_0^D \left| \int_x^D g'(t) \left(\frac{v(t)}{h(t)}\right)^{1/p} \left(\frac{h(t)}{v(t)}\right)^{1/p} dt \right|^p \mu(dx) \\ &\leq \int_0^D \int_x^D \frac{v(t)}{h(t)} |g'(t)|^p dt \left[ \int_x^D \left(\frac{h(s)}{v(s)}\right)^{p^*-1} ds \right]^{p-1} \mu(dx) \\ &\quad \text{(by Hölder's inequality)} \\ &= \int_0^D \frac{v(t)}{h(t)} |g'(t)|^p dt \int_0^t \left[ \int_x^D \left(\frac{h(s)}{v(s)}\right)^{p^*-1} ds \right]^{p-1} \mu(dx) \\ &\quad \text{(by Fubini's Theorem)} \\ &\leq D_p(g) \sup_{t \in (0, D)} H(t), \end{aligned}$$

where

$$H(t) = \frac{1}{h(t)} \int_0^t \left[ \int_x^D \left(\frac{h(s)}{v(s)}\right)^{p^*-1} ds \right]^{p-1} \mu(dx).$$

For  $f \in \mathcal{F}_H$  with  $\sup_{x \in (0, D)} H(f)(x) < \infty$ , let

$$h(t) = \int_0^t f^{p-1}(s)u(s)ds.$$

Then  $h' = f^{p-1}u$ . By Cauchy's mean-value theorem, we have

$$\sup_{x \in (0, D)} H(x) \leq \sup_{x \in (0, D)} H(f)(x).$$

Thus,  $\lambda_p \geq \inf_{x \in (0, D)} H(f)(x)^{-1}$ . The assertion then follows by making the supremum with respect to  $f \in \mathcal{F}_H$ .

**Step 2** Prove that

$$\sup_{f \in \mathcal{F}_H} \inf_{x \in (0, D)} H(f)(x)^{-1} = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} H(f)(x)^{-1} = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1}.$$

(a) We prove the part ' $\geq$ '. Since  $\mathcal{F}_I \subset \mathcal{F}_H$ , it suffices to show that

$$\sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} H(f)(x)^{-1} \geq \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1}$$

for  $f \in \mathcal{F}_I$  with  $\sup_{x \in (0, D)} I(f) < \infty$ . Since  $f(D) \geq 0$ , by replacing  $f$  in the denominator of  $H(f)$  with  $-\int_x^D f'(s)ds$  and using Cauchy's mean-value theorem, we have

$$\sup_{x \in (0, D)} H(f)(x) \leq \sup_{x \in (0, D)} I(f)(x) < \infty.$$

So the assertion holds by making the supremum with respect to  $f \in \mathcal{F}_I$ .

(b) To prove the equality, it suffices to show that

$$\sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1} \geq \sup_{f \in \mathcal{F}_II} \inf_{x \in (0, D)} II(f)(x)^{-1}.$$

For  $f \in \mathcal{F}_II$ , without loss of generality, assume that  $\inf_{x \in (0, D)} II(f)(x)^{-1} > 0$ . Let  $g = f[II(f)]^{p^*-1}$ . Then  $g \in \mathcal{F}_I$ . Moreover,

$$v(x)(-g'(x))^{p-1} = \int_0^x f^{p-1} d\mu \geq \int_0^x g^{p-1} d\mu \inf_{t \in (0, x)} \frac{f^{p-1}(t)}{g^{p-1}(t)},$$

i.e.,

$$I(g)(x)^{-1} \geq \inf_{x \in (0, D)} II(f)(x)^{-1}.$$

Hence,

$$\sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1} \geq \inf_{x \in (0, D)} I(g)(x)^{-1} \geq \inf_{x \in (0, D)} II(f)(x)^{-1}$$

and the assertion holds since  $f \in \mathcal{F}_II$  is arbitrary.

Then there is another method to prove the equality: prove that

$$\sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1} \geq \lambda_p.$$

Let  $g$  be an a.e. eigenfunction corresponding to  $\lambda_p$ . Then  $g$  is positive and strictly decreasing. It is easy to check that  $g \in \mathcal{F}_I$ . By (9), we have

$$\lambda_p = \inf_{x \in (0, D)} I(g)(x)^{-1} \leq \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1}.$$

**Step 3** When  $u$  and  $v'$  are continuous, we prove that

$$\sup_{f \in \mathcal{F}_II} \inf_{x \in (0, D)} II(f)(x)^{-1} \geq \sup_{h \in \mathcal{H}} \inf_{x \in (0, D)} R(h)(x).$$

First, we change the form of  $R(h)$ . Let  $g$  with  $g(D) = 0$  be a positive function on  $[0, D)$  such that  $h = g'/g$  (see the arguments after Lemma 3.6). Then

$$R(h) = -u^{-1} \{ |h|^{p-2} [v'h + (p-1)v(h^2 + h')] \} = -\frac{1}{ug^{p-1}} L_p g.$$

Now, we turn to our main text. It suffices to show that

$$\sup_{f \in \mathcal{F}_II} \inf_{x \in (0, D)} II(f)(x)^{-1} \geq \inf_{x \in (0, D)} R(h)(x) \quad \text{for every } h \in \mathcal{H}.$$

