

# Unified representation of formulas for single birth processes

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**Abstract** Based on a new explicit representation of the solution to the Poisson equation with respect to single birth processes, the unified treatment for various criteria on classical problems (including uniqueness, recurrence, ergodicity, exponential ergodicity, strong ergodicity, as well as extinction probability, etc.) for the processes are presented.

**Keywords** Single birth process, Poisson equation, uniqueness, recurrence, ergodicity, moments of return time

**MSC** 60J60

## 1 Introduction

Consider a continuous-time homogeneous Markov chains  $\{X(t): t \geq 0\}$ , on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , with transition probability matrix  $P(t) = (p_{ij}(t))$  on a countable state space  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ . We call  $\{X(t): t \geq 0\}$  a single birth process if its transition rate (density) matrix  $Q = (q_{ij}: i, j \in \mathbb{Z}_+)$  is irreducible and satisfies that  $q_{i,i+1} > 0$ ,  $q_{i,i+j} = 0$  for all  $i \in \mathbb{Z}_+$  and  $j \geq 2$ . Such a matrix  $Q = (q_{ij})$  with  $\sum_j q_{ij} = 0$  for every  $i$  (conservativity) is called a single birth  $Q$ -matrix. Refer to [15]. In the literature, the single birth process is also called upwardly skip-free process, or birth and death process with catastrophes (cf. [1–3] for instance).

The single birth process, as a natural extension of birth and death process which is a simplest  $Q$ -process (Markov chain), has its own origins in practice, refer to the earlier papers [2,13,15], for instance. The exit boundary of the process consists at most one single extremal point and so the single birth process is nearly the largest class for which the explicit criteria on classical problems can be expected. Actually, the study on the object is quite fruited and relatively

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completed (cf. [4–6,15–17]). Based on this advantage, the single birth process becomes a fundamental comparison tool in studying more complex processes, such as infinite-dimensional reaction-diffusion processes. Refer to [6; Chapters 3 and 4, Part III] and [15]. Usually, the single birth process is non-symmetric and hence it is regarded as a representative one of the non-symmetric processes. For non-symmetric processes, comparing with the symmetric ones, our knowledge is much limited, except for single birth processes to which much results are known as just mentioned. Up to now, the known results are all presented in some recursive forms. This paper introduces a single unified representation, as well as a unified treatment, of various formulas for single birth processes.

Throughout the paper, we consider only the single birth  $Q$ -matrix  $Q = (q_{ij})$ . Set  $q_i = -q_{ii}$  for each  $i \in \mathbb{Z}_+$ . For a given function  $c$  (to be fixed in this and the next sections, and then to be specified case by case), define an operator  $\Omega$  as follows:

$$\Omega g = Qg + cg,$$

where

$$(Qg)_i = \sum_j q_{ij}(g_j - g_i).$$

Clearly, if  $c \leq 0$ , then  $\Omega$  is an operator corresponding to a single birth process with killing rates  $(-c_i)$ .

The following sequences are used throughout this paper:

$$\tilde{F}_i^{(i)} = 1, \quad \tilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \tilde{F}_k^{(i)}, \quad n > i \geq 0, \quad (1.1)$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^k q_{nj} - c_n, \quad 0 \leq k < n. \quad (1.2)$$

Note that if  $c \leq 0$ , then  $\tilde{q}_n^{(k)} \geq 0$  and then  $\tilde{F}_n^{(k)} \geq 0$  for every  $n > k \geq 0$ . In what follows, we omit the superscript ‘ $\sim$ ’ everywhere in  $\tilde{F}$  and  $\tilde{q}$  once  $c_i \equiv 0$ , and often use the convention that  $\sum_{\emptyset} = 0$ .

Here is the first of our main results.

**Theorem 1.1** *Given a single-birth  $Q$ -matrix  $Q = (q_{ij})$  and functions  $c$  and  $f$ , the solution  $g$  to the Poisson equation*

$$\Omega g = f \quad (1.3)$$

*has the following representation:*

$$g_n = g_0 + \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \quad n \geq 0. \quad (1.4)$$

*In particular, the harmonic function  $g$  of  $\Omega$  (i.e.,  $\Omega g = 0$ ) can be represented as*

$$g_n = g_0 \left( 1 - \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad n \geq 0.$$

Conversely, for each boundary/initial value  $g_0 \in \mathbb{R}$ , the function  $(g_n)$  defined by (1.4) is a solution to (1.3).

For single birth processes, almost all problems we concerned with are related to the solutions to some specific Poisson equation. Here, we unify these equations as (1.3) with different functions  $c$  and  $f$  which are listed as in Table 1.

