Front. Math. China 2014, 9(6): 1261–1292
DOI 10.1007/s11464-014-0374-6

RESEARCH ARTICLE

Mixed eigenvalues of discrete p-Laplacian

Mu-Fa CHEN[1], **Lingdi WANG**[1,2], **Yuhui ZHANG**[1]

1 School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China
2 School of Mathematics and Information Sciences, Henan University, Kaifeng 475004, China

© Higher Education Press and Springer-Verlag Berlin Heidelberg 2014

Abstract This paper deals with the principal eigenvalue of discrete p-Laplacian on the set of nonnegative integers. Alternatively, it is studying the optimal constant of a class of weighted Hardy inequalities. The main goal is the quantitative estimates of the eigenvalue. The paper begins with the case having reflecting boundary at origin and absorbing boundary at infinity. Several variational formulas are presented in different formulation: the difference form, the single summation form, and the double summation form. As their applications, some explicit lower and upper estimates, a criterion for positivity (which was known years ago), as well as an approximating procedure for the eigenvalue are obtained. Similarly, the dual case having absorbing boundary at origin and reflecting boundary at infinity is also studied. Two examples are presented at the end of Section 2 to illustrate the value of the investigation.

Keywords Discrete p-Laplacian, mixed eigenvalue, variational formula, explicit estimate, positivity criterion, approximating procedure
MSC 60J60, 34L15

1 Introduction

In the past years, we have been interested in various aspects of stability speed, such as exponentially ergodic rate, exponential decay rate, algebraic convergence speed, exponential convergence speeds. The convergence speeds are often described by principal eigenvalues or the optimal constants in different types of inequalities. Having a great effort on the L^2-case (refer to [1–4] and references therein), we now come to a more general setup, studying the nonlinear p-Laplacian, especially on the discrete space $E := \{0, 1, \cdots, N\}$ $(N \leqslant +\infty)$ in this paper. This is a typical topic in harmonic analysis (cf. [8]). The method adopted in this paper is analytic rather than probabilistic. Let us presume that $N < +\infty$ for a moment. Following the classification given in

Received August 13, 2013; accepted March 3, 2014
Corresponding author: Lingdi WANG, E-mail: wanglingdi@mail.bnu.edu.cn

[4,6], where $p=2$ was treated, we have four types of boundary conditions: DD, DN, ND, and NN, according to Dirichlet (code 'D') or Neumann (code 'N') boundary at each of the endpoints. For instance, the Neumann condition at the right endpoint means that $f_{N+1}=f_N$. For Dirichlet condition, it means that $f_{N+1}=0$. In the continuous context, the DN-case was partially studied in [7] by Jin and Mao. For the NN- and DD-cases, one may refer to [5]. Based on [4,6], here, we study the mixed eigenvalues (i.e., the ND- and DN-cases) of discrete p-Laplacian. Certainly, the above classification of the boundaries for p-Laplacian remains meaningful even if $N=+\infty$.

This paper is organized as follows. In Section 2, we study the ND-case. First, we introduce three groups of variational formulas for the eigenvalue. As a consequence, we obtain the basic estimates (i.e., the ratio of the upper and the lower bounds is a constant) of the eigenvalue. Furthermore, an approximating procedure and some improved estimates are presented. Except the basic estimates, when $p\neq 2$, the other results seem to be new. To illustrate the power of our main results, two examples are included at the end of Section 2. Usually, the nonlinear case (here, it means $p\neq 2$) is much harder than the linear one ($p=2$). We are lucky in the present situation since most of ideas developed in [4] are still suitable in the present general setup. This saves us a lot of spaces. Thus, we do not need to publish all details, but emphasize some key points and the difference to [4]. The sketched proofs are presented in Section 3. In Section 4, the corresponding results for the DN-case are presented.

2 ND-case

Throughout the paper, denoted by $\mathscr{C}_K$ the set of functions having compact support. In this section, let $E=\{i\colon 0\leqslant i<N+1\}$ ($N\leqslant+\infty$). The discrete p-Laplacian is defined as follows:

$$\Omega_p f(k)=\nu_k|f_k-f_{k+1}|^{p-2}(f_{k+1}-f_k)-\nu_{k-1}|f_{k-1}-f_k|^{p-2}(f_k-f_{k-1}),\quad p>1,$$

where $\{\nu_k\colon k\in E\}$ is a positive sequence with boundary condition $\nu_{-1}=0$ (and $f_{-1}=f_0$). Alternatively, we may rewrite Ω_p as

$$\Omega_p f(k)=\nu_k|f_k-f_{k+1}|^{p-1}\mathrm{sgn}(f_{k+1}-f_k)-\nu_{k-1}|f_{k-1}-f_k|^{p-1}\mathrm{sgn}(f_k-f_{k-1}),$$

especially when $p\in(1,2)$. Then we have the following discrete version of the p-Laplacian eigenvalue problem with ND-boundary conditions:

$$\text{'Eigenequation'}\colon\ \Omega_p g(k)=-\lambda\mu_k|g_k|^{p-2}g_k,\quad k\in E; \tag{1}$$

$$\text{ND-boundary conditions: } 0\neq g_0=g_{-1}\text{ and } g_{N+1}=0 \text{ if } N<+\infty. \tag{2}$$

If (λ,g) is a solution to the eigenvalue problem, then λ is called an 'eigenvalue' and g is its eigenfunction. Especially, when $p=2$, the first (or principal) eigenvalue corresponds to the exponential decay rate for birth-death process on half line, where $\{\mu_k\}$ is just the invariant measure of the birth-death process and $\{\nu_k\}$ is a quantity related to the recurrence criterion of the process ([4; Sections 2 and 3]).

Define

$$D_p(f) = \sum_{k\in E} \nu_k |f_k - f_{k+1}|^p, \quad p \geqslant 1,\ f \in \mathscr{C}_K,$$

and the ordinary inner product

$$(f, g) = \sum_{k\in E} f_k g_k.$$

Then we have

$$D_p(f) = (-\Omega_p f, f).$$

Actually,

$$\begin{aligned}(-\Omega_p f, f) = &\sum_{k=0}^{N} \nu_k f_k |f_k - f_{k+1}|^{p-2}(f_k - f_{k+1}) \\ &+ \sum_{k=0}^{N} \nu_{k-1} f_k |f_k - f_{k-1}|^{p-2}(f_k - f_{k-1}).\end{aligned}$$

Since $\nu_{-1} = 0$, one may rewrite the second term as $\sum_{k=1}^{N}$ and then as $\sum_{k=0}^{N-1}$ by a change of the index. Combining the resulting sum with the first one, we get

$$\begin{aligned}(-\Omega_p f, f) &= \sum_{k=0}^{N-1} \nu_k |f_k - f_{k+1}|^{p-2}(f_k - f_{k+1})^2 + \nu_N |f_N|^p \\ &= \sum_{k\in E} \nu_k |f_k - f_{k+1}|^p \quad (\text{since } f_{N+1} = 0).\end{aligned}$$

In this section, we are interested in the principal eigenvalue defined by the following classical variational formula:

$$\lambda_p = \inf\{D_p(f)\colon \mu(|f|^p) = 1,\ f \in \mathscr{C}_K\}, \tag{3}$$

where $\mu(f) = \sum_{k\in E} \mu_k f_k$. We mention that the Neumann boundary at left endpoint is described by $f_0 = f_{-1}$ or $\nu_{-1} = 0$. The Dirichlet boundary condition at right endpoint is described by $f_{N+1} = 0$ if $N < +\infty$. Actually, the condition also holds even if $N = +\infty$ ($f_N := \lim_{i\to N} f_i$ provided $N = +\infty$) as will be proved in Proposition 3.4 below. Formula (3) can be rewritten as the following weighted Hardy inequality:

$$\mu(|f|^p) \leqslant A D_p(f), \quad f \in \mathscr{C}_K,$$

with optimal constant $A = \lambda_p^{-1}$. This explains the relationship between the p-Laplacian eigenvalue and the Hardy's inequality. Throughout this paper, we concentrate on $p \in (1, +\infty)$ since the degenerated cases that $p = 1$ or $+\infty$ are often easier (cf. [11; Lemmas 5.4, 5.6]).

2.1 Main results

To state our main results, we need some notations. For $p > 1$, let p^* be its conjugate number (i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$). Define $\hat{\nu}_j = \nu_j^{1-p^*}$ and three operators which are parallel to those introduced in [4], as follows:

$$I_i(f) = \frac{1}{\nu_i (f_i - f_{i+1})^{p-1}} \sum_{j=0}^{i} \mu_j f_j^{p-1} \quad \text{(single summation form)},$$

$$II_i(f) = \frac{1}{f_i^{p-1}} \left[\sum_{j \in \mathrm{supp}(f) \cap [i,N]} \hat{\nu}_j \left(\sum_{k=0}^{j} \mu_k f_k^{p-1} \right)^{p^*-1} \right]^{p-1}$$

$$\text{(double summation form)},$$

$$R_i(w) = \mu_i^{-1} [\nu_i (1 - w_i)^{p-1} - \nu_{i-1} (w_{i-1}^{-1} - 1)^{p-1}] \quad \text{(difference form)}.$$

We make a convention that $w_{-1} > 0$ is free and $w_N = 0$ if $N < +\infty$. For the lower estimates to be studied below, their domains are defined, respectively, as follows:

$$\mathscr{F}_I = \{f \colon f > 0 \text{ and } f \text{ is strictly decreasing}\},$$
$$\mathscr{F}_{II} = \{f \colon f > 0 \text{ on } E\},$$
$$\mathscr{W} = \left\{ w \colon w_i \in (0,1) \text{ if } \sum_{j \in E} \hat{\nu}_j < +\infty \text{ and } w_i \in (0,1] \text{ if } \sum_{j \in E} \hat{\nu}_j = +\infty \right\}.$$

For the upper estimates, some modifications are needed to avoid the non-summable problem:

$$\widetilde{\mathscr{F}}_I = \{f \colon f \text{ is strictly decreasing on some } [n,m], \; 0 \leqslant n < m < N+1,\; f_\cdot = f_{\cdot \vee n} \mathbb{1}_{\cdot \leqslant m}\},$$
$$\widetilde{\mathscr{F}}_{II} = \{f \colon f_i > 0 \text{ up to some } m \in [1, N+1) \text{ and then vanishes}\},$$
$$\widetilde{\mathscr{W}} = \{w \colon \exists\, m \in [1, N+1) \text{ such that } w_i > 0 \text{ up to } m-1,\; w_m = 0,\; w_i < 1 - (\nu_{i-1}/\nu_i)^{p-1} (w_{i-1}^{-1} - 1) \text{ for } i = 0, 1, \ldots, m\}.$$

In some extent, these functions are imitated of eigenfunction corresponding to λ_p. Each part of Theorem 2.1 below plays a different role in our study. Operator I is used to deduce the basic estimates (Theorem 2.3) and operator II is a tool to produce our approximating procedure (Theorem 2.4). In comparing with these two operators, the operator R is easier in the computation. Noting that for each $f \in \mathscr{F}_I$, the term $\inf_{i \in E} I_i(f)^{-1}$ given in part (i) below is a lower bound of λ_p, it indicates that the formulas on the right-hand side of each term in Theorem 2.1 are mainly used for the lower estimates. Similarly, the formulas on the left-hand side are used for the upper estimates.

Theorem 2.1 *For λ_p $(p > 1)$, we have*

(i) *single summation forms:*

$$\inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{i\in E} I_i(f)^{-1} = \lambda_p = \sup_{f\in\mathscr{F}_I}\inf_{i\in E} I_i(f)^{-1};$$

(ii) *double summation forms:*

$$\inf_{f\in\widetilde{\mathscr{F}}_{II}}\sup_{i\in\mathrm{supp}(f)} II_i(f)^{-1} = \lambda_p = \sup_{f\in\mathscr{F}_{II}}\inf_{i\in E} II_i(f)^{-1};$$

(iii) *difference forms:*

$$\inf_{w\in\widetilde{\mathscr{W}}}\sup_{i\in E} R_i(w) = \lambda_p = \sup_{w\in\mathscr{W}}\inf_{i\in E} R_i(w).$$

Moreover, the supremum on the right-hand sides of the three above formulas can be attained.

The next proposition adds some additional sets of test functions for operators I and II. For simplicity, in what follows, we use $\downarrow$ (resp. $\downarrow\downarrow$) to denote decreasing (resp. strictly decreasing). In parallel, we also use the notation $\uparrow$ and $\uparrow\uparrow$.