Without loss of generality, assume that  $\inf_{x \in (0,D)} R(h)(x) > 0$ , which implies  $R(h) > 0$  on  $(0, D)$ . Let  $f = g(R(h))^{p^*-1}$  ( $g$  is the function just specified). Since  $u, v'$  are continuous, we have  $f \in \mathcal{F}_H$  and

$$u(x)f^{p-1}(x) = -L_p g(x), \quad x \in (0, D).$$

Moreover, by (10), we have

$$g(y) - g(D) = \int_y^D \hat{v}(x) \left( \int_0^x f^{p-1} d\mu \right)^{p^*-1} dx.$$

So  $g^{p-1}/f^{p-1} \geq H(f)$  on  $(0, D)$  and

$$\inf_{(0,D)} R(h) = \inf_{(0,D)} \frac{f^{p-1}}{g^{p-1}} \leq \inf_{(0,D)} H(f)^{-1} \leq \sup_{f \in \mathcal{F}_H} \inf_{x \in (0,D)} H(f)(x)^{-1}.$$

Hence, the required assertion holds.

**Step 4** Prove that

$$\sup_{h \in \mathcal{H}} \inf_{x \in (0,D)} R(h)(x) \geq \lambda_p$$

when  $u$  and  $v'$  are continuous.

Note that

$$\hat{v}(x, D) \left( \int_0^x f^{p-1} d\mu \right)^{p^*-1} \leq f H(f)(x)^{p^*-1} \leq \hat{v}(x, D) \left( \int_0^D f^{p-1} d\mu \right)^{p^*-1}.$$

If  $\hat{v}(0, D) < \infty$ , then choose  $f \in L^{p-1}(\mu)$  to be a positive function such that  $g = f H(f)^{p^*-1} < \infty$ . Set  $\bar{h} = g'/g$ . Then  $\bar{h} \in \mathcal{H}$  since  $u$  and  $v'$  are continuous. Moreover,  $L_p g = -u f^{p-1}$  and

$$R(\bar{h}) = -\frac{1}{u g^{p-1}} L_p g = \frac{f^{p-1}}{g^{p-1}} > 0.$$

If  $\hat{v}(0, D) = \infty$ , then set  $\bar{h} = 0$ . So  $R(\bar{h}) = 0$ . In other words, we always have

$$\sup_{h \in \mathcal{H}} \inf_{x \in (0,D)} R(h)(x) \geq 0.$$

Without loss of generality, assume that  $\lambda_p > 0$  and  $g$  is an eigenfunction of  $\lambda_p$ , i.e.,

$$L_p g = -\lambda_p u |g|^{p-2} g.$$

Let  $h = g'/g \in \mathcal{H}$ . Then  $R(h) = \lambda_p$  and the assertion holds.

**Step 5** Prove that the supremum in the lower estimates can be attained.

Since

$$0 = \lambda_p \geq \inf_{x \in (0,D)} H(f)(x)^{-1} \geq 0, \quad 0 = \lambda_p \geq \inf_{x \in (0,D)} I(f)(x)^{-1} \geq 0$$

for every  $f$  in the set defining  $\lambda_p$ , the assertion is clear for the case that  $\lambda_p = 0$ . Similarly, the conclusion holds for operator  $R$  as seen from the preceding proof in Step 4. For the case that  $\lambda_p > 0$ , assume that  $g$  is an eigenfunction corresponding to  $\lambda_p$ . Let  $\bar{h} = g'/g \in \mathcal{H}$ . Then  $R(\bar{h}) = \lambda_p$ ,  $I(g)^{-1} \equiv \lambda_p$  by letting  $f = \lambda_p^{p^*-1}g$  in (9) and  $II(g)^{-1} \equiv \lambda_p$  by letting  $f = \lambda_p^{p^*-1}g$  in (10) whenever  $g(D) = 0$ .

Now, it remains to show that the vanishing property of eigenfunction at  $D$ , which is proved in the following proposition by using the variational formula proved in Step 1 above.

**Proposition 3.7** *Let  $g$  be an a.e. eigenfunction of  $\lambda_p > 0$ . Then  $g(D) = 0$ .*

*Proof* Let  $f = g - g(D)$ . Then  $f \in \mathcal{F}_II$ . By (10), we have

$$f(x) = \lambda_p^{p^*-1} \int_x^D \hat{v}(t) \left( \int_0^t g^{p-1} d\mu \right)^{p^*-1} dt.$$

We prove the proposition by dividing it into two cases. Denote

$$M(x) = \int_x^D \hat{v}(t) \left( \int_0^t d\mu \right)^{p^*-1} dt.$$

(a) If  $M(x) = \infty$ , then  $f(x) = g(x) - g(D) < \infty$  and

$$\lambda_p^{1-p^*} f(x) = \int_x^D \hat{v}(t) \left( \int_0^t g^{p-1} d\mu \right)^{p^*-1} dt > g(D)M(x) = \infty$$

once  $g(D) \neq 0$ . So there is a contradiction.