Table 1

Problem	$c_i \in \mathbb{R}$	$f_i \in \mathbb{R}$
Harmonic function	$c_i \in \mathbb{R}$	$f_i \equiv 0$
Uniqueness	$c_i \equiv -\lambda < 0$	$f_i \equiv 0$
Recurrence	$c_i \equiv 0$	$f_i = q_{i0}(1 - \delta_{i0})$
Extinction/return probability	$c_i \equiv 0$	$f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$
Ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1 - \delta_{i0})g_0 - 1$
Strong ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1 - \delta_{i0})g_0 - 1$
Polynomial moment	$c_i \equiv 0$	$f_i^{(\ell)} = q_{ii_0}(1 - \delta_{ii_0})g_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}$
Exponential moment/ergodicity	$c_i \equiv \lambda > 0$	$f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$
Laplace transform of return time	$c_i \equiv -\lambda < 0$	$f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$

We remark that in the two cases for ergodicity and strong ergodicity, even though the Poisson equation and the functions  $c$  and  $f$  are the same, but their solutions are required to be finite and bounded, respectively.

This paper is organized as follows. The proof of Theorem 1.1 is given in the next section, using a lemma on the representation of solution to a class of linear equations. Then, Sections 3–7 are devoted, respectively, to the criteria on the problems listed in Table 1, and related problems to be specific subsequently. Roughly speaking, the unified treatment presented in the paper consists of the following three steps.

- (a) Find out the Poisson equation corresponding to the problem we are interested in.
- (b) Apply Theorem 1.1 to get the solution to the Poisson equation.
- (c) Work out a criterion for the problem using the solution obtained in (b).

Step (a) is more or less known from the previous study; step (b) is now automatic; hence, our main work is spent on step (c).

For the reader’s convenience, several key formulas used often in the proofs are collected into Appendix in a single page which consists the last page of the paper (so that it can be printed out separately).

## 2 Poisson equation

In this section, we consider the solutions of the Poisson equation (1.3) for single birth processes. Let us begin with a simple result for the solution to a class of linear equations.

**Lemma 2.1** For given real numbers  $(\alpha_{nk})_{n-1 \geq k \geq 0}$  and  $(f_n)_{n \geq 0}$ , the solution  $(g_n)_{n \geq 0}$  to the recursive inhomogeneous equations

$$g_n = \sum_{0 \leq k \leq n-1} \alpha_{nk} g_k + f_n, \quad n \geq 0, \quad (2.1)$$

can be represented as

$$g_n = \sum_{0 \leq k \leq n} \gamma_{nk} f_k, \quad n \geq 0, \quad (2.2)$$

where for fixed  $k \geq 0$ ,  $(\gamma_{nk})_{n \geq k}$  with  $\gamma_{kk} = 1$  is the solution to the recursive equations

$$\gamma_{nk} = \sum_{k \leq j \leq n-1} \alpha_{nj} \gamma_{jk}, \quad n > k. \quad (2.3)$$

*Proof* Use induction. For  $n = 0$ , we have

$$g_0 = f_0 = \gamma_{00} f_0 = \sum_{0 \leq k \leq 0} \gamma_{0k} f_k.$$

Assume that (2.2) holds for all  $n \leq m$ . When  $n = m + 1$ , from (2.1), we see that

$$\begin{aligned} g_{m+1} &= \sum_{0 \leq k \leq m} \alpha_{m+1,k} g_k + f_{m+1} \\ &= \sum_{0 \leq k \leq m} \alpha_{m+1,k} \sum_{0 \leq \ell \leq k} \gamma_{k\ell} f_\ell + f_{m+1} \\ &= \sum_{0 \leq \ell \leq m} \left( \sum_{\ell \leq k \leq m} \alpha_{m+1,k} \gamma_{k\ell} \right) f_\ell + f_{m+1} \\ &= \sum_{0 \leq \ell \leq m} \gamma_{m+1,\ell} f_\ell + f_{m+1} \\ &= \sum_{0 \leq \ell \leq m+1} \gamma_{m+1,\ell} f_\ell. \end{aligned}$$

Hence, (2.2) holds for  $n = m + 1$ . By induction, the representation (2.2) holds for all  $n \geq 0$ .  $\square$

Note that the coefficients  $(\alpha_{nk})$  are often fixed and so are  $(\gamma_{nk})$ . Then Lemma 2.1 says that once replacing  $(\alpha_{nk})$  by  $(\gamma_{nk})$ , the solution to (2.1) has a complete representation (2.2), mainly in terms of the inhomogeneous term  $(f_n)$  in (2.1).