Proposition 2.2 *For λ_p $(p > 1)$, we have*

$$\inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{i\in\mathrm{supp}(f)} II_i(f)^{-1} = \lambda_p = \sup_{f\in\mathscr{F}_I}\inf_{i\in E} II_i(f)^{-1},$$
$$\lambda_p = \inf_{f\in\widetilde{\mathscr{F}}'_{II}\cup\widetilde{\mathscr{F}}_{II}}\sup_{i\in\mathrm{supp}(f)} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}'_I}\sup_{i\in E} I_i(f)^{-1},$$

where

$$\widetilde{\mathscr{F}}'_I = \{f\colon f\downarrow\downarrow,\ f \text{ is positive up to some } m\in[1,N+1), \text{ then vanishes}\}\subset\widetilde{\mathscr{F}_I},$$

$$\widetilde{\mathscr{F}}'_{II} = \{f\colon f>0 \text{ and } f\,II(f)^{p^*-1}\in L^p(\mu)\}.$$

Throughout the paper, we write $\widetilde{\mu}[m,n] = \sum_{j=m}^{n}\widetilde{\mu}_j$ for a measure $\widetilde{\mu}$ and define $k(p) = pp^{*p-1}$ (Fig. 1). Next, define

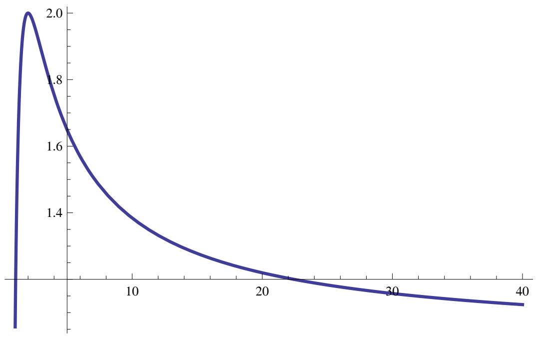

Fig. 1 Function $p\to k(p)^{1/p}$ is unimodal with maximum 2 at $p=2$

$$\sigma_p = \sup_{n\in E}(\mu[0,n]\,\hat\nu[n,N]^{p-1}).$$

Applying $f=\hat\nu[\cdot,D]^{r(p-1)}$ ($r=1/2$ or 1) to Theorem 2.1 (i), we obtain the basic estimates given in Theorem 2.3 below. This result was known in 1990s (cf. [8; Theorem 7]). See also [10].

Theorem 2.3 (Basic estimates) *For $p>1$, we have $\lambda_p>0$ if and only if $\sigma_p<+\infty$. More precisely,*

$$(k(p)\sigma_p)^{-1}\leqslant\lambda_p\leqslant\sigma_p^{-1}.$$

In particular, when $N=+\infty$, we have

$$\lambda_p\begin{cases}=0, & \hat\nu[1,+\infty)=+\infty,\\ >0, & \displaystyle\sum_{k=0}^{+\infty}\hat\nu_k\mu[0,k]^{p^*-1}<+\infty.\end{cases}$$

As an application of variational formulas in Theorem 2.1 (ii), we obtain an approximating procedure in the next theorem. This approach can improve the above basic estimates step by step. Noticing that λ_p is trivial once $\sigma_p=+\infty$ by Theorem 2.3, we may assume that $\sigma_p<+\infty$ in the study on the approximating procedure.

Theorem 2.4 (Approximating procedure) *Assume that $\sigma_p<+\infty$. Let $p>1$.*

(i) *When $\hat\nu[0,N]<+\infty$, define*

$$f_1=\hat\nu[\cdot,N]^{1/p^*},\quad f_n=f_{n-1}I\!I(f_{n-1})^{p^*-1}\ (n\geqslant 2),\quad \delta_n=\sup_{i\in E}I\!I_i(f_n).$$

Otherwise, define $\delta_n=+\infty$. Then δ_n is decreasing in n (denote its limit by δ_∞) and

$$\lambda_p\geqslant\delta_\infty^{-1}\geqslant\cdots\geqslant\delta_1^{-1}\geqslant(k(p)\sigma_p)^{-1}.$$

(ii) *For fixed $\ell,m\in E$, $\ell<m$, define*

$$f_1^{(\ell,m)}=\hat\nu[\cdot\vee\ell,m]\,\mathbb{1}_{\leqslant m},\quad f_n^{(\ell,m)}=f_{n-1}^{(\ell,m)}(I\!I(f_{n-1}^{(\ell,m)}))^{p^*-1}\mathbb{1}_{\leqslant m}\ (n\geqslant 2),$$

where $\mathbb{1}_{\leqslant m}$ is the indicator of the set $\{0,1,\ldots,m\}$ and then define

$$\delta_n'=\sup_{\ell,m:\,\ell<m}\min_{i\leqslant m}I\!I_i(f_n^{(\ell,m)}).$$

Then δ_n' is increasing in n (denote its limit by δ_∞') and

$$\sigma_p^{-1}\geqslant\delta_1'^{-1}\geqslant\cdots\geqslant\delta_\infty'^{-1}\geqslant\lambda_p.$$

Next, define

$$\overline{\delta}_n=\sup_{\ell,m:\,\ell<m}\frac{\mu(|f_n^{(\ell,m)}|^p)}{D_p(f_n^{(\ell,m)})},\quad n\geqslant 1.$$

Then $\overline{\delta}_n^{-1} \geqslant \lambda_p$ *and* $\overline{\delta}_{n+1} \geqslant \delta_n'$ *for* $n \geqslant 1$.

The next result is a consequence of Theorem 2.4.

Corollary 2.5 (Improved estimates) *For* $p > 1$, *we have*

$$\sigma_p^{-1} \geqslant \delta_1'^{-1} \geqslant \lambda_p \geqslant \delta_1^{-1} \geqslant (k(p)\sigma_p)^{-1},$$

where

$$\delta_1 = \sup_{i\in E}\left[\frac{1}{\hat\nu[i,N]^{1/p^*}}\sum_{j=i}^{N}\hat\nu_j\left(\sum_{k=0}^{j}\mu_k\hat\nu[k,N]^{(p-1)/p^*}\right)^{p^*-1}\right]^{p-1},$$

$$\delta_1' = \sup_{\ell\in E}\frac{1}{\hat\nu[\ell,N]^{p-1}}\left[\sum_{j=\ell}^{N}\hat\nu_j\left(\sum_{k=0}^{j}\mu_k\hat\nu[k\vee\ell,N]^{p-1}\right)^{p^*-1}\right]^{p-1}.$$

Moreover,

$$\overline{\delta}_1 = \sup_{m\in E}\frac{1}{\hat\nu[m,N]}\sum_{j=0}^{N}\mu_j\hat\nu[j\vee m,N]^p \in [\sigma_p, p\sigma_p],$$

and $\overline{\delta}_1 \leqslant \delta_1'$ *for* $1 < p \leqslant 2$, $\overline{\delta}_1 \geqslant \delta_1'$ *for* $p \geqslant 2$.

An remarkable point of Corollary 2.5 is its last assertion which is comparable with the known result that $\overline{\delta}_1 = \delta_1'$ when $p = 2$ (cf. [4; Theorem 3.2]). This indicates that some additional work is necessary for general p than the specific one $p = 2$.

2.2 Examples

In the worst case that $p = 2$ (cf. Fig. 1), the ratio $k^{1/p}(p)$ of the upper and lower estimates is no more than 2 which can be improved (no more than $\sqrt{2}$) by the improved estimates as shown by a large number of examples (cf. [4]). The same conclusion should also be true for general p as shown by two examples below. Actually, the effectiveness of the improved bounds δ_1 and $\overline{\delta}_1$ shown by the examples is quite unexpected.

Example 2.6 Assume that $E = \{0, 1, \ldots, N\}$, $a > 0$, and $r > 1$. Let $\mu_k = r^k$, $\nu_k = ar^{k+1}$ for $k \in E$. Then

$$\sigma_p = \frac{r}{a(r-1)(r^{p^*-1}-1)^{p-1}},$$

$$\delta_1 = \frac{1}{ar(r^{1/p}-1)}\sup_{i\in E}\left\{\sum_{j=i}^{N}\left(r^{i-j+[(j-i+1)/p]} - r^{i-j-(i/p)}\right)^{p^*-1}\right\}^{p-1},$$

$$\overline{\delta}_1 = \frac{r^{p^*}-1}{a(r^{p^*-1}-1)^p(r-1)},$$

$$\delta_1' = \frac{1}{ar}\sup_{\ell\in E}\left\{\sum_{j=\ell}^{N}\left(\frac{r^{\ell+1}-1}{r^j(r-1)} + (j-\ell)r^{\ell-j}\right)^{p^*-1}\right\}^{p-1}.$$

The improved estimates given in Corollary 2.5 are shown in Figure 2.

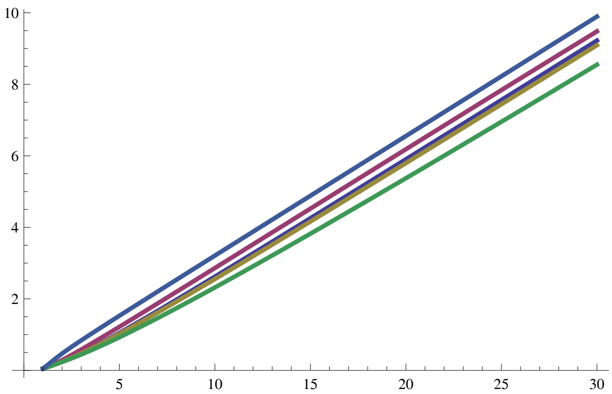

Fig. 2 Let $N = 80$, $a = 1$, $r = 20$, and let p vary over $(1.001, 30.001)$ avoiding the singularity at $p = 1$. Viewing from right-hand side, curves from top to bottom are $(k(p)\sigma_p)^{1/p}$, ${\delta_1}^{1/p}$, $\overline{\delta}_1^{1/p}$, ${\delta_1'}^{1/p}$, and $\sigma_p^{1/p}$, respectively. Note that lower bounds $\overline{\delta}_1^{1/p}$ and ${\delta_1'}^{1/p}$ of $\lambda_p^{-1/p}$ are nearly overlapped.

The ratio between δ_1 and δ_1' (or $\overline{\delta}_1$) is obvious smaller than the basic estimates $k(p)$ obtained in Theorem 2.3. When $p = 2$, $\overline{\delta}_1 = \delta_1'$ which is known as just mentioned.

Example 2.7 Assume that $E = \{0, 1, \ldots, N\}$, $N < +\infty$. Let $\mu_k = 1$ and $\nu_k = 1$ for $k \in E$. Then

$$\sigma_p = \sup_{n \in E} \big[(n+1)(N-n+1)^{p-1}\big],$$

$$\delta_1 = \sup_{i \in E} \left[\frac{1}{(N-i+1)^{(p-1)/p}} \sum_{j=i}^{N} \left(\sum_{k=0}^{j} (N-k+1)^{(p-1)/p^*}\right)^{p^*-1}\right]^{p-1},$$

$$\overline{\delta}_1 = \sup_{m \in E} \left(m(N-m+1)^{p-1} + \frac{1}{N-m+1} \sum_{j=m}^{N} (N-j+1)^p\right),$$

$$\delta_1' = \sup_{\ell \in E} \left\{\frac{1}{N-\ell+1} \sum_{j=\ell}^{N} \left[\ell(N-\ell+1)^{p-1} + \sum_{k=\ell}^{j} (N-k+1)^{p-1}\right]^{p^*-1}\right\}^{p-1}.$$

Surprisingly, the improved estimates δ_1, δ_1', and $\overline{\delta}_1$ are nearly overlapped as shown in Figure 3.

3 Proofs of main results in Section 2

This section is organized as follows. Some preparations are collected in Subsection 3.1. The preparations may not be needed completely for our proofs here, but they are useful for the study in a more general setup. The proofs of the main results are presented in Subsection 3.2.

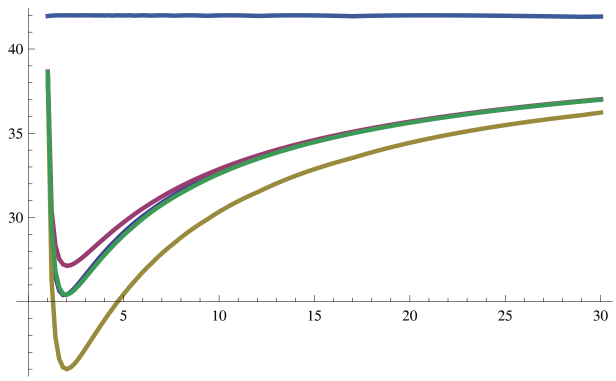

Fig. 3 Let $N = 40$, and let p vary over $(1.0175, 30.0175)$ avoiding singularity at $p = 1$. Viewing from right-hand side, curves from top to bottom are again $(k(p)\sigma_p)^{1/p}$, ${\delta_1}^{1/p}$, $\overline{\delta}_1^{1/p}$, ${\delta_1'}^{1/p}$, and $\sigma_p^{1/p}$, respectively. Note that $\overline{\delta}_1^{1/p}$ and ${\delta_1'}^{1/p}$ (lower bounds of $\lambda_p^{-1/p}$), as well as ${\delta_1}^{1/p}$ (upper bound) are nearly overlapped, except in a small neighborhood of $p = 2$. For this example, exact λ_p is unknown except that $\lambda_p = \sin^2 \frac{\pi}{2(N+2)}$ when $p = 2$.