(b) If  $M(x) < \infty$ , then

$$fII(f)(x)^{p^*-1} = \int_x^D \hat{v}(t) \left( \int_0^t (g - g(D))^{p-1} d\mu \right)^{p^*-1} dt < g(0)M(0) < \infty.$$

Replacing  $f$  in the denominator of  $II(f)$  with this term and using Cauchy's mean-value theorem twice, we have

$$\sup_{(0,D)} II(f) \leq \frac{1}{\lambda_p} \sup_{(0,D)} \frac{f^{p-1}}{g^{p-1}} = \frac{1}{\lambda_p} \sup_{x \in (0,D)} \left( 1 - \frac{g(D)}{g(x)} \right)^{p-1} = \frac{1}{\lambda_p} \left( 1 - \frac{g(D)}{g(0)} \right)^{p-1}.$$

The last equality comes from the fact that  $g \downarrow \downarrow$ . If  $g(D) > 0$ , then

$$\lambda_p^{-1} \leq \inf_{f \in \mathcal{F}_II} \sup_{x \in (0,D)} II(f)(x) \leq \sup_{x \in (0,D)} II(f)(x) < \lambda_p^{-1},$$

which is a contradiction. Therefore, we must have  $g(D) = 0$ .  $\square$

By now, we have finished the proof of the lower estimates of  $\lambda_p$ . Dually, one can prove the upper estimates without too much difficulty. We ignore the details here.



The following lemma or its variants have been used many times before (cf. [3; Proof of Theorem 3.1], [2; p. 97], or [7], and the earlier publications therein). It is essentially an application of the integration by parts formula, and is a key to the proof of Theorem 2.3.

**Lemma 3.8** *Assume that  $m$  and  $n$  are two non-negative locally integrable functions. For  $p > 1$ , define*

$$S(x) = \left( \int_x^D n(y) dy \right)^{p-1}, \quad M(x) = \int_0^x m(y) dy,$$

and

$$c_0 = \sup_{x \in (0, D)} S(x)M(x) < \infty.$$

Then

$$\int_0^x m(y) S(y)^{p^*r/p} dy \leq \frac{c_0}{1 - \frac{p^*r}{p}} S(x)^{(p^*r/p)-1}, \quad r \in \left(0, \frac{p}{p^*}\right).$$

*Proof of Theorem 2.3* First, we prove that  $\lambda_p \geq (k(p)\sigma_p)^{-1}$ . Fixing  $r \in (0, p/p^*)$ , let  $f(x) = \hat{v}(x, D)^{p^*r/p}$ . Applying  $m(x) = u(x)$ ,  $n(x) = \hat{v}(x)$  to Lemma 3.8, we have

$$M(x) = \mu(0, x), \quad S(x) = \hat{v}(x, D)^{p-1}, \quad c_0 = \sigma_p,$$

and

$$\int_0^x \hat{v}(y, D)^r \mu(dy) \leq \frac{\sigma_p}{1 - \frac{p^*r}{p}} \hat{v}(x, D)^{r-(p/p^*)}.$$

Since

$$|f'|^{p-2} f' = - \left( \frac{p^*r}{p} \hat{v}(\cdot, D)^{(p^*r/p)-1} \hat{v}(\cdot) \right)^{p-1},$$

we have

$$\sup_{x \in (0, D)} I(f)(x) \leq \frac{[p/(p^*r)]^{p-1}}{1 - \frac{p^*r}{p}} \sigma_p. \quad (11)$$

By Theorem 2.1 (i), (11), and an optimization with respect to  $r \in (0, p/p^*)$ , we obtain

$$\lambda_p^{-1} \leq \left( \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1} \right)^{-1} \leq pp^{*p-1} \sigma_p = k(p)\sigma_p.$$

Now, we prove that  $\lambda_p \leq \sigma_p^{-1}$ . For fixed  $x_0, x_1 \in (0, D)$  with  $x_0 < x_1$ , let

$$f(x) = \hat{v}(x \vee x_0, D) \mathbb{1}_{[0, x_1]}(x).$$

Then

$$I(f)(x) = \hat{v}(x_0, D)^{p-1} \mu(0, x_0) + \int_{x_0}^x \hat{v}(t, D)^{p-1} \mu(dt), \quad x \in (x_0, x_1),$$

and  $I(f)(x) = \infty$  on  $[0, x_0] \cup [x_1, D]$  by convention  $1/0 = \infty$ . Combining with Theorem 2.1 (i), we have

$$\lambda_p^{-1} \geq \inf_{x < x_1} I(f)(x) = \hat{v}(x_0, D)^{p-1} \mu(0, x_0), \quad x_0 < x_1.$$

Thereby the assertion that  $\lambda_p \leq \sigma_p^{-1}$  follows by letting  $x_1 \rightarrow D$ . Since

$$\mu(0, x)^{p^*-1} \hat{v}(x, D) \leq \int_x^D \mu(0, s)^{p^*-1} \hat{v}(s) ds \leq \int_0^D \mu(0, s)^{p^*-1} \hat{v}(s) ds,$$

the assertions hold. □

From the proof above, it is easy to understand why we choose the test function as  $f = \hat{v}[\cdot, D]^{1/p^*}$  in [5; Proof of Theorem 2.3 (a)] in the discrete case.