Without condition  $\gamma_{kk} = 1$ , equation (2.3) is clearly homogeneous. However, it becomes inhomogeneous under condition  $\gamma_{kk} \neq 0$  (then one may assume that  $\gamma_{kk} = 1$ ):

$$\gamma_{nk} = \sum_{k+1 \leq j \leq n-1} \alpha_{nj} \gamma_{jk} + \alpha_{nk} \gamma_{kk}, \quad n \geq k + 1,$$

provided  $\alpha_{k+1,k} \neq 0$ . Otherwise, once  $\alpha_{k+1,k} = 0$ , by induction, we actually have  $\gamma_{nk} = 0$  for all  $n \geq k + 1$ . Thus, under condition  $\gamma_{kk} = 1$ , by Lemma 2.1 (for fixed  $k$ ), we have the following alternative representation of  $(\gamma_{nk})$ :

$$\gamma_{nk} = \sum_{k+1 \leq j \leq n} \gamma_{nj} \alpha_{jk}, \quad n \geq k + 1.$$

In what follows, we will use the following variant of Lemma 2.1. Replacing the initial 0 by  $i$  and the coefficient  $(\alpha_{nk})$  by  $(\alpha_{nk}\beta_k)$ , respectively, for some non-zero sequence  $(\beta_n)$ , and set  $h_n = g_n/\beta_n$  ( $n \geq i$ ), we obtain the following result.

**Corollary 2.2** *The solution  $(h_n)_{n \geq i}$  to the recursive equations*

$$h_n = \frac{1}{\beta_n} \left( \sum_{i \leq k \leq n-1} \alpha_{nk} h_k + f_n \right), \quad n \geq i, \tag{2.4}$$

can be represented as

$$h_n = \sum_{i \leq k \leq n} \frac{\gamma_{nk}}{\beta_k} f_k, \quad n \geq i, \tag{2.5}$$

where for each fixed  $i$ ,  $(\gamma_{ni})_{n \geq i}$  with  $\gamma_{ii} = 1$  is the solution to the equations

$$\gamma_{ni} = \frac{1}{\beta_n} \sum_{i \leq k \leq n-1} \alpha_{nk} \gamma_{ki}, \quad n > i.$$

Equivalently,

$$\gamma_{ii} = 1, \quad \gamma_{ni} = \sum_{i+1 \leq k \leq n} \frac{\gamma_{nk}}{\beta_k} \alpha_{ki}, \quad n \geq i + 1. \tag{2.6}$$

Specifying  $\beta_n = q_{n,n+1}$  and  $\alpha_{nk} = \tilde{q}_n^{(k)}$  in Corollary 2.2 and using the successive formula of  $\tilde{F}_n^{(k)}$  defined in (1.1), we obtain the following result.

**Corollary 2.3** *For given  $f$ , the sequence  $(h_n)$  defined successively by*

$$h_n = \frac{1}{q_{n,n+1}} \left( f_n + \sum_{i \leq k \leq n-1} \tilde{q}_n^{(k)} h_k \right), \quad n \geq i,$$

has a unified expression as follows:

$$h_n = \sum_{k=i}^n \frac{\tilde{F}_n^{(k)}}{q_{k,k+1}} f_k, \quad n \geq i.$$

In particular, the sequence  $(\tilde{F}_n^{(k)})$  defined in (1.1) has the following expression:

$$\tilde{F}_i^{(i)} = 1, \quad \tilde{F}_n^{(i)} = \sum_{k=i+1}^n \frac{\tilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, \quad n \geq i + 1. \tag{2.7}$$

Before moving further, let us mention a comparison result for different  $\gamma_{nj}$ , which may be useful elsewhere but not in this paper.

**Proposition 2.4** *For each triple  $n \geq i > j$ , the following assertion holds:*

$$\gamma_{nj} = \sum_{i \leq k \leq n} \frac{\gamma_{nk}}{\beta_k} \sum_{j \leq \ell \leq i-1} \alpha_{k\ell} \gamma_{\ell j}. \tag{2.8}$$

Furthermore, if  $\alpha_{nk} \geq 0$  and  $\beta_n > 0$  for all  $n > k$ , then  $\gamma_{ni} \gamma_{ij} \leq \gamma_{nj}$  for all  $n \geq i \geq j$ .

*Proof* The first assertion is simply a consequence of Corollary 2.2. In fact, for fixed  $i > j$ , take

$$f_n = \sum_{j \leq \ell \leq i-1} \alpha_{n\ell} \gamma_{\ell j}, \quad n \geq i.$$

Then

$$\gamma_{nj} = \frac{1}{\beta_n} \left[ \sum_{i \leq \ell \leq n-1} \alpha_{n,\ell} \gamma_{\ell j} + \sum_{j \leq \ell \leq i-1} \alpha_{n\ell} \gamma_{\ell j} \right] = \frac{1}{\beta_n} \left[ \sum_{i \leq \ell \leq n-1} \alpha_{n\ell} \gamma_{\ell j} + f_n \right], \quad n \geq i.$$