3.1 Some preparations

A large part of the results stated in Section 2 depend on the properties of the eigenfunction g of λ_p. The goal of this subsection is studying these properties.

Define an operator

$$\overline{\Omega}_p f(k) = \Omega_p f(k) - \mu_k d_k |f_k|^{p-2} f_k, \quad k \in E,\ p > 1,$$

where $\{d_k\}_{k\in E}$ is a fixed nonnegative sequence. Then there is an extended equation of (1):

$$\overline{\Omega}_p f(k) = -\overline{\lambda}\mu_k |f_k|^{p-2} f_k, \quad k \in E, \tag{4}$$

which coincides with equation (1) for $\overline{\lambda} = \lambda$ if $d_k = 0$ for every $k \in E$.

Proposition 3.1 *Define*

$$\overline{D}_p(f) = D_p(f) + \sum_{k\in E} d_k \mu_k |f_k|^p, \quad f \in \mathscr{C}_K.$$

Let

$$\overline{\lambda} = \inf\{\overline{D}_p(f) \colon \mu(|f|^p) = 1,\ f \in \mathscr{C}_K \text{ and } f_{N+1} = 0 \text{ if } N < +\infty\}. \tag{5}$$

Then the solution, say g, to equation (4) with ND-boundaries is either positive or negative. In particular, the assertion holds for the eigenfunction of λ_p.

Proof Since $g_{-1} = g_0$, by making summation from 0 to $i \in E$ with respect to k on both sides of (4), we get

$$\nu_i |g_i - g_{i+1}|^{p-2}(g_i - g_{i+1}) = \sum_{k=0}^{i} (\overline{\lambda} - d_k)\mu_k |g_k|^{p-2} g_k, \quad i \in E. \tag{6}$$

If $\overline{\lambda} = 0$, then the assertion is obvious by (6) and induction. If $\overline{\lambda} > 0$, then $g_0 \neq 0$ (otherwise, $g \equiv 0$). Without loss of generality, assume that $g_0 = 1$ (if not, replace g with g/g_0). Suppose that there exists k_0, $1 \leqslant k_0 < N$, such that $g_i > 0$ for $i < k_0$ and $g_{k_0} \leqslant 0$. Let

$$f_i = g_i \mathbb{1}_{i<k_0} + \varepsilon \mathbb{1}_{i=k_0}$$

for some $0 < \varepsilon < g_{k_0-1}$. Then f belongs to the setting defining $\overline{\lambda}$ (cf. (5)). Since $g_{k_0} \leqslant 0 < \varepsilon < g_{k_0-1}$ and $|\varepsilon - g_{k_0-1}| < |g_{k_0} - g_{k_0-1}|$, we have

$$\begin{aligned}\overline{\Omega}_p f(k_0-1) &= \overline{\Omega}_p g(k_0-1) + \nu_{k_0-1}|g_{k_0-1} - g_{k_0}|^{p-2}(g_{k_0-1} - g_{k_0})\\ &\quad - \nu_{k_0-1}|\varepsilon - g_{k_0-1}|^{p-2}(g_{k_0-1} - \varepsilon)\\ &\geqslant \overline{\Omega}_p g(k_0-1) - \nu_{k_0-1}|\varepsilon - g_{k_0-1}|^{p-2}[(g_{k_0-1} - \varepsilon) - (g_{k_0-1} - g_{k_0})]\\ &= \overline{\Omega}_p g(k_0-1) - \nu_{k_0-1}(g_{k_0-1} - \varepsilon)^{p-2}(g_{k_0} - \varepsilon)\\ &> \overline{\Omega}_p g(k_0-1),\\ \overline{\Omega}_p f(k_0) &= -\nu_{k_0}\varepsilon^{p-1} + \nu_{k_0-1}|g_{k_0-1} - \varepsilon|^{p-2}(g_{k_0-1} - \varepsilon) - \mu_{k_0} d_{k_0}\varepsilon^{p-1}.\end{aligned}$$

Hence,

$$\begin{aligned}\overline{D}_p(f) &= (-\overline{\Omega}_p f, f)\\ &= -\sum_{i=0}^{k_0-2} f_i \overline{\Omega}_p f(i) - f_{k_0-1}\overline{\Omega}_p f(k_0-1) - \varepsilon \overline{\Omega}_p f(k_0)\\ &< -\sum_{i=0}^{k_0-2} g_i \overline{\Omega}_p g(i) - g_{k_0-1}\overline{\Omega}_p g(k_0-1) - \varepsilon \overline{\Omega}_p f(k_0)\\ &= \overline{\lambda}\sum_{i=0}^{k_0-1} \mu_i |g_i|^{p-2} g_i^2 + \varepsilon[(\nu_{k_0} + \mu_{k_0} d_{k_0})\varepsilon^{p-1} - \nu_{k_0-1}(g_{k_0-1} - \varepsilon)^{p-1}].\end{aligned}$$

In the second equality, we have used the fact that

$$\nu_i |g_i - g_{i+1}|^{p-2}(g_i - g_{i+1}) g_i = \sum_{k=0}^{i} (\overline{\lambda} - d_k)\mu_k |g_k|^p - \sum_{k=0}^{i-1} \nu_k |g_k - g_{k+1}|^p, \quad i \in E, \tag{7}$$

which can be obtained from (4), by a computation similar to that of $(-\Omega_p f, f)$ given above (3). Noticing that

$$\mu(|f|^p) = \sum_{i=0}^{k_0-1} \mu_i |g_i|^{p-2} g_i^2 + \mu_{k_0}\varepsilon^p$$

and

$$\nu_{k_0} + \mu_{k_0} d_{k_0} - \overline{\lambda}\mu_{k_0} < \nu_{k_0-1}\Big(\frac{g_{k_0-1}}{\varepsilon} - 1\Big)^{p-1}$$

for small enough ε, we obtain a contradiction to (5):

$$\frac{\overline{D}_p(f)}{\mu(|f|^p)} < \overline{\lambda} \leqslant \frac{\overline{D}_p(f)}{\mu(|f|^p)}.$$

This proves the first assertion and then the second one is obvious. □

Before moving further, we introduce an equation which is somehow more general than eigenequation:

$$\text{Poisson equation:} \quad \Omega_p g(k) = -\mu_k |f_k|^{p-2} f_k. \tag{8}$$

By putting $f = \lambda_p^{p^*-1} g$, we return to eigenequation. From (8), for $i, j \in E$ with $i < j$, we obtain

$$\nu_j |g_j - g_{j+1}|^{p-2}(g_j - g_{j+1}) - \nu_{i-1}|g_{i-1} - g_i|^{p-2}(g_{i-1} - g_i) = \sum_{k=i}^{j} \mu_k |f_k|^{p-2} f_k. \tag{9}$$

Moreover, if g is positive and decreasing, then

$$g_n - g_{N+1} = \sum_{j=n}^{N} \left(\frac{1}{\nu_j} \sum_{k=0}^{j} \mu_k |f_k|^{p-2} f_k \right)^{p^*-1}, \quad n \in E. \tag{10}$$

Besides Proposition 3.1, two more propositions are needed. One describes the monotonicity of the eigenfunction presented in the next proposition, and the other one is about the vanishing property to be presented in Proposition 3.4.

Proposition 3.2 *Assume that* (λ_p, g) *is a solution to* (1) *with ND-boundaries and* $\lambda_p > 0$. *Then the eigenfunction* g *is strictly monotone. Furthermore,*

$$\frac{1}{\lambda_p} = \left[\frac{1}{g_n - g_{N+1}} \sum_{k=n}^{N} \left(\frac{1}{\nu_k} \sum_{i=0}^{k} \mu_i g_i^{p-1} \right)^{p^*-1} \right]^{p-1}, \quad n \in E. \tag{11}$$

Proof Without loss of generality, assume that $g_0 = 1$. The first assertion follows by letting $i = 0$ and $f = \lambda_p^{p^*-1} g$ in (9). Moreover, it is clear that g is strictly decreasing. Formula (11) then follows from (10) by letting $f = \lambda_p^{p^*-1} g$. □

As mentioned above, with $f = \lambda_p^{p^*-1} g$, (9) and (10) are simple variants of eigenequation (1). However, for general test function f, the left-hand side of the function g defined by (10) may be far away from the eigenfunction of λ_p. Nevertheless, we regard the resulting function (assuming $g_{N+1} = 0$) as a mimic of the eigenfunction. This explains where the operator II comes from: it is regarded as an approximation of λ_p^{-1} since $II(\lambda_p^{p^*-1} g) \equiv \lambda_p^{-1}$. Next, write $II_i(f)$ as u_i/v_i. Then $I_i(f)$ is defined by

$$I_i(f) = \frac{u_i - u_{i+1}}{v_i - v_{i+1}}.$$

In other words, the operator I comes from II in the use of proportional property. The operator R also comes from the eigenequation by setting $w_i = g_{i+1}/g_i$.

Remark 3.3 Define

$$\widetilde{\lambda}_p = \inf\{D_p(f)\colon \mu(|f|^p) = 1 \text{ and } f_{N+1} = 0\}. \tag{12}$$

(i) It is easy to check that the assertions in Propositions 3.1 and 3.2 also hold for $\widetilde{\lambda}_p$ defined by (12).

(ii) Define

$$\lambda_p^{(n)} = \inf\{D_p(f)\colon \mu(|f|^p) = 1,\ f = f_{\leqslant n}\}.$$

We have $\lambda_p = \widetilde{\lambda}_p$ and $\lambda_p^{(n)} \downarrow \lambda_p$ as $n \uparrow N$. Indeed, it is clear that

$$\lambda_p^{(n)} \geqslant \lambda_p \geqslant \widetilde{\lambda}_p.$$

By definition of $\widetilde{\lambda}_p$, for any fixed $\varepsilon > 0$, there exists $\overline{f} \in L^p(\mu)$ such that $f_{N+1} = 0$ and

$$\frac{D_p(\overline{f})}{\mu(|\overline{f}|^p)} \leqslant \widetilde{\lambda}_p + \varepsilon.$$

Define $f^{(n)} = \overline{f}\mathbb{1}_{[0,n)}$. Then $f^{(n)} \in L^p(\mu)$ and

$$D_p(f^{(n)}) \uparrow D_p(\overline{f}), \quad \mu(|f^{(n)}|^p) \uparrow \mu(|\overline{f}|^p), \quad n \to N.$$

Next, for large enough n, we have

$$\lambda_p \leqslant \lambda_p^{(n)} \leqslant \frac{D_p(f^{(n)})}{\mu(|f^{(n)}|^p)} \leqslant \frac{D_p(\overline{f})}{\mu(|\overline{f}|^p)} + \varepsilon \leqslant \widetilde{\lambda}_p + 2\varepsilon \leqslant \lambda_p + 2\varepsilon.$$

By letting $n \to N$ first and then $\varepsilon \downarrow 0$, it follows that $\lambda_p = \widetilde{\lambda}_p$. Actually, we have $\lambda_p^{(n)} \downarrow \lambda_p$.

(iii) The test functions in (12) are described by $f_{N+1} = 0$, which can be seen as the imitations of eigenfunction, so the assertion in item (ii) above also implies the vanishing property of eigenfunction to some extent.

From now on, in this section, without loss of generality, we assume that **the eigenfunction g corresponding to λ_p (or $\widetilde{\lambda}_p$) is nonnegative and strictly decreasing.** The monotone property is important to our study. For example, it guarantees the meaning of the operator I defined in Section 2 since the denominator is $f_i - f_{i+1}$ there.

3.2 Sketch proof of main results in ND-case

Since a large part of proofs are analogies of the case that $p = 2$, we need only to show some keys and some difference between the general p and the specific $p = 2$.