*Proof of Theorem 2.4* Using Cauchy’s mean-value theorem and definitions of  $\delta'_n, \delta_n, \bar{\delta}_n$ , and  $\lambda_p$ , it is not hard to show the most of the results except that  $\bar{\delta}_{n+1} \geq \delta'_n$ . Put  $f = f_n^{x_0, x_1}$  and  $g = f_{n+1}^{x_0, x_1}$ . Then  $g = fII(f)^{p^*-1}$ . By simple calculation, we have

$$D_p(g) = \int_0^{x_1} |g'|^{p-1} |g'| (x) v(x) dx = \int_0^{x_1} v(x)^{-1} \int_0^x f^{p-1} d\mu |g'(x)| v(x) dx.$$

Exchanging the order of the integrals, we have

$$\begin{aligned} D_p(g) &= - \int_0^{x_1} f^{p-1}(t) \mu(dt) \int_t^{x_1} g'(x) dx \quad (\text{by Fubini's Theorem}) \\ &\leq \int_0^{x_1} f^{p-1}(t) g(t) \mu(dt) \quad (\text{since } g(x_1) \geq 0) \\ &\leq \int_0^{x_1} g^p d\mu \sup_{t \in (0, x_1)} \left( \frac{f(t)}{g(t)} \right)^{p-1} \\ &\leq \mu(|g|^p) \sup_{x \in (0, x_1)} II(f)(x)^{-1}. \end{aligned}$$

So the required assertion holds. □

*Proof of Corollary 2.5* (a) The calculation of  $\delta_1$  is simple. We compute  $\delta'_1$  first. Consider the term  $\inf_{x < x_1} II(f_1^{x_0, x_1})(x)$ . By calculation, we obtain that for  $x \in (x_0, x_1)$ , the numerator of  $(II(f_1^{x_0, x_1})(x))^{p^*-1}|_x$  equals

$$\hat{v}(x) \left[ \int_x^{x_1} \hat{v}(s) \left( \int_0^s (f_1^{x_0, x_1})^{p-1} d\mu \right)^{p^*-1} ds - \hat{v}(x, x_1) \left( \int_0^x (f_1^{x_0, x_1})^{p-1} d\mu \right)^{p^*-1} \right],$$

which is obviously non-negative. So

$$II(f_1^{x_0, x_1})(x) = \left[ \frac{1}{\hat{v}(x, x_1)} \int_x^{x_1} \hat{v}(s) \left( \int_0^s \hat{v}(t \vee x_0, x_1)^{p-1} \mu(dt) \right)^{p^*-1} ds \right]^{p-1}$$

is increasing in  $x \in (x_0, x_1)$ . Hence,

$$\begin{aligned} \delta'_1 &= \sup_{x_0 < x_1} \left[ \frac{1}{\hat{\nu}(x_0, x_1)} \int_{x_0}^{x_1} \hat{\nu}(s) \left( \int_0^s \hat{\nu}(t \vee x_0, x_1)^{p-1} \mu(dt) \right)^{p^*-1} ds \right]^{p-1} \\ &= \sup_{x_0 \in (0, D)} \frac{1}{\hat{\nu}(x_0, D)^{p-1}} \left[ \int_{x_0}^D \hat{\nu}(s) \left( \int_0^s \hat{\nu}(t \vee x_0, D)^{p-1} \mu(dt) \right)^{p^*-1} ds \right]^{p-1}. \end{aligned}$$

In the last equality, we have used the fact that  $H(f_1^{x_0, x_1})(x_0)$  is increasing in  $x_1 \in [x_0, D]$ . Indeed, let

$$N_k(s, y) = \int_{x_0}^s \hat{\nu}(t, y)^k \mu(dt), \quad f(s, y) = \hat{\nu}(s) N_{p-1}(s, y)^{p^*-1}.$$

Then

$$\begin{aligned} H(f_1^{(x_0, y)})(x_0)^{p^*-1} &= \frac{1}{\hat{\nu}(x_0, y)} \left[ \int_{x_0}^y f(s, y) ds + \int_{x_0}^y \hat{\nu}(s) ds \int_0^{x_0} \hat{\nu}(x_0, y)^{p-1} \mu(dt) \right] \\ &= \frac{1}{\hat{\nu}(x_0, y)} \int_{x_0}^y f(s, y) ds + \mu(0, x_0) \hat{\nu}(x_0, y)^{p-1} \\ &=: H_1(y) + H_2(y), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial y} N_{p-1}(s, y) &= \int_{x_0}^s (p-1) \hat{\nu}(t, y)^{p-2} \hat{\nu}(y) \mu(dt) = (p-1) \hat{\nu}(y) N_{p-2}(s, y), \\ \frac{\partial}{\partial y} f(s, y) &= (p^* - 1) \hat{\nu}(s) N_{p-1}(s, y)^{p^*-2} \frac{\partial}{\partial y} N_{p-1}(s, y), \\ \frac{\partial}{\partial y} \int_{x_0}^y f(s, y) ds &= \int_{x_0}^y \frac{\partial}{\partial y} f(s, y) ds + f(y, y). \end{aligned}$$

Hence, the numerator of  $dH_1/dy$  equals

$$\begin{aligned} &\left( \frac{\partial}{\partial y} \int_{x_0}^y f(s, y) ds \right) \hat{\nu}(x_0, y) - \hat{\nu}(y) \int_{x_0}^y f(s, y) ds \\ &= \hat{\nu}(x_0, y) \hat{\nu}(y) \int_{x_0}^y \hat{\nu}(s) N_{p-1}(s, y)^{p^*-2} N_{p-2}(s, y) ds \\ &\quad + \hat{\nu}(x_0, y) f(y, y) - \hat{\nu}(y) \int_{x_0}^y f(s, y) ds \\ &= \hat{\nu}(y) \left( \hat{\nu}(x_0, y) \int_{x_0}^y \hat{\nu}(s) N_{p-1}(s, y)^{p^*-2} N_{p-2}(s, y) ds \right. \\ &\quad \left. - \int_{x_0}^y \hat{\nu}(s) N_{p-1}(s, y)^{p^*-1} ds \right) + \hat{\nu}(x_0, y) f(y, y). \end{aligned}$$

Since

$$\hat{\nu}(x_0, y) N_{p-2}(s, y) - N_{p-1}(s, y) \geq 0, \quad s \in [x_0, y],$$

we see that  $dH_1/dy$  is positive. It is obvious that  $dH_2/dy$  is positive. So  $\Pi(f_1^{x_0,y})(x_0)$  is increasing in  $y$  and the required assertion holds.

(b) Compute  $\bar{\delta}_1$ . By definition of  $\bar{\delta}_1$ , we have

$$\begin{aligned} \|f_1^{x_0,x_1}\|_p^p &= \int_0^{x_1} \left( \int_{x_0 \vee x}^{x_1} \hat{v}(s) ds \right)^p \mu(dx) \\ &= \mu(0, x_0) \hat{v}(x_0, x_1)^p + \int_{x_0}^{x_1} \left( \int_x^{x_1} \hat{v}(t) dt \right)^p \mu(dx), \\ D_p(f_1^{x_0,x_1}) &= \int_{x_0}^{x_1} \hat{v}(t)^p v(t) dt = \hat{v}(x_0, x_1). \end{aligned}$$

Hence,

$$\begin{aligned} \bar{\delta}_1 &= \sup_{x_0 < x_1} \left( \mu(0, x_0) \hat{v}(x_0, x_1)^{p-1} + \frac{1}{\hat{v}(x_0, x_1)} \int_{x_0}^{x_1} \hat{v}(s, x_1)^p \mu(ds) \right) \\ &= \sup_{x_0 \in (0, D)} \left( \mu(0, x_0) \hat{v}(x_0, D)^{p-1} + \frac{1}{\hat{v}(x_0, D)} \int_{x_0}^D \hat{v}(s, D)^p \mu(ds) \right). \end{aligned}$$

In the second equality, we have used the fact that

$$\mu(0, x_0) \hat{v}(x_0, x_1)^{p-1} + \frac{1}{\hat{v}(x_0, x_1)} \int_{x_0}^{x_1} \hat{v}(s, x_1)^p \mu(ds) \uparrow \quad \text{in } x_1.$$

Indeed, it suffices to show that

$$\frac{1}{\hat{v}(x_0, x)} \int_{x_0}^x \hat{v}(s, x)^p \mu(ds) \leq \frac{1}{\hat{v}(x_0, y)} \int_{x_0}^y \hat{v}(s, y)^p \mu(ds), \quad x_0 \leq x < y,$$

which is equivalent to

$$\frac{1}{\hat{v}(x_0, y)} \int_x^y \hat{v}(s, y)^p \mu(ds) + \int_{x_0}^x \left( \frac{\hat{v}(s, y)^p}{\hat{v}(x_0, y)} - \frac{\hat{v}(s, x)^p}{\hat{v}(x_0, x)} \right) \mu(ds) \geq 0.$$

Since  $p > 1$  and  $\hat{v}(t, x) \leq \hat{v}(x_0, x)$  for  $x \geq t \geq x_0$ , we have

$$\frac{\hat{v}(t, y)^p}{\hat{v}(t, x)^p} = \left[ \frac{\hat{v}(t, x) + \hat{v}(x, y)}{\hat{v}(t, x)} \right]^p \geq 1 + \frac{\hat{v}(x, y)}{\hat{v}(t, x)} \geq 1 + \frac{\hat{v}(x, y)}{\hat{v}(x_0, x)} = \frac{\hat{v}(x_0, y)}{\hat{v}(x_0, x)}$$

for  $t \geq x_0$  and the required assertion holds.

(c) Comparing  $\delta'_1$  and  $\bar{\delta}_1$ . It is easy to see that

$$\begin{aligned} \int_x^D \hat{v}(\cdot, D)^p d\mu &= \int_x^D \hat{v}(t, D)^{p-1} \int_t^D \hat{v}(s) ds \mu(dt) \\ &= \int_x^D \hat{v}(s) \int_x^s \hat{v}(t, D)^{p-1} \mu(dt) ds, \end{aligned}$$

$$\mu(0, x)\hat{\nu}(x, D)^p = \int_x^D \hat{\nu}(s) \int_0^x \hat{\nu}(x, D)^{p-1} \mu(dt) ds.$$

Let  $a_x(s) = \hat{\nu}(s)/\hat{\nu}(x, D)$  for  $s \in (x, D)$ . Noticing that  $a_x$  is a probability on  $(x, D)$ , by the increasing property of moments  $\mathbb{E}(|X|^s)^{1/s}$  in  $s > 0$  and combining the preceding assertions (a) and (b), we have

$$\begin{aligned} \bar{\delta}_1 &= \sup_{x \in (0, D)} \int_x^D a_x(s) \int_0^s \hat{\nu}(t \vee x, D)^{p-1} \mu(dt) ds \\ &\leq \sup_{x \in (0, D)} \left[ \int_x^D a_x(s) \left( \int_0^s \hat{\nu}(t \vee x, D)^{p-1} \mu(dt) \right)^{p^*-1} ds \right]^{p-1} \quad (\text{if } p^* - 1 > 1) \\ &= \delta'_1. \end{aligned}$$

Similarly, if  $p^* - 1 < 1$  (i.e.,  $p > 2$ ), then  $\bar{\delta}_1 \geq \delta'_1$ .