Proof of Theorem 2.1 We adopt the following circle arguments for the lower estimates:

$$\begin{aligned}\lambda_p &\geqslant \sup_{f\in\mathscr{F}_{II}}\inf_{i\in E} II_i(f)^{-1}\\ &= \sup_{f\in\mathscr{F}_{I}}\inf_{i\in E} II_i(f)^{-1}\\ &= \sup_{f\in\mathscr{F}_{I}}\inf_{i\in E} I_i(f)^{-1}\\ &\geqslant \sup_{w\in\mathscr{W}}\inf_{i\in E} R_i(w)\\ &\geqslant \lambda_p.\end{aligned}$$

Step 1 Prove that

$$\lambda_p \geqslant \sup_{f\in\mathscr{F}_{II}}\inf_{i\in E} II_i(f)^{-1}.$$

Clearly, we have $\lambda_p \geqslant \widetilde{\lambda}_p$ by Remark 3.3 (ii). Let $\{h_k\colon k\in E\}$ be a positive sequence, and let g satisfy $\mu(|g|^p)=1$ and $g_{N+1}=0$. Then

$$\begin{aligned}1 &= \mu(|g|^p)\\ &\leqslant \sum_{i=0}^N \mu_i\bigg(\sum_{k=i}^N |g_k-g_{k+1}|\bigg)^p \quad (\text{since } g_{N+1}=0)\\ &= \sum_{i=0}^N \mu_i\bigg(\sum_{k=i}^N |g_k-g_{k+1}|\Big(\frac{\nu_k}{h_k}\Big)^{1/p}\Big(\frac{h_k}{\nu_k}\Big)^{1/p}\bigg)^p\\ &\leqslant \sum_{i=0}^N \mu_i\bigg[\bigg(\sum_{k=i}^N |g_k-g_{k+1}|^p\,\frac{\nu_k}{h_k}\bigg)^{1/p}\bigg(\sum_{j=i}^N\Big(\frac{h_j}{\nu_j}\Big)^{p^*/p}\bigg)^{1/p^*}\bigg]^p\\ &\qquad\qquad (\text{by Hölder's inequality})\\ &= \sum_{k=0}^N \frac{\nu_k}{h_k}|g_k-g_{k+1}|^p\sum_{i=0}^k\mu_i\bigg(\sum_{j=i}^N \hat{\nu}_j h_j^{p^*-1}\bigg)^{p-1}\\ &\qquad (\text{by exchanging the order of the sums})\\ &\leqslant D_p(g)\sup_{k\in E} H_k,\end{aligned}$$

where

$$H_k = \frac{1}{h_k}\sum_{i=0}^k \mu_i\bigg(\sum_{j=i}^N \hat{\nu}_j h_j^{p^*-1}\bigg)^{p-1}.$$

For every $f\in\mathscr{F}_{II}$ with $\sup_{i\in E} II_i(f)<+\infty$, let

$$h_k=\sum_{j=0}^k \mu_j f_j^{p-1}.$$

Then

$$\sup_{k\in E} H_k \leqslant \sup_{k\in E} II_k(f)$$

by the proportional property. Hence,

$$D_p(g) \geqslant \inf_{k\in E} II_k(f)^{-1}$$

for every g with $\mu(|g|^p) = 1$, $g_{N+1} = 0$, and $f \in \mathscr{F}_{II}$. By making the supremum with respect to $f \in \mathscr{F}_{II}$ first and then the infimum with respect to g with $g_{N+1} = 0$ and $\mu(|g|^p) = 1$, it follows

$$\lambda_p \geqslant \widetilde{\lambda}_p \geqslant \sup_{f\in\mathscr{F}_{II}} \inf_{i\in E} II_i(f)^{-1}.$$

Step 2 Prove that

$$\sup_{f\in\mathscr{F}_{II}} \inf_{i\in E} II_i(f)^{-1} = \sup_{f\in\mathscr{F}_{I}} \inf_{i\in E} II_i(f)^{-1} = \sup_{f\in\mathscr{F}_{I}} \inf_{i\in E} I_i(f)^{-1}.$$

Using the proportional property, on the one hand, for any fixed $f \in \mathscr{F}_I$, we have

$$\begin{aligned}
\sup_{i\in E} II_i(f) &= \sup_{i\in E} \bigg[\frac{1}{f_i}\sum_{j=i}^{N} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k f_k^{p-1}\bigg)^{p^*-1}\bigg]^{p-1} \\
&= \sup_{i\in E} \bigg\{\bigg[\sum_{j=i}^{N} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k f_k^{p-1}\bigg)^{p^*-1}\bigg]\bigg[\sum_{j=i}^{N}(f_j - f_{j+1}) + f_{N+1}\bigg]^{-1}\bigg\}^{p-1} \\
&\leqslant \sup_{j\in E} \frac{1}{\nu_j (f_j - f_{j+1})^{p-1}} \sum_{k=0}^{j} \mu_k f_k^{p-1} \quad (\text{since } \hat{\nu}_j = {\nu_j}^{1-p^*}) \\
&= \sup_{j\in E} I_i(f).
\end{aligned}$$

Since $\mathscr{F}_I \subset \mathscr{F}_{II}$, by making the infimum with respect to $f \in \mathscr{F}_I$ on both sides of the inequality above, we have

$$\sup_{f\in\mathscr{F}_{II}} \inf_{i\in E} II_i(f)^{-1} \geqslant \sup_{f\in\mathscr{F}_{I}} \inf_{i\in E} II_i(f)^{-1} \geqslant \sup_{f\in\mathscr{F}_{I}} \inf_{i\in E} I_i(f)^{-1}.$$

On the other hand, for any fixed $f \in \mathscr{F}_{II}$, let

$$g = f\,II(f)^{p^*-1} \in \mathscr{F}_I.$$

Similar to the proof above, we have

$$\sup_{i\in E} I_i(g) \leqslant \sup_{i\in E} II_i(f).$$

Therefore,

$$\sup_{f\in\mathscr{F}_I}\inf_{i\in E} I_i(f)^{-1} \geqslant \sup_{f\in\mathscr{F}_{II}}\inf_{i\in E} II_i(f)^{-1},$$

and the required assertion holds.

Step 3 Prove that

$$\sup_{f\in\mathscr{F}_{II}}\inf_{i\in E} II_i(f)^{-1} \geqslant \sup_{w\in\mathscr{W}}\inf_{i\in E} R_i(w).$$

First, we change the form of $R_i(w)$. Given $w\in\mathscr{W}$, let $u_{i+1}=w_iw_{i-1}\cdots w_0$ for $i\geqslant 0$, $u_0=1$. Then u is positive, strictly decreasing, and $w_i=u_{i+1}/u_i$. By a simple rearrangement, we get

$$R_i(w)=\frac{1}{\mu_i}\Big[\nu_i\Big(1-\frac{u_{i+1}}{u_i}\Big)^{p-1}-\nu_{i-1}\Big(\frac{u_{i-1}}{u_i}-1\Big)^{p-1}\Big]=-\frac{1}{\mu_i u_i^{p-1}}\,\Omega_p u(i).$$

Next, we prove the main assertion. Without loss of generality, assume that $\inf_{i\in E} R_i(w)>0$. Let u be the function constructed above and let $f=uR(w)^{p^*-1}>0$. Then $\Omega_p u=-\mu f^{p-1}$. Since $\nu_{-1}=0$, u is decreasing and $u>0$. By (10), we have

$$u_i-u_{N+1}=\sum_{k=i}^{N}\bigg(\frac{1}{\nu_k}\sum_{j=0}^{k}\mu_j f_j^{p-1}\bigg)^{p^*-1}.$$

Hence,

$$R_i(w)^{1-p^*}=\frac{u_i}{f_i}\geqslant\frac{1}{f_i}\sum_{k=i}^{N}\bigg(\frac{1}{\nu_k}\sum_{j=0}^{k}\mu_j f_j^{p-1}\bigg)^{p^*-1}=II_i(f)^{p^*-1}.$$

Then

$$\sup_{f\in\mathscr{F}_{II}}\inf_{i\in E} II_i(f)^{-1}\geqslant\inf_{i\in E}R_i(w)$$

holds for every $w\in\mathscr{W}$ and the assertion follows immediately.

Step 4 Prove that

$$\sup_{w\in\mathscr{W}}\inf_{i\in E}R_i(w)\geqslant\lambda_p.$$

If $\sum_{i\in E}\hat{\nu}_i<+\infty$, then choose f to be a positive function satisfying $h=fII(f)^{p^*-1}<+\infty$. We have

$$h_i=\sum_{k=i}^{N}\hat{\nu}_k\bigg(\sum_{j=0}^{k}\mu_j f_j^{p-1}\bigg)^{p^*-1},\quad h\downarrow\downarrow,\quad h_i-h_{i+1}=\hat{\nu}_i\bigg(\sum_{j=0}^{i}\mu_j f_j^{p-1}\bigg)^{p^*-1}.$$

Let $\overline{w}_i=h_{i+1}/h_i$ for $i\in E$. By a simple calculation, we obtain

$$R_i(\overline{w})=\frac{-\Omega_p h(i)}{\mu_i h_i^{p-1}}=\frac{f_i^{p-1}}{h_i^{p-1}}>0.$$

If $\sum_{i\in E}\hat\nu_i=+\infty$, then set $\overline{w}\equiv 1$. We have $R_i(\overline{w})=0$. So

$$\sup_{w\in\mathscr{W}}\inf_{i\in E}R_i(w)\geqslant 0.$$

Without loss of generality, assume that $\lambda_p>0$. By Proposition 3.1, the eigenfunction g of λ_p is positive and strictly decreasing. Let $\overline{w}_i=g_{i+1}/g_i\in\mathscr{W}$. Then the assertion follows from the fact that $R_i(\overline{w})=\lambda_p$ for every $i\in E$.

Step 5 We prove that the supremum in the circle arguments can be attained.

As an application of the circle arguments before Step 1, the assertion is easy in the case of $\lambda_p=0$ since

$$0=\lambda_p\geqslant\inf_{i\in E}I\!I_i(f)^{-1}\geqslant 0,\quad 0=\lambda_p\geqslant\inf_{i\in E}I_i(f)^{-1}\geqslant 0$$

for every f in the set defining λ_p and

$$\lambda_p\geqslant\sup_{w\in\mathscr{W}}\inf_{i\in E}R_i(w)\geqslant\inf_{i\in E}R_i(\overline{w})\geqslant 0$$

for $\overline{w}$ used in Step 4 above. In the case that $\lambda_p>0$ with eigenfunction g satisfying $g_0=1$, let $\overline{w}_i=g_{i+1}/g_i$. Then $R_i(\overline{w})=\lambda_p$ as seen from the remarks after Proposition 3.2 and $I_i(g)=\lambda_p$ by letting $f=\lambda_p^{p^*-1}g$ in (9). Moreover, we have $I\!I_i(g)=\lambda_p$ for $i\in E$ by letting $f=\lambda_p^{p^*-1}g$ in (10) whenever $g_{N+1}=0$.

It remains to rule out the probability that $g_{N+1}>0$. The Proposition 3.4 below, which is proved by the variational formulas verified in Step 1, gives us the positive answer.

Proposition 3.4 *Assume that $\lambda_p>0$ and $p>1$. Let g be an eigenfunction corresponding to λ_p. Then*

$$g_{N+1}:=\lim_{i\to N+1}g_i=0.$$

Proof Let $f=g-g_{N+1}$. Then $f\in\mathscr{F}_{I\!I}$. By (11), we have

$$\lambda_p^{1-p^*}f_i=\sum_{j=i}^N\hat\nu_j\bigg(\sum_{k=0}^j\mu_k {g_k}^{p-1}\bigg)^{p^*-1}.$$

Denote

$$M_i=\sum_{j=i}^N\hat\nu_j\bigg(\sum_{k=0}^j\mu_k\bigg)^{p^*-1}.$$

If $M_i=+\infty$, then

$$\lambda_p^{1-p^*}f_i=\sum_{j=i}^N\hat\nu_j\bigg(\sum_{k=0}^j\mu_k {g_k}^{p-1}\bigg)^{p^*-1}>M_ig_{N+1}.$$

There is a contradiction once $g_{N+1} \neq 0$. If $M_i < +\infty$, then

$$\begin{aligned}
\sup_{i\in E} II_i(f) &= \sup_{i\in E} \frac{1}{(g_i - g_{N+1})^{p-1}} \left[\sum_{j=i}^{N} \hat{\nu}_j \left(\sum_{k=0}^{j} \mu_k (g_k - g_{N+1})^{p-1} \right)^{p^*-1} \right]^{p-1} \\
&= \sup_{i\in E} \frac{1}{\lambda_p} \left\{ \frac{\sum_{j=i}^{N} \hat{\nu}_j [\sum_{k=0}^{j} \mu_k (g_k - g_{N+1})^{p-1}]^{p^*-1}}{\sum_{j=i}^{N} \hat{\nu}_j [\sum_{k=0}^{j} \mu_k g_k^{p-1}]^{p^*-1}} \right\}^{p-1} \quad \text{(by (11))} \\
&\leqslant \frac{1}{\lambda_p} \sup_{k\in E} \left(1 - \frac{g_{N+1}}{g_k}\right)^{p-1} \quad \text{(by the proportional property)} \\
&= \frac{1}{\lambda_p} \left(1 - \frac{g_{N+1}}{g_0}\right)^{p-1} \quad (\text{since } g \downarrow\downarrow).
\end{aligned}$$

If $g_{N+1} > 0$, then by the variational formula for lower estimates proved in Step 1 above, we have

$$\lambda_p^{-1} \leqslant \inf_{f\in \mathscr{F}_{II}} \sup_{i\in E} II_i(f) \leqslant \sup_{i\in E} II_i(f) < \lambda_p^{-1},$$

which is a contradiction. So we must have $g_{N+1} = 0$. □

By now, we have finished the proof for the lower estimates. From this proposition, we see that the vanishing property of eigenfunction holds naturally. So the classification also holds for $N = +\infty$. Combining with (10), the vanishing property also further explains where the operator II comes from. Then, we come back to the main proof of Theorem 2.1.