(d) Prove that  $\bar{\delta}_1 \leq p\sigma_p$ . Using the integration by parts formula, we have

$$\begin{aligned} \int_{x_0}^x \hat{\nu}(y, D)^p \mu(dy) &= \hat{\nu}(y, D)^p \mu(0, y)|_{x_0}^x + p \int_{x_0}^x \hat{\nu}(y, D)^{p-1} \hat{\nu}(y) \mu(0, y) dy \\ &\leq \sigma_p \hat{\nu}(x, D) - \hat{\nu}(x_0, D)^p \mu(0, x_0) + p\sigma_p \int_{x_0}^x \hat{\nu}(y) dy. \end{aligned}$$

Since  $\hat{\nu}(x, D) < \infty$ , letting  $x \rightarrow D$ , we have

$$\begin{aligned} \bar{\delta}_1 &= \sup_{x_0 \in (0, D)} \left( \mu(0, x_0) \hat{\nu}(x_0, D)^{p-1} + \frac{1}{\hat{\nu}(x_0, D)} \int_{x_0}^D \hat{\nu}(\cdot, D)^p d\mu \right) \\ &\leq \sup_{x_0 \in (0, D)} \left[ \mu(0, x_0) \hat{\nu}(x_0, D)^{p-1} \right. \\ &\quad \left. + \frac{1}{\hat{\nu}(x_0, D)} \left( -\hat{\nu}(x_0, D)^p \mu(0, x_0) + p\sigma_p \int_{x_0}^D \hat{\nu}(y) dy \right) \right] \\ &= p\sigma_p, \end{aligned}$$

and the required assertion holds. □

#### 4 DN-case

From now on, we concern on  $p$ -Laplacian eigenvalue with DN-boundaries. We use the same notation as the previous ND-case since they play the similar role but have different meaning in different context. Let  $D \leq \infty$ ,  $p > 1$ . The  $p$ -Laplacian eigenvalue problem with DN-boundary conditions is

$$\begin{cases} \text{Eigenequation:} & L_p g(x) = -\lambda u(x) |g|^{p-2} g(x); \\ \text{DN-boundaries:} & g(0) = 0, \quad g'(D) = 0 \text{ if } D < \infty. \end{cases} \quad (12)$$

The first eigenvalue  $\lambda_p$  has the following classical variational formula:

$$\lambda_p = \inf \left\{ \frac{D_p(f)}{\mu(|f|^p)} : f(0) = 0, f \neq 0, f \in \mathcal{C}[0, D], \right. \\ \left. v^{p^*-1} f' \in \mathcal{C}(0, D), D_p(f) < \infty \right\}. \tag{13}$$

Correspondingly, we are also estimating the optimal constant  $A := \lambda_p^{-1}$  in the *weighted Hardy inequality*:

$$\mu(|f|^p) \leq AD_p(f), \quad f(0) = 0, \quad f \in \mathcal{D}(D_p).$$

For  $p > 1$ , define  $\hat{v} = v^{1-p^*}$  and  $\hat{\nu}(dx) = \hat{v}(x)dx$ . We use the following operators:

$$I(f)(x) = \frac{1}{(vf'|f'|^{p-2})(x)} \int_x^D f^{p-1} d\mu \quad (\text{single integral form}),$$

$$II(f)(x) = \frac{1}{f^{p-1}(x)} \left[ \int_0^x \hat{v}(s) \left( \int_s^D f^{p-1} d\mu \right)^{p^*-1} ds \right]^{p-1} \\ (\text{double integral form}),$$

$$R(h)(x) = -u^{-1} \{ |h|^{p-2} [v'h + (p-1)v(h^2 + h')] \}(x) \quad (\text{differential form}).$$

The three operators above have domains, respectively, as follows:

$$\mathcal{F}_I = \{ f \in \mathcal{C}[0, D] : v^{p^*-1} f' \in \mathcal{C}(0, D), f(0) = 0 \text{ and } f'|_{(0,D)} > 0 \}, \\ \mathcal{F}_{II} = \{ f : f \in \mathcal{C}[0, D], f(0) = 0 \text{ and } f|_{(0,D)} > 0 \}, \\ \mathcal{H} = \left\{ h : h \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], h|_{(0,D)} > 0 \text{ and } \int_{0+} h(u)du = \infty \right\},$$

where  $\int_{0+}$  means  $\int_0^\varepsilon$  for sufficiently small  $\varepsilon > 0$ . Some modifications are needed when studying the upper estimates:

$$\widetilde{\mathcal{F}}_I = \{ f \in \mathcal{C}[0, x_0] : f(0) = 0, v^{p^*-1} f' \in \mathcal{C}(0, x_0), f'|_{(0,x_0)} > 0 \\ \text{for some } x_0 \in (0, D), \text{ and } f = f(\cdot \wedge x_0) \}, \\ \widetilde{\mathcal{F}}_{II} = \{ f : f(0) = 0, \exists x_0 \in (0, D) \text{ such that } f = f(\cdot \wedge x_0) > 0 \\ \text{and } f \in \mathcal{C}[0, x_0] \}, \\ \widetilde{\mathcal{H}} = \left\{ h : \exists x_0 \in (0, D) \text{ such that } h \in \mathcal{C}[0, x_0] \cap \mathcal{C}^1(0, x_0), h|_{(0,x_0)} > 0, \right. \\ \left. h|_{[x_0,D]} = 0, \int_{0+} h(u)du = \infty, \text{ and } \sup_{(0,x_0)} [v'h + (p-1)(h^2 + h')v] < 0 \right\}.$$

When  $D = \infty$ , replace  $[0, D]$  and  $(0, D]$  with  $[0, D)$  and  $(0, D)$ , respectively. Besides, we also need the following notation:

$$\widetilde{\mathcal{F}}'_{II} = \{ f : f(0) = 0, f \in \mathcal{C}[0, D] \text{ and } fII(f) \in L^p(\mu) \}.$$

If  $\mu(0, D) = \infty$ , then  $\lambda_p$  defined by (13) is trivial. Indeed, let

$$f = \mathbb{1}_{(\delta, D]} + h\mathbb{1}_{[0, \delta]},$$

where  $h$  is chosen such that  $h(0) = 0$  and  $f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]$  (for example,  $h(x) = -x^2 \cdot \delta^{-2} + 2x \cdot \delta^{-1}$ ). Then  $D_p(f) \in (0, \infty)$  and  $\mu(|f|^p) = \infty$ . It follows that  $\lambda_p = 0$ .

Otherwise,  $\mu(0, D) < \infty$ . Then for every  $f$  with  $\mu(|f|^p) = \infty$ , by setting

$$f^{(x_0)} = f(\cdot \wedge x_0) \in L^p(\mu),$$

we have

$$\infty > D(f^{(x_0)}) \rightarrow D(f), \quad \infty > \mu(|f^{(x_0)}|^p) \rightarrow \mu(|f|^p) \quad \text{as } x_0 \rightarrow D.$$

In other words, for  $f \notin L^p(\mu)$ , both  $\mu(|f|^p)$  and  $D_p(f)$  can be approximated by a sequence of functions belonging to  $L^p(\mu)$ . Hence, we can rewrite  $\lambda_p$  as follows:

$$\lambda_p = \inf\{D_p(f) : \mu(|f|^p) = 1, f(0) = 0, \text{ and } f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]\}. \quad (14)$$

In this case, we also have

$$\lambda_p = \inf\{D_p(f) : \mu(|f|^p) = 1, f(0) = 0, f = f(\cdot \wedge x_0), \\ f \in \mathcal{C}^1(0, x_0) \cap \mathcal{C}[0, x_0] \text{ for some } x_0 \in (0, D)\}.$$

We are now ready to state the main results in the present context.

**Theorem 4.1** *Assume that  $\mu(0, D) < \infty$ . For  $p > 1$ , the following variational formulas hold for  $\lambda_p$  defined by (14) (equivalently, (13)):*

(i) *single integral forms:*

$$\inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0, D)} I(f)(x)^{-1} = \lambda_p = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1};$$

(ii) *double integral forms:*

$$\lambda_p = \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0, D)} II(f)(x)^{-1} = \inf_{f \in \tilde{\mathcal{F}}_{II} \cup \tilde{\mathcal{F}}'_{II}} \sup_{x \in (0, D)} II(f)(x)^{-1}, \\ \lambda_p = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} II(f)(x)^{-1} = \sup_{f \in \mathcal{F}_{II}} \inf_{x \in (0, D)} II(f)(x)^{-1}.$$

Moreover, if  $u$  and  $v'$  are continuous, then we have additionally

(iii) *differential forms:*

$$\inf_{h \in \tilde{\mathcal{H}}} \sup_{x \in (0, D)} R(h)(x) = \lambda_p = \sup_{h \in \mathcal{H}} \inf_{x \in (0, D)} R(h)(x).$$

Define

$$k(p) = pp^{*p-1}, \quad \sigma_p = \sup_{x \in (0, D)} \mu(x, D)\hat{\nu}(0, x)^{p-1}.$$

As an application of the variational formulas in Theorem 4.1 (i), we have the following theorem which was also known in 1990s (cf. [11; Lemmas 3.2, 3.4]).

**Theorem 4.2** (Criterion and basic estimates) *For  $p > 1$ ,  $\lambda_p > 0$  if and only if  $\sigma_p < \infty$ . Moreover,*

$$(k(p)\sigma_p)^{-1} \leq \lambda_p \leq \sigma_p^{-1}.$$

*In particular, we have  $\lambda_p = 0$  if  $\mu(0, D) = \infty$  and  $\lambda_p > 0$  if*

$$\int_0^D \mu(s, D)^{p^*-1} \hat{\nu}(ds) < \infty.$$

The next result is an application of the variational formulas in Theorem 4.1 (ii).