For the upper estimates, we adopt the following circle arguments:

$$\begin{aligned}
\lambda_p &\leqslant \inf_{f\in \widetilde{\mathscr{F}}'_{II} \cup \widetilde{\mathscr{F}}_{II}} \sup_{i\in \mathrm{supp}(f)} II_i(f)^{-1} \\
&\leqslant \inf_{f\in \widetilde{\mathscr{F}}_{II}} \sup_{i\in \mathrm{supp}(f)} II_i(f)^{-1} \\
&= \inf_{f\in \widetilde{\mathscr{F}}_{I}} \sup_{i\in \mathrm{supp}(f)} II_i(f)^{-1} \\
&= \inf_{f\in \widetilde{\mathscr{F}}_{I}} \sup_{i\in E} I_i(f)^{-1} \\
&\leqslant \inf_{f\in \widetilde{\mathscr{F}}'_{I}} \sup_{i\in E} I_i(f)^{-1} \\
&\leqslant \inf_{w\in \widetilde{\mathscr{W}}} \sup_{i\in E} R_i(w) \\
&\leqslant \lambda_p.
\end{aligned}$$

Since the proofs are parallel to that of the lower bounds part, we ignore most of the details here and only mention a technique when proving

$$\inf_{f\in \widetilde{\mathscr{F}}_{II}} \sup_{i\in \mathrm{supp}(f)} II_i(f)^{-1} = \inf_{f\in \widetilde{\mathscr{F}}_{I}} \sup_{i\in \mathrm{supp}(f)} II_i(f)^{-1} = \inf_{f\in \widetilde{\mathscr{F}}_{I}} \sup_{i\in E} I_i(f)^{-1}.$$

To see this, we adopt a small circle arguments below:

$$\lambda_p \leqslant \inf_{f\in\widetilde{\mathscr{F}}_{II}} \sup_{i\in\text{supp}(f)} II_i(f)^{-1} \leqslant \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{i\in\text{supp}(f)} II_i(f)^{-1} \leqslant \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{i\in E} I_i(f)^{-1} \leqslant \lambda_p.$$

The technique is about an approximating procedure, which is used to prove the last inequality above. Recall that

$$\lambda_p^{(m)} = \inf\{D_p(f)\colon \mu(|f|^p) = 1,\ f = f\mathbb{1}_{\cdot\leqslant m}\}$$

and $\lambda_p^{(m)} \downarrow \lambda_p$ as $m \uparrow N$ (see Remark 3.3). Let g be an eigenfunction of $\lambda_p^{(m)} > 0$ with $g_0 = 1$. Then $\{g_i\}_{i=0}^m$ is strictly decreasing and $g_{m+1} = 0$ by letting E be $E^{(m)} := [0, m] \cap E$ in Proposition 3.2. Extend g to E with $g_i = 0$ for $i \geqslant m+1$, we have

$$g \in \widetilde{\mathscr{F}}_I', \quad \text{supp}(g) = \{0, 1, \ldots, m\}, \quad \lambda_p^{(m)} = I_k(g)^{-1}\ (k \leqslant m).$$

Hence,

$$\lambda_p^{(m)} = \sup_{i\leqslant m} I_i(g)^{-1} \geqslant \inf_{f\in\widetilde{\mathscr{F}}_I', \text{supp}(f)=E^{(m)}} \sup_{k\in E} I_k(f)^{-1} \geqslant \inf_{f\in\widetilde{\mathscr{F}}_I'} \sup_{k\in E} I_k(f)^{-1}.$$

Since $\widetilde{\mathscr{F}}_I' \subset \widetilde{\mathscr{F}}_I$, the right-hand side of the formula above is bounded below by $\inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{k\in E} I_k(f)^{-1}$. So the required assertion follows by letting $m \to N$. □

Instead of the approximating with finite state space used in the proof of the upper bound above, it seems more natural to use the truncating procedure for the 'eigenfunction' g. However, the next result, which is easy to check by (9) and Proposition 3.2, shows that the procedure is not practical in general.

Remark 3.5 Let (λ_p, g) be a non-trivial solution to eigenequation (1) and (2) with $\lambda_p > 0$. Define $g^{(m)} = g\mathbb{1}_{\leqslant m}$. Then

$$\min_{i\in\text{supp}(g^{(m)})} II_i(g^{(m)}) = \frac{(1 - \frac{g_{m+1}}{g_m})^{p-1}}{\lambda_p}.$$

In particular, the sequence $\{\min_{i\in\text{supp}(g^{(m)})} II_i(g^{(m)})\}_{m\geqslant 1}$ may not converge to λ_p^{-1} as $m \uparrow +\infty$.

Proof The proof is simply an application of $f = \lambda_p^{1/(p-1)} g$ to (9), based on Proposition 3.2. □

For simplicity, we write $\varphi_i = \hat{\nu}[i, N]^{p-1}$ in the proofs of Theorems 2.3, 2.4, and Corollary 2.5 below.

Proof of Theorem 2.3 (a) First, we prove that $\lambda_p \geqslant (k(p)\sigma_p)^{-1}$. Without loss of generality, assume that $\varphi_0 < +\infty$ (otherwise, $\sigma_p = +\infty$). Let $f = \varphi^{1/p} <$

$+\infty$. Using the summation by parts formula, we have

$$\begin{aligned}\sum_{j=0}^{i}\mu_j f_j^{p-1} &= \mu[0,i]\varphi_i^{1/p^*}+\sum_{j=0}^{i-1}\mu[0,j](\varphi_j^{1/p^*}-\varphi_{j+1}^{1/p^*})\\ &\leqslant \sigma_p\bigg[\varphi_i^{-1/p}+\sum_{j=0}^{i-1}\frac{1}{\varphi_j}(\varphi_j^{1/p^*}-\varphi_{j+1}^{1/p^*})\bigg]\\ &\leqslant p\sigma_p\,\varphi_i^{-1/p}.\end{aligned}$$

In the last inequality, we have used the fact that

$$\sum_{j=0}^{i-1}\frac{1}{\varphi_j}(\varphi_j^{1/p^*}-\varphi_{j+1}^{1/p^*})\leqslant (p-1)\varphi_i^{-1/p}.$$

To see this, since $\varphi_0>0$, it suffices to show that

$$\varphi_j^{1/p^*}-\varphi_{j+1}^{1/p^*}\leqslant (p-1)\varphi_j(\varphi_{j+1}^{-1/p}-\varphi_j^{-1/p}).$$

Multiplying $\varphi_{j+1}^{1/p}$ on both sides, this is equivalent to

$$p\varphi_j^{1/p^*}\varphi_{j+1}^{1/p}\leqslant (p-1)\varphi_j+\varphi_{j+1}^{1/p^*}\varphi_{j+1}^{1/p},$$

which is now obvious by Young's inequality:

$$\varphi_j^{1/p^*}\varphi_{j+1}^{1/p}\leqslant \frac{1}{p^*}(\varphi_j^{1/p^*})^{p^*}+\frac{1}{p}(\varphi_{j+1}^{1/p})^{p}.$$

Since

$$\frac{1}{\nu_i}=\hat{\nu}_i^{p-1}=(\varphi_i^{p^*-1}-\varphi_{i+1}^{p^*-1})^{p-1},\quad \varphi_i^{\frac{1}{p(p-1)}}\varphi_{i+1}^{1/p}\leqslant\frac{1}{p}\varphi_i^{p^*-1}+\frac{1}{p^*}\varphi_{i+1}^{p^*-1},$$

we have

$$\begin{aligned}I_i(f)&=\frac{1}{\nu_i(\varphi_i^{1/p}-\varphi_{i+1}^{1/p})^{p-1}}\sum_{j=0}^{i}\mu_j\varphi_j^{1/p^*}\\ &\leqslant\frac{p\sigma_p\varphi_i^{-1/p}}{\nu_i(\varphi_i^{1/p}-\varphi_{i+1}^{1/p})^{p-1}}\\ &=p\sigma_p\bigg(\frac{\varphi_i^{p^*-1}-\varphi_{i+1}^{p^*-1}}{\varphi_i^{p^*-1}-\varphi_i^{1/(p(p-1))}\varphi_{i+1}^{1/p}}\bigg)^{p-1}\\ &\leqslant pp^{*p-1}\sigma_p.\end{aligned}\tag{13}$$

Then the required assertion follows by Theorem 2.1 (i).

(b) Next, we prove that $\lambda_p \geqslant \sigma_p^{-1}$. Let $f = \hat{\nu}[\cdot \vee n, m]\mathbb{1}_{\cdot \leqslant m}$ for some $m, n \in E$ with $n < m$. Then $f \in \widetilde{\mathscr{F}_I}$ and $f_i - f_{i+1} = \hat{\nu}_i \mathbb{1}_{n \leqslant i \leqslant m}$. By convention $1/0 = +\infty$, we have

$$I_i(f) = \bigg(\sum_{k=0}^{n} \mu_k \hat{\nu}[n, m]^{p-1} + \sum_{k=n+1}^{i} \mu_k \hat{\nu}[k, m]^{p-1}\bigg)\mathbb{1}_{[n,m]} + \infty \mathbb{1}_{[n,m]^c}.$$

So

$$\begin{aligned}
\lambda_p^{-1} &= \sup_{f \in \widetilde{\mathscr{F}}_I} \inf_{i \in E} I_i(f) \\
&\geqslant \inf_{i \in E} I_i(f) \\
&= \inf_{n \leqslant i \leqslant m} I_i(f) \\
&= \inf_{n \leqslant i \leqslant m} \bigg(\sum_{k=0}^{n} \mu_k \hat{\nu}[n, m]^{p-1} + \sum_{k=n+1}^{i} \mu_k \hat{\nu}[k, m]^{p-1}\bigg) \\
&= \sum_{k=0}^{n} \mu_k \hat{\nu}[n, m]^{p-1}, \quad m > n.
\end{aligned}$$

The assertion that $\lambda_p^{-1} \geqslant \sigma_p$ follows by letting $m \to N$.

(c) At last, if $\hat{\nu}[1, +\infty) = +\infty$, then $\lambda_p = 0$ is obvious. If

$$\sum_{k=1}^{+\infty} \hat{\nu}_k \mu[0, k]^{p^*-1} < +\infty,$$

then

$$\varphi_n \mu[0, n] = \bigg(\sum_{k=n}^{+\infty} \hat{\nu}_k \mu[0, n]^{p^*-1}\bigg)^{p-1} \leqslant \bigg(\sum_{k=1}^{+\infty} \hat{\nu}_k \mu[0, k]^{p^*-1}\bigg)^{p-1} < +\infty.$$

So $\sigma_p = +\infty$ and $\lambda_p = 0$. □

Proof of Theorem 2.4 By definitions of $\{\overline{\delta}_n\}$ and $\{\delta_n'\}$, using the proportional property, it is not hard to prove most of the results except that $\overline{\delta}_{n+1} \geqslant \delta_n'$ ($n \geqslant 1$). Put $g = f_{n+1}^{(\ell,m)}$ and $f = f_n^{(\ell,m)}$. Then $g = f II(f)^{p^*-1}\mathbb{1}_{\cdot \leqslant m}$. By a simple calculation, we have

$$(g_i - g_{i+1})^{p-1} = \frac{1}{\nu_i} \sum_{k=0}^{i} \mu_k f_k^{p-1}, \quad i \leqslant m.$$

Inserting this term into $D_p(g)$, we obtain

$$D_p(g) = \sum_{i=0}^{m} \nu_i (g_i - g_{i+1})^{p-1}(g_i - g_{i+1}) = \sum_{i=0}^{m} \sum_{k=0}^{i} \mu_k f_k^{p-1}(g_i - g_{i+1}).$$

Noticing $g_{m+1} = 0$ and exchanging the order of the sums, we obtain

$$D_p(g) = \sum_{k=0}^{m} \mu_k f_k^{p-1} \sum_{i=k}^{m} (g_i - g_{i+1}) = \sum_{k=0}^{m} \mu_k f_k^{p-1} g_k \leqslant \sum_{k=0}^{m} \mu_k g_k^p \max_{0\leqslant i\leqslant m} \frac{f_k^{p-1}}{g_k^{p-1}},$$

i.e.,

$$D_p(g) \leqslant \mu(|g|^p) \sup_{0\leqslant i\leqslant m} II_i(f)^{-1}.$$

So the required assertion follows by definitions of $\overline{\delta}_{n+1}$ and δ_n'. □

Most of the results in Corollary 2.5 can be obtained from Theorem 2.4 directly. Here, we study only those assertions concerning δ_1' and $\overline{\delta}_1$.