**Theorem 4.3** (Approximating procedure) *Assume that  $\mu(0, D) < \infty$  and  $\sigma_p < \infty$ .*

(i) *Let*

$$f_1 = \hat{\nu}(0, \cdot)^{1/p^*}, \quad f_{n+1} = f_n \Pi(f_n)^{p^*-1}, \quad \delta_n = \sup_{x \in (0, D)} \Pi(f_n)(x), \quad n \geq 1.$$

*Then  $\delta_n$  is decreasing in  $n$  and*

$$\lambda_p \geq \delta_n^{-1} \geq (k(p)\sigma_p)^{-1}.$$

(ii) *For fixed  $x_0 \in (0, D)$ , let*

$$f_1^{(x_0)} = \hat{\nu}(0, \cdot \wedge x_0), \quad f_n^{(x_0)} = f_{n-1}^{(x_0)} \Pi(f_{n-1}^{(x_0)})(\cdot \wedge x_0)^{p^*-1},$$

$$\delta'_n = \sup_{x_0 \in (0, D)} \inf_{x \in (0, D)} \Pi(f_n^{(x_0)})(x), \quad n \geq 1.$$

*Then  $\delta'_n$  is increasing in  $n$  and*

$$\sigma_p^{-1} \geq \delta'_n \geq \lambda_p.$$

*Moreover, define*

$$\bar{\delta}_n = \sup_{x_0 \in (0, D)} \frac{\|f_n^{(x_0)}\|_p^p}{D_p(f_n^{(x_0)})}, \quad n \geq 1.$$

*Then*

$$\bar{\delta}_n^{-1} \geq \lambda_p, \quad \bar{\delta}_{n+1} \geq \delta'_n, \quad n \geq 1.$$

Most of the results in Corollary 4.4 below can be obtained directly from Theorem 4.3.

**Corollary 4.4** (Improved estimates) *Assume that  $\mu(0, D) < \infty$  and  $\lambda_p > 0$ . We have*

$$\sigma_p^{-1} \geq \delta_1^{-1} \geq \lambda_p \geq \delta_1^{-1} \geq (k(p)\sigma_p)^{-1},$$



where

$$\delta_1 = \sup_{x \in (0, D)} \left[ \frac{1}{\hat{\nu}(0, x)^{1/p^*}} \int_0^x \hat{\nu}(s) \left( \int_s^D \hat{\nu}(0, t)^{p/p^*2} \mu(dt) \right)^{p^*-1} ds \right]^{p-1},$$

$$\delta'_1 = \sup_{x \in (0, D)} \frac{1}{\hat{\nu}(0, x)^{p-1}} \left[ \int_0^x \hat{\nu}(s) \left( \int_s^D \hat{\nu}(0, t \wedge x)^{p-1} \mu(dt) \right)^{p^*-1} ds \right]^{p-1}.$$

Moreover,

$$\bar{\delta}_1 = \sup_{x \in (0, D)} \left( \mu(x, D) \hat{\nu}(0, x)^{p-1} + \frac{1}{\hat{\nu}(0, x)} \int_0^x \hat{\nu}(0, t)^p \mu(dt) \right) \in [\sigma_p, p\sigma_p],$$

$\bar{\delta}_1 \geq \delta_1$  for  $p \geq 2$  and  $\bar{\delta}_1 \leq \delta'_1$  for  $1 < p \leq 2$ .

When  $p = 2$ , the equality  $\delta_1 = \bar{\delta}_1$  was proved in [4; Theorem 6].

Most of the results in this section are parallel to that in Section 2. One may follow Section 3 or [4,7] to complete the proofs without too many difficulties. The details are omitted here. Instead, we prove some properties of the eigenfunction  $g$ , which are used in choosing the test functions for the operators.

**Lemma 4.5** *Let  $(\lambda_p, g)$  be a solution to (12),  $g \neq 0$ . Then  $g'$  does not change sign, and so does  $g$ .*

*Proof* First, the solution provided by Lemma 3.1 is trivial:  $g = 0$ , if the given constants  $A$  and  $B$  are zero. Because we are in the situation that  $g(0) = 0$ , we can assume that  $g'(0) \neq 0$ . Next, we prove that  $g'$  does not change sign by seeking a contradiction. If there exists  $x_0 \in (0, D)$  such that  $g'(x_0) = 0$ , then  $g(x_0) \neq 0$  by [12; Lemma 2.3]. Let

$$\bar{g} = g \mathbb{1}_{[0, x_0]} + g(x_0) \mathbb{1}_{(x_0, D]}.$$

By simple calculation, we obtain

$$D_p(\bar{g}) = (-L_p \bar{g}, \bar{g})_\mu = \lambda_p \mu_{0, x_0}(|g|^p).$$

So

$$\lambda_p \leq \frac{D_p(\bar{g})}{\mu(|\bar{g}|^p)} = \frac{\lambda_p \mu_{0, x_0}(|g|^p)}{\mu_{0, x_0}(|g|^p) + \mu(x_0, D)|g(x_0)|^p} < \lambda_p,$$

which is a contradiction. Therefore,  $g'$  does not change sign. Since  $g(0) = 0$ , the second assertion holds naturally.  $\square$

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