Proof of Corollary 2.5 (a) We compute δ_1' first. Since $p > 1$ and

$$\frac{1}{\hat{\nu}[i,m]} \sum_{j=i}^{m} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \hat{\nu}[k\vee\ell, m]^{p-1} \bigg)^{p^*-1}$$

is increasing in $i \in [\ell, m]$ (not hard to check), we have

$$\min_{i\leqslant m} II_i(f_1^{(\ell,m)}) = \bigg[\frac{1}{\hat{\nu}[\ell,m]} \sum_{j=\ell}^{m} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \hat{\nu}[k\vee\ell, m]^{p-1} \bigg)^{p^*-1} \bigg]^{p-1}.$$

We claim that

$$\delta_1' = \sup_{\ell\in E} \frac{1}{\varphi_\ell} \bigg[\sum_{j=\ell}^{N} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \varphi_{k\vee\ell} \bigg)^{p^*-1} \bigg]^{p-1}$$

because

$$\frac{1}{\hat{\nu}[\ell,m]^{p-1}} \bigg[\sum_{j=\ell}^{m} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \hat{\nu}[k\vee\ell, m]^{p-1} \bigg)^{p^*-1} \bigg]^{p-1}$$

is increasing in m ($m > \ell$). To see this, it suffices to show that

$$\frac{1}{\hat{\nu}[\ell,m+1]} \sum_{j=\ell}^{m+1} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \hat{\nu}[k\vee\ell, m+1]^{p-1} \bigg)^{p^*-1} \geqslant \frac{1}{\hat{\nu}[\ell,m]} \sum_{j=\ell}^{m} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \hat{\nu}[k\vee\ell, m]^{p-1} \bigg)^{p^*-1}.$$

Equivalently,

$$\sum_{j=\ell}^{m+1} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \frac{\hat{\nu}[k\vee\ell, m+1]^{p-1}}{\hat{\nu}[\ell,m+1]^{p-1}} \bigg)^{p^*-1} \geqslant \sum_{j=\ell}^{m} \hat{\nu}_j \bigg(\sum_{k=0}^{j} \mu_k \frac{\hat{\nu}[k\vee\ell, m]^{p-1}}{\hat{\nu}[\ell,m]^{p-1}} \bigg)^{p^*-1}.$$

It suffices to show that

$$\frac{\hat\nu[k\vee\ell, m+1]}{\hat\nu[\ell, m+1]} \geqslant \frac{\hat\nu[k\vee\ell, m]}{\hat\nu[\ell, m]}, \quad k\in E.$$

When $k\leqslant \ell$, the required assertion is obvious. When $k>\ell$, the inequality is just

$$\frac{\hat\nu[k, m+1]}{\hat\nu[k, m]} \geqslant \frac{\hat\nu[\ell, m+1]}{\hat\nu[\ell, m]}.$$

Noticing that $\hat\nu[i, m+1]=\hat\nu_{m+1}+\hat\nu[i,m]$ for any fixed $i\leqslant m$ and $\hat\nu[k,m]<\hat\nu[\ell,m]$ for $k>\ell$, we have

$$\frac{\hat\nu[k, m+1]}{\hat\nu[k, m]} = 1+\frac{\hat\nu_{m+1}}{\hat\nu[k,m]} > 1+\frac{\hat\nu_{m+1}}{\hat\nu[\ell,m]} = \frac{\hat\nu[\ell, m+1]}{\hat\nu[\ell, m]},$$

and then the required monotone property follows.

(b) Computing $\overline{\delta}_1$. Since

$$\mu(|f_1^{(\ell,m)}|^p)=\sum_{j=0}^m \mu_j\hat\nu[\ell\vee j, m]^p = \mu[0,\ell]\hat\nu[\ell,m]^p+\sum_{j=\ell+1}^m \mu_j\hat\nu[j,m]^p$$

and

$$\begin{aligned} D_p(f_1^{(\ell,m)}) &= \sum_{j=0}^m \nu_j(f_1^{(\ell,m)}(j)-f_1^{(\ell,m)}(j+1))^p\\ &= \sum_{j=\ell}^m \nu_j\hat\nu_j^p\\ &= \hat\nu[\ell,m] \quad (\text{since } \hat\nu_k=\nu_k^{1-p^*}),\end{aligned}$$

we have

$$\frac{\mu(|f_1^{(\ell,m)}|^p)}{D_p(f_1^{(\ell,m)})} = \hat\nu[\ell,m]^{p-1}\mu[0,\ell]+\frac{1}{\hat\nu[\ell,m]}\sum_{k=\ell+1}^m \mu_k\hat\nu[k,m]^p.$$

So

$$\overline{\delta}_1=\sup_{\ell,m\in E:\,\ell<m}\bigg(\hat\nu[\ell,m]^{p-1}\mu[0,\ell]+\frac{1}{\hat\nu[\ell,m]}\sum_{k=\ell+1}^m \mu_k\hat\nu[k,m]^p\bigg).$$

The assertion on $\overline{\delta}_1$ follows immediately once we show that

$$\hat\nu[\ell,m]^{p-1}\mu[0,\ell]+\frac{1}{\hat\nu[\ell,m]}\sum_{k=\ell+1}^m \mu_k\hat\nu[k,m]^p$$

is increasing in m ($\ell<m$). To see this, it suffices to show that

$$\frac{1}{\hat\nu[\ell,m]}\sum_{k=\ell+1}^m \mu_k\hat\nu[k,m]^p \leqslant \frac{1}{\hat\nu[\ell,m+1]}\sum_{k=\ell+1}^{m+1} \mu_k\hat\nu[k,m+1]^p,$$

or equivalently,

$$\frac{\mu_{m+1}}{\hat{\nu}[\ell,m+1]}\,\hat{\nu}_{m+1}^{p}+\sum_{k=\ell+1}^{m}\mu_k\bigg(\frac{\hat{\nu}[k,m+1]^p}{\hat{\nu}[\ell,m+1]}-\frac{\hat{\nu}[k,m]^p}{\hat{\nu}[\ell,m]}\bigg)\geqslant 0.$$

Since $p>1$ and $k>\ell$, we have

$$\bigg(\frac{\hat{\nu}[k,m+1]}{\hat{\nu}[k,m]}\bigg)^p>\frac{\hat{\nu}[k,m+1]}{\hat{\nu}[k,m]}=1+\frac{\hat{\nu}_{m+1}}{\hat{\nu}[k,m]}>1+\frac{\hat{\nu}_{m+1}}{\hat{\nu}[\ell,m]}=\frac{\hat{\nu}[\ell,m+1]}{\hat{\nu}[\ell,m]}.$$

So the required assertion holds.

(c) We compare $\overline{\delta}_1$ with σ_p and δ_1'.

For the convenience of comparison of $\overline{\delta}_1$ with σ_p and δ_1', we rewrite $\overline{\delta}_1$ as follows:

$$\overline{\delta}_1=\sup_{l\in E}\bigg(\varphi_\ell\mu[0,\ell]+\frac{1}{\varphi_\ell^{p^*-1}}\sum_{k=\ell+1}^{N}\mu_k\varphi_k^{p(p^*-1)}\bigg).$$

By definition of σ_p, it is clear that $\overline{\delta}_1\geqslant\sigma_p$. To compare $\overline{\delta}_1$ with δ_1', we further change the form of $\overline{\delta}_1$. By definition of φ, we have

$$\begin{aligned}\sum_{j=0}^{N}\mu_j\varphi_{j\vee m}^{p^*}&=\sum_{j=0}^{m-1}\mu_j\varphi_m\sum_{k=m}^{N}\hat{\nu}_k+\sum_{j=m}^{N}\mu_j\varphi_j\sum_{k=j}^{N}\hat{\nu}_k\\&=\sum_{k=m}^{N}\hat{\nu}_k\sum_{j=0}^{m-1}\mu_j\varphi_m+\sum_{k=m}^{N}\hat{\nu}_k\sum_{j=m}^{k}\mu_j\varphi_j.\end{aligned}$$

So

$$\overline{\delta}_1=\sup_{l\in E}\bigg(\frac{1}{\varphi_\ell^{p^*-1}}\sum_{k=0}^{N}\mu_k\varphi_{k\vee\ell}^{p^*}\bigg)=\sup_{m\in E}\bigg(\frac{1}{\varphi_m^{p^*-1}}\sum_{k=m}^{N}\hat{\nu}_k\sum_{j=0}^{k}\mu_j\varphi_{m\vee j}\bigg).$$

Denote $a_\ell(k)=\hat{\nu}_k/\varphi_\ell^{p^*-1}$. Then $\sum_{k=\ell}^{N}a_\ell(k)=1$ (i.e., $\{a_\ell(k)\colon k=\ell,\dots,N\}$ is a probability measure on $\{\ell,\ell+1,\dots,N\}$). By the increasing property of the moments $\mathbb{E}(|X|^s)^{1/s}$ in $s>0$, it follows that

$$\begin{aligned}\delta_1'&=\sup_{\ell\in E}\bigg[\sum_{j=\ell}^{N}a_\ell(j)\bigg(\sum_{k=0}^{j}\mu_k\varphi_{k\vee\ell}\bigg)^{p^*-1}\bigg]^{p-1}\\&\geqslant\sup_{\ell\in E}\sum_{j=\ell}^{N}a_\ell(j)\sum_{k=0}^{j}\mu_k\varphi_{k\vee\ell}\quad(\text{if }p^*-1>1)\\&=\overline{\delta}_1.\end{aligned}$$

Hence, $\overline{\delta}_1\leqslant\delta_1'$ for $1<p\leqslant 2$. Otherwise, $\overline{\delta}_1\leqslant\delta_1'$ for $p\geqslant 2$.

(d) At last, we prove that $\overline{\delta}_1 \leqslant p\sigma_p$. Using the summation by parts formula, we have

$$\sum_{j=0}^{N} \mu_j \varphi_{j\vee\ell}^{p^*} = \sum_{j=\ell}^{N} (\varphi_j^{p^*} - \varphi_{j+1}^{p^*})\mu[0,j].$$

Hence,

$$\begin{aligned}\frac{1}{\varphi_m^{p^*-1}} \sum_{j=0}^{N} \mu_j \varphi_{j\vee m}^{p^*} &= \frac{1}{\varphi_m^{p^*-1}} \sum_{j=m}^{N} (\varphi_j^{p^*} - \varphi_{j+1}^{p^*})\mu[0,j] \\ &\leqslant \sigma_p \frac{1}{\varphi_m^{p^*-1}} \sum_{j=m}^{N} \frac{1}{\varphi_j} (\varphi_j^{p^*} - \varphi_{j+1}^{p^*}) \\ &\leqslant \frac{\sigma_p \sum_{j=m}^{N} \frac{1}{\varphi_j} (\varphi_j^{p^*} - \varphi_{j+1}^{p^*})}{\sum_{j=m}^{N} (\varphi_j^{p^*-1} - \varphi_{j+1}^{p^*-1})} \quad (\text{since } \varphi_{N+1} = 0).\end{aligned}$$

By Young's inequality, we have

$$\varphi_j \varphi_{j+1}^{p^*-1} \leqslant \frac{1}{p^*} \varphi_j^{p^*} + \frac{1}{p} \varphi_{j+1}^{p^*}.$$

Combining this inequality with the proportional property, we obtain

$$\begin{aligned}\frac{1}{\varphi_m^{p^*-1}} \sum_{j=0}^{N} \mu_j \varphi_{j\vee m}^{p^*} &\leqslant \sigma_p \sup_{j\in E} \frac{\varphi_j^{p^*} - \varphi_{j+1}^{p^*}}{\varphi_j (\varphi_j^{p^*-1} - \varphi_{j+1}^{p^*-1})} \\ &= \sigma_p \sup_{j\in E} \frac{\varphi_j^{p^*} - \varphi_{j+1}^{p^*}}{\varphi_j^{p^*} - \varphi_j \varphi_{j+1}^{p^*-1}} \\ &\leqslant p\sigma_p.\end{aligned}$$

So the assertion holds. □

4 DN-case

In this section, we use the same notations as the last section because they play the same role. However, they have different meaning in different sections. Set $E = \{i \in \mathbb{N}\colon 1 \leqslant i < N+1\}$. Let $\{\mu_i\}_{i\in E}$ and $\{\nu_i\}_{i\in E}$ be two positive sequences. Similar to the ND-case, we have the discrete version of p-Laplacian eigenvalue problem with DN-boundaries:

$$\begin{aligned}\text{'Eigenequation': } &\Omega_p\, g(k) = -\lambda \mu_k |g_k|^{p-2} g_k, \quad k \in E; \\ \text{DN-boundary conditions: } &g_0 = 0 \text{ and } g_{N+1} = g_N \text{ if } N < +\infty,\end{aligned}$$

where

$$\Omega_p f(k) = \nu_{k+1} |f_{k+1} - f_k|^{p-2} (f_{k+1} - f_k) - \nu_k |f_k - f_{k-1}|^{p-2} (f_k - f_{k-1}), \quad p > 1,$$

$\nu_{N+1} := 0$ if $N < +\infty$ and $\nu_{N+1} := \lim_{i\to+\infty} \nu_i$ if $N = +\infty$. Let λ_p denote the first eigenvalue. Then

$$\lambda_p = \inf\left\{\frac{D_p(f)}{\mu(|f|^p)} \colon f \neq 0,\ D_p(f) < +\infty\right\}. \tag{14}$$

where

$$\mu(f) = \sum_{k\in E} \mu_k f_k \leqslant +\infty, \quad D_p(f) = \sum_{k\in E} \nu_k |f_k - f_{k-1}|^p, \quad f_0 = 0.$$

The constant λ_p describes the optimal constant $A = \lambda_p^{-1}$ in the following *weighted Hardy inequality*:

$$\mu(|f|^p) \leqslant A D_p(f), \quad f(0) = 0,$$

or equivalently,

$$\|f\|_{L^p(\mu)} \leqslant A^{1/p} \|\partial^- f\|_{L^p(\nu)}, \quad f(0) = 0,$$

where $\partial^- f(k) = f_{k-1} - f_k$. In other words, we are studying again the weighted Hardy inequality in this section. In view of this, by a duality [9; p. 13], the optimal constant $\lambda_p^{-1/p}$ in the last inequality coincides with $\lambda_{p^*}^{-1/p^*}$, which is the optimal constant in the inequality

$$\|f\|_{L^{p^*}(\nu^{1-p^*})} \leqslant {A'}^{1/p^*} \|\partial^+ f\|_{L^{p^*}(\mu^{1-p^*})}, \quad f(N+1) = 0,$$

where $\partial^+ f(k) = f_{k+1} - f_k$, studied in Section 2. However, due to the difference of boundaries in these two cases, the variational formulas and the approximating procedure are different (cf. [4]). Therefore, it is worthy to present some details here. Similar notations as Section 2 are defined as follows. Define $\hat{\nu}_j = \nu_j^{1-p^*}$ for $j \in E$, and

$$I_i(f) = \frac{1}{\nu_i (f_i - f_{i-1})^{p-1}} \sum_{j=i}^{N} \mu_j f_j^{p-1} \quad \text{(single summation form)},$$

$$II_i(f) = \frac{1}{f_i^{p-1}} \left[\sum_{j=1}^{i} \hat{\nu}_j \left(\sum_{k=j}^{N} \mu_k f_k^{p-1}\right)^{p^*-1}\right]^{p-1} \quad \text{(double summation form)},$$

$$R_i(w) = \mu_i^{-1}[\nu_i (1 - w_{i-1}^{-1})^{p-1} - \nu_{i+1}(w_i - 1)^{p-1}],\ w_0 := +\infty \ \text{(difference form)}.$$

For the lower bounds, the domains of the operators are defined, respectively, as follows:

$$\mathscr{F}_I = \{f \colon f > 0 \text{ and is strictly increasing on } E\},$$
$$\mathscr{F}_{II} = \{f \colon f > 0 \text{ on } E\},$$
$$\mathscr{W} = \{w \colon w_i > 1 \text{ for } i \in E\}.$$

Note that the test function given in $\mathscr{F}_I$ is different from that given in Section 2. Again, this is due to the property of eigenfunction (which is proved in Proposition 4.6 later). For the upper bounds, we need modify these sets as follows:

$$\begin{aligned}\widetilde{\mathscr{F}}_I &= \{f\colon \exists\, m\in E \text{ such that } f \text{ is strictly increasing on } \{1,\ldots,m\},\\ &\qquad \text{and } f_\cdot = f_{\cdot\wedge m}\},\\ \widetilde{\mathscr{F}}_{II} &= \{f\colon f_\cdot = f_{\cdot\wedge m}>0 \text{ for some } m\in E\},\\ \widetilde{\mathscr{W}} &= \bigcup_{m\in E}\{w\colon 1<w_i<1+\nu_i^{p^*-1}(1-w_{i-1}^{-1})\nu_{i+1}^{1-p^*} \text{ for } 1\leqslant i\leqslant m-1\\ &\qquad\qquad \text{and } w_i=1 \text{ for } i\geqslant m\}.\end{aligned}$$

Define $\widetilde{R}$ acting on $\widetilde{\mathscr{W}}$ as a modified form of R by replacing μ_m with $\widetilde{\mu}_m := \sum_{k=m}^N \mu_k$ in $R_i(w)$ for the same m in $\widetilde{\mathscr{W}}$. The change of μ_m is due to the Neumann boundary at right endpoint. Note that if $w_i=1$ for every $i\geqslant m$, then

$$\widetilde{R}_i(w)=R_i(w)=0,\quad i>m.$$

Besides, we also need the following set:

$$\widetilde{\mathscr{F}}'_{II} = \{f\colon f>0 \text{ on } E \text{ and } fII(f)^{p^*-1}\in L^p(\mu)\}.$$

If $\sum_{i\in E}\mu_i=+\infty$, let $f_i=1$ for $i\in E$ and $f_0=0$. Then

$$D_p(f)=\sum_{k=1}^N \nu_k|f_k-f_{k-1}|^p=\nu_1<+\infty \qquad\text{and}\qquad \mu(|f|^p)=+\infty.$$

So $\lambda_p=0$ by (14). If $\sum_{i\in E}\mu_i<+\infty$, then as we will prove later (Lemma 4.5) that λ_p coincides with

$$\lambda_p^{[1]} := \inf\{D_p(f)\colon \mu(|f|^p)=1\}.$$

Actually, the later is also coincides with

$$\lambda_p^{[1]} = \inf\{D_p(f)\colon \mu(|f|^p)=1,\ f_i=f_{i\wedge m} \text{ for some } m\in E\}.$$

Now, we introduce the main results, many of which are parallel to that in Section 2. However, the exchange of boundary conditions 'D' and 'N' makes many difference. For example, the results related to $\widetilde{R}$, the definition of σ_p (see Theorem 4.2 later), and so on.

Theorem 4.1 *Assume that $p>1$ and $\sum_{k=1}^N \mu_i<+\infty$. Then the following variational formulas holds for λ_p (equivalently, $\lambda_p^{[1]}$ or $\lambda_p^{[2]}$).*

(i) *Single summation forms*:

$$\inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{i\in E} I_i(f)^{-1} = \lambda_p = \sup_{f\in\mathscr{F}_I}\inf_{i\in E} I_i(f)^{-1}.$$

(ii) *Double summation forms*:

$$\lambda_p = \inf_{f\in\widetilde{\mathscr F}_{II}} \sup_{i\in E} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr F}_{I}} \sup_{i\in E} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr F}'_{II}\cup\widetilde{\mathscr F}_{II}} \sup_{i\in E} II_i(f)^{-1},$$

$$\lambda_p = \sup_{f\in\mathscr F_{II}} \inf_{i\in E} II_i(f)^{-1} = \sup_{f\in\mathscr F_{I}} \inf_{i\in E} II_i(f)^{-1}.$$

(iii) *Difference forms*:

$$\inf_{w\in\widetilde{\mathscr W}} \sup_{i\in E} \widetilde R_i(w) = \lambda_p = \sup_{w\in\mathscr W} \inf_{i\in E} R_i(w).$$

As an application of the variational formulas in Theorem 4.1 (i), we have the following theorem. This result was known in 1990s (cf. [8; Theorem 7] plus the duality technique, cf. [9; p. 13]). See also [10]. It can be regarded as a dual of Theorem 2.3.

Theorem 4.2 (Basic estimates) *For $p>1$, we have λ_p (or equivalently, $\lambda_p^{[1]}$ or $\lambda_p^{[2]}$ provided $\sum_{k\in E}\mu_k<+\infty$) is positive if and only if $\sigma_p<+\infty$, where*

$$\sigma_p = \sup_{n\in E}(\mu[n,N]\hat\nu[1,n]^{p-1}).$$

More precisely,

$$(k(p)\sigma_p)^{-1} \leqslant \lambda_p \leqslant \sigma_p^{-1},$$

where $k(p)=pp^{*p-1}$. In particular, we have $\lambda_p=0$ if $\sum_{i\in E}\mu_i=+\infty$ and $\lambda_p>0$ if $N<+\infty$, or $\sum_{k=1}^{+\infty}\mu[k,N]^{p^*-1}\nu_k<+\infty$, or $\sum_{k=1}^{+\infty}(\mu_k+\hat\nu_k)<+\infty$.

The next result is an application of the variational formulas in Theorem 4.1 (ii). It is interesting that the result is not a direct dual of Theorem 2.4.

Theorem 4.3 (Approximating procedure) *Assume that $p>1$, $\sum_{k\in E}\mu_k<+\infty$, and $\sigma_p<+\infty$. Then the following assertions hold.*

(i) *Define*

$$f_1=\hat\nu[1,\cdot]^{1/p^*},\quad f_n=f_{n-1}II(f_{n-1})^{p^*-1},\quad \delta_n=\sup_{i\in E} II_i(f_n).$$

Then δ_n is decreasing in n and

$$\lambda_p \geqslant \delta_\infty^{-1} \geqslant \cdots \geqslant \delta_1^{-1} \geqslant (k(p)\sigma_p)^{-1}.$$

(ii) *For fixed $m\in E$, define*

$$f_1^{(m)}=\hat\nu[1,\cdot\wedge m],\quad f_n^{(m)}=f_{n-1}^{(m)}II(f_{n-1}^{(m)})(\cdot\wedge m)^{p^*-1},\ n\geqslant 2,$$

and

$$\delta'_n=\sup_{m\in E}\inf_{i\in E} II_i(f_n^{(m)}).$$

Then δ'_n is increasing in n and

$$\sigma_p^{-1} \geqslant {\delta'_1}^{-1} \geqslant \cdots \geqslant {\delta'_\infty}^{-1} \geqslant \lambda_p.$$

Next, define

$$\overline{\delta}_n = \sup_{m\in E} \frac{\mu(f_n^{(m)p})}{D_p(f_n^{(m)})}, \quad n \in E.$$

Then $\overline{\delta}_n \geqslant \lambda_p$ and $\overline{\delta}_{n+1} \geqslant \delta'_n$ for every $n \geqslant 1$.

Corollary 4.4 (Improved estimates) *Assume that $\sum_{k\in E}\mu_k < +\infty$. For $p > 1$, we have*

$$\sigma_p^{-1} \geqslant {\delta'_1}^{-1} \geqslant \lambda_p^{-1} \geqslant \delta_1^{-1} \geqslant (k(p)\sigma_p)^{-1},$$

where

$$\delta_1 = \sup_{i\in E}\left[\hat\nu[1,i]^{-1/p^*}\sum_{j=1}^{i}\hat\nu_j\left(\sum_{k=j}^{N}\mu_k\hat\nu[1,k]^{(p-1)/p^*}\right)^{p^*-1}\right]^{p-1},$$

$$\delta'_1 = \sup_{m\in E}\frac{1}{\hat\nu[1,m]^{p-1}}\left[\sum_{j=1}^{m}\hat\nu_j\left(\sum_{k=j}^{N}\mu_k\hat\nu[1,k\wedge m]^{p-1}\right)^{p^*-1}\right]^{p-1}.$$

Moreover,

$$\overline{\delta}_1 = \sup_{m\in E}\frac{1}{\hat\nu[1,m]}\sum_{j=1}^{N}\mu_j\hat\nu[1,j\wedge m]^p \in [\sigma_p, p\sigma_p],$$

and $\overline{\delta}_1 \geqslant \delta'_1$ for $p \geqslant 2$, $\overline{\delta}_1 \leqslant \delta'_1$ for $1 < p \leqslant 2$.

When $p = 2$, the result that $\delta'_1 = \overline{\delta}_1$ is also known (see [4; Theorem 4.3]).

4.1 Partial proofs of main results

Before moving to the proofs of the main results, we give some more descriptions of λ_p. Define

$$\lambda_p^{(m)} = \inf\{D_p(f)\colon \mu(|f|^p) = 1,\ f_i = f_{i\wedge m},\ i \in E\}.$$

Let

$$\widetilde{D}_p(f) = \sum_{i=1}^{m}\widetilde{\nu}_i|f_i - f_{i-1}|^p, \quad \widetilde{\mu}(f) = \sum_{i=1}^{m}\widetilde{\mu}_i|f_i|^p,$$

where $\widetilde{\nu}$ and $\widetilde{\mu}$ are defined as follows:

$$\widetilde{\nu}_i = \nu_i\ (i \leqslant m); \quad \widetilde{\mu}_i = \mu_i\ (i \leqslant m-1), \quad \widetilde{\mu}_m = \sum_{k=m}^{N}\mu_k.$$

For $f = f_{\cdot\wedge m}$, we have

$$\widetilde{D}_p(f) = D_p(f), \quad \widetilde{\mu}(|f|^p) = \mu(|f|^p).$$

So $\lambda_p^{(m)}$ is the first eigenvalue of the local Dirichlet form $(\widetilde{D}, \mathscr{D}(\widetilde{D}))$ on $E^{(m)} := \{1, 2, \ldots, m\}$ with reflecting (Neumann) boundary at $m+1$ and absorbing (Dirichlet) boundary at 0. Furthermore, we have the following fact.

Lemma 4.5 *If $\sum_{i\in E} \mu_i < +\infty$, then $\lambda_p = \lambda_p^{[1]} = \lambda_p^{[2]}$. Moreover, $\lambda_p^{(m)} \downarrow \lambda_p^{[2]}$ as $m \to N$.*

Proof Since each f with $\mu(|f|^p) = +\infty$ can be approximated by $f_i^{(m)} = f_{i\wedge m}$ ($m \in E$) with respect to norm $\|\cdot\|^p = D_p(\cdot) + \mu(|\cdot|^p)$, it is clear that $\lambda_p = \lambda_p^{[1]}$.

We now prove that $\lambda_p^{[1]} = \lambda_p^{[2]}$. It is clear that $\lambda_p^{[2]} \geqslant \lambda_p^{[1]}$ since $\sum_{k\in E} \mu_k < +\infty$. For any fixed $\varepsilon > 0$, there exists $f \in L^p(\mu)$ such that

$$\frac{D_p(f)}{\mu(|f|^p)} \leqslant \lambda_p^{[1]} + \varepsilon.$$

Let $f^{(n)} = f_{\cdot\wedge n}$. Then

$$D_p(f^{(n)}) \to D_p(f), \quad \mu(|f^{(n)}|^p) \to \mu(|f|^p), \quad n \to N.$$

By definitions of $\lambda_p^{(n)}$ and $\lambda_p^{[2]}$, for large enough $n \in E$, we have

$$\lambda_p^{[2]} \leqslant \lambda_p^{(n)} \leqslant \frac{D_p(f^{(n)})}{\mu(|f^{(n)}|^p)} \leqslant \frac{D_p(f)}{\mu(|f|^p)} + \varepsilon \leqslant \lambda_p^{[1]} + 2\varepsilon \leqslant \lambda_p^{[2]} + 2\varepsilon.$$

Hence, $\lambda_p^{[1]} = \lambda_p^{[2]}$ and $\lambda_p^{(m)} \downarrow \lambda_p^{[2]}$. □

Proof of Theorem 4.1 In parallel to the ND-case, we also adopt two circle arguments to prove the theorem. For instance, the circle argument below is adopted for the upper estimates:

$$\begin{aligned}
\lambda_p &\leqslant \inf_{f\in\widetilde{\mathscr{F}}'_{II}\cup\widetilde{\mathscr{F}}_{II}} \sup_{i\in E} II_i(f)^{-1} \\
&\leqslant \inf_{f\in\widetilde{\mathscr{F}}_{II}} \sup_{i\in E} II_i(f)^{-1} \\
&= \inf_{f\in\widetilde{\mathscr{F}}_{I}} \sup_{i\in E} II_i(f)^{-1} \\
&= \inf_{f\in\widetilde{\mathscr{F}}_{I}} \sup_{i\in E} I_i(f)^{-1} \\
&\leqslant \inf_{w\in\widetilde{\mathscr{W}}} \sup_{i\in E} \widetilde{R}_i(w) \\
&\leqslant \lambda_p.
\end{aligned}$$

The proofs are similar to the ND-case. Here, we present the proofs of the assertions related to the operator $\widetilde{R}$ in the above circle argument, which are obvious different from that in Section 2 due to the boundary conditions.

(i) We first prove

$$\inf_{f\in\widetilde{\mathscr{F}}_{II}} \sup_{i\in E} II_i(f)^{-1} \leqslant \inf_{w\in\widetilde{\mathscr{W}}} \sup_{i\in E} \widetilde{R}_i(w).$$

For $w \in \widetilde{\mathscr{W}}$, it is easy to check that $\widetilde{R}_i(w) > 0$. Let $u_0 = 0$ and $u_i = u_{i\wedge m} > 0$ for $i \in E$ such that $w_i\ (= u_{i+1}/u_i) \in \widetilde{\mathscr{W}}$. Then u_i is strictly increasing on $[0, m]$. Put

$$f_i^{p-1} = \begin{cases} \mu_i^{-1}[\nu_i(u_i - u_{i-1})^{p-1} - \nu_{i+1}(u_{i+1} - u_i)^{p-1}], & i \leqslant m-1, \\ \widetilde{\mu}_m^{-1}\nu_m(u_m - u_{m-1})^{p-1}, & i \geqslant m. \end{cases}$$

We have $f \in \widetilde{\mathscr{F}}_{II}$ and

$$\Omega_p u(k) = -\mu_k f_k^{p-1}, \quad k \leqslant m-1,$$
$$\nu_m(u_m - u_{m-1})^{p-1} = \sum_{j=m}^{N} \mu_j f_m^{p-1} = \sum_{j=m}^{N} \mu_j f_j^{p-1}.$$

By a simple reorganization and making summation from 1 to i $(\leqslant m)$ with respect to k, we obtain

$$\sum_{k=1}^{i} \hat{\nu}_k \bigg(\sum_{j=k}^{N} \mu_j f_j^{p-1}\bigg)^{p^*-1} = u_i - u_0 = u_i, \quad i \leqslant m.$$

Therefore,

$$\widetilde{R}_i(w)^{p^*-1} = \frac{f_i}{u_i} = f_i\bigg[\sum_{k=1}^{i} \hat{\nu}_k \bigg(\sum_{j=k}^{N} \mu_j f_j^{p-1}\bigg)^{p^*-1}\bigg]^{-1} = \mathit{II}_i(f)^{1-p^*}, \quad i \leqslant m,$$

and then

$$\sup_{i \in E} \widetilde{R}_i(w) \geqslant \max_{i \leqslant m} \widetilde{R}_i(w) \geqslant \sup_{i \in E} \mathit{II}_i(f)^{-1} \geqslant \inf_{f \in \widetilde{\mathscr{F}}_{II}} \sup_{i \in \mathrm{supp}(f)} \mathit{II}_i(f)^{-1}.$$

(ii) We prove that

$$\inf_{w \in \widetilde{\mathscr{W}}} \sup_{i \in E} \widetilde{R}_i(w) \leqslant \lambda_p.$$

Let g denote the solution to the equation

$$-\Omega_p g(i) = \lambda_p^{(m)} \widetilde{\mu}_i |g_i|^{p-2} g_i, \quad g_0 = 0, \quad g_{m+1} := g_m, \quad i \in E^{(m)} := \{0, 1, \ldots, m\}.$$

Without loss of generality, assume $g_1 > 0$. Then g is strictly increasing (by Proposition 4.6 below) and

$$-\nu_{i+1}(g_{i+1} - g_i)^{p-1} + \nu_i(g_i - g_{i-1})^{p-1} = \lambda_p^{(m)} \mu_i g_i^{p-1}, \quad i \leqslant m-1,$$
$$\nu_m(g_m - g_{m-1})^{p-1} = \lambda_p^{(m)} \widetilde{\mu}_m g_m^{p-1}.$$

That is,

$$\nu_i\Big(1 - \frac{g_{i-1}}{g_i}\Big)^{p-1} - \nu_{i+1}\Big(\frac{g_{i+1}}{g_i} - 1\Big)^{p-1} = \lambda_p^{(m)} \mu_i, \quad i \leqslant m-1;$$

$$\nu_m\Big(1-\frac{g_{m-1}}{g_m}\Big)^{p-1}=\lambda_p^{(m)}\widetilde{\mu}_m.$$

Let $\overline{w}_i = g_{i+1}/g_i$ for $i \leqslant m-1$ and $\overline{w}_i = 1$ for $i \geqslant m$. Then $\overline{w} \in \widetilde{\mathscr{W}}$ and $\widetilde{R}_i(w) = \lambda_p^{(m)}$ for every $i \leqslant m$. Therefore,

$$\begin{aligned}\lambda_p^{(m)} &= \max_{0\leqslant i\leqslant m}\widetilde{R}_i(\overline{w})\\ &\geqslant \inf_{w\in\widetilde{\mathscr{W}}:\ w_i=w_{i\wedge m}}\ \sup_{0\leqslant i\leqslant m}\widetilde{R}_i(w)\\ &\geqslant \inf_{w\in\widetilde{\mathscr{W}}:\ w_i=w_{i\wedge n}\text{ for some }n\in E}\ \sup_{i\in E}\widetilde{R}_i(w)\\ &\geqslant \inf_{w\in\widetilde{\mathscr{W}}}\sup_{i\in E}\widetilde{R}_i(w).\end{aligned}$$

We obtain the required assertion by letting $m \to N$. □

Noticing the difference between the ND- and the DN-cases, one may finish the proofs of other theorems in this section without much difficulties following Section 2 or [4; Section 4]. So we ignore the details here but present one proposition below, which is essential to our study and is used to verify the last inequalities related to R or $\widetilde{R}$ in the two circle arguments. The proposition, whose proof is independent of the other assertions in this paper, provides the basis for imitating the eigenfunction to construct the corresponding test functions of the operators.

Proposition 4.6 *Assume that g is a nontrivial solution to p-Laplacian problem with DN-boundary conditions. Then g is monotone. Moreover, g is increasing provided $g_1 > 0$.*

Proof The proof is parallel to that in [3; Proposition 3.4]. We give the skeleton of the proof. The proof is quite easy in the case of $\lambda_p = 0$. For the case that $\lambda_p > 0$, suppose that there exists $n \in E$ such that $g_0 < g_1 < \cdots < g_n \geqslant g_{n+1}$. Then define $\overline{g}_i = g_{i\wedge n}$. By a simple calculation and (14), we obtain

$$\lambda_p \leqslant \frac{D_p(\overline{g})}{\mu(|\overline{g}|^p)} = \frac{\lambda_p\sum_{k=1}^{n-1}\mu_k|g_k|^p + \nu_n|g_n-g_{n-1}|^{p-2}(g_n-g_{n-1})g_n}{\sum_{k=1}^{n-1}\mu_k|g_k|^p + |g_n|^p\sum_{k=n}^{N}\mu_k} < \lambda_p.$$

In the last inequality, we have used the following fact:

$$\nu_n|g_n-g_{n-1}|^{p-2}(g_n-g_{n-1})g_n \leqslant -g(n)\Omega_p g(n) = \lambda_p\mu_n|g_n|^p < \lambda_p|g_n|^p\sum_{k=n}^{N}\mu_k$$

for $n < N$. Therefore, there is a contradiction and so the required assertion holds. □

Acknowledgements The authors would like to thank Professors Yonghua Mao and Yutao Ma for their helpful comments and suggestions. This work was supported in part by the National Natural Science Foundation of China (Grant No. 11131003), the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20100003110005), the '985' project from the Ministry of Education in China, and the Fundamental Research Funds for the Central Universities.

References

1. Chen M F. Explicit bounds of the first eigenvalue. Sci China Ser A, 2000, 43(10): 1051–1059
2. Chen M F. Variational formulas and approximation theorems for the first eigenvalue in dimension one. Sci China Ser A, 2001, 44(4): 409–418
3. Chen M F. Eigenvalues, Inequalities, and Ergodic Theory. New York: Springer, 2005
4. Chen M F. Speed of stability for birth-death process. Front Math China, 2010, 5(3): 379–516
5. Chen M F. Bilateral Hardy-type inequalities. Acta Math Sin (Engl Ser), 2013, 29(1): 1–32
6. Chen M F, Wang L D, Zhang Y H. Mixed principal eigenvalues in dimension one. Front Math China, 2013, 8(2): 317–343
7. Jin H Y, Mao Y H. Estimation of the optimal constants in the L^p-Poincaré inequalities on the half line. Acta Math Sinica (Chin Ser), 2012, 55(1): 169–178 (in Chinese)
8. Kufner A, Maligranda L, Persson L E. The Hardy Inequality: About Its History and Some Related Results. Plzen: Vydavatelsky Servis Publishing House, 2007
9. Kufner A, Persson L E. Weighted Inequalities of Hardy Type. Singapore: World Sci, 2003
10. Mao Y H. Nash inequalities for Markov processes in dimension one. Acta Math Sin (Engl Ser), 2002, 18(1): 147–156
11. Opic B, Kufner A. Hardy Type Inequalities. Harlow: Longman Scientific and Technical, 1990