Hence, by Corollary 2.2, we get

$$\gamma_{nj} = \sum_{i \leq k \leq n} \frac{\gamma_{nk}}{\beta_k} f_k = \sum_{i \leq k \leq n} \frac{\gamma_{nk}}{\beta_k} \sum_{j \leq \ell \leq i-1} \alpha_{k\ell} \gamma_{\ell j}, \quad n \geq i.$$

If  $\alpha_{nk} \geq 0$  and  $\beta_n > 0$  for all  $n$  and  $k$ , then from (2.8), it follows that for all  $n > i > j$ ,

$$\gamma_{nj} = \gamma_{ni} \gamma_{ij} + \sum_{i+1 \leq k \leq n} \frac{\gamma_{nk}}{\beta_k} \sum_{j \leq \ell \leq i-1} \alpha_{k\ell} \gamma_{\ell j} \geq \gamma_{ni} \gamma_{ij}.$$

In the cases of  $n = i$  or  $i = j$ , the conclusion is trivial. □

Now, we turn to prove our first result.

*Proof of Theorem 1.1* For each  $i \geq 0$ , we have

$$\begin{aligned} (\Omega g)_i &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} q_{ij} \sum_{k=j}^{i-1} (g_{k+1} - g_k) + c_i g_i \\ &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq k \leq i-1} \sum_{j=0}^k q_{ij} (g_{k+1} - g_k) + c_i g_i \\ &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq k \leq i-1} \left( \sum_{j=0}^k q_{ij} - c_i \right) (g_{k+1} - g_k) + c_i g_0 \\ &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + c_i g_0. \end{aligned} \tag{2.9}$$

Denote  $g_{k+1} - g_k$  by  $w_k$  for  $k \geq 0$ . Then

$$(\Omega g)_i = q_{i,i+1}w_i - \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)}w_k + c_i g_0, \quad i \geq 0.$$

Now, we rewrite the Poisson equation (1.3) as

$$w_i = \frac{1}{q_{i,i+1}} \left( \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)}w_k + \tilde{f}_i \right), \quad i \geq 0,$$

where  $\tilde{f}_i = f_i - c_i g_0$  for  $i \geq 0$ . By Corollary 2.3, we obtain

$$w_i = \sum_{j=0}^i \frac{\tilde{F}_i^{(j)} \tilde{f}_j}{q_{j,j+1}}, \quad i \geq 0.$$

So the solution of the Poisson equation (1.3) satisfies

$$g_i = g_0 + \sum_{k=0}^{i-1} w_k = g_0 + \sum_{k=0}^{i-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)} \tilde{f}_j}{q_{j,j+1}}, \quad i \geq 1.$$

The first assertion is proven. The second assertion is simply a consequence of the first one.

To prove the last assertion of the theorem, noting that by (1.4), we have

$$g_{n+1} - g_n = \sum_{j=0}^n \frac{\tilde{F}_n^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \quad n \geq 0.$$

Thus, from (2.9), it follows for each  $i \geq 0$  that

$$(\Omega g)_i = q_{i,i+1} \sum_{j=0}^i \frac{\tilde{F}_i^{(j)}(f_j - c_j g_0)}{q_{j,j+1}} - \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}} + c_i g_0.$$

Because (by exchanging the order of sums and using (1.1))

$$\begin{aligned} \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}} &= \sum_{0 \leq j \leq i-1} \frac{f_j - c_j g_0}{q_{j,j+1}} \sum_{k=j}^{i-1} \tilde{q}_i^{(k)} \tilde{F}_k^{(j)} \\ &= q_{i,i+1} \sum_{0 \leq j \leq i-1} \frac{\tilde{F}_i^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \end{aligned}$$

we obtain  $\Omega g = f$  as required. □

**Remark 2.5** (1) One may obtain  $(\tilde{q}_n^{(k)}, \tilde{F}_n^{(k)})$  from  $(q_n^{(k)}, F_n^{(k)})$  easily replacing the original  $Q = (q_{ij})$  by  $\tilde{Q} = (\tilde{q}_{ij})$ :

$$\begin{cases} \tilde{q}_{i0} = q_{i0} - c_i, \\ \tilde{q}_{ij} = q_{ij}, \quad j \neq 0, i \in E. \end{cases}$$

In other words, only the first column of  $Q = (q_{ij})$  is modified. Then the original Poisson equation  $\Omega g = f$  can be rewritten as  $\tilde{Q}g = \tilde{f}$  with  $\tilde{f}_i = f_i - c_i g_0$ .

(2) Alternatively, one may enlarge the space  $E$  by adding a point, say  $-1$  for instance. Then introduce suitable  $\bar{q}_{-1,i}, \bar{q}_{i,-1}, \bar{g}_{-1}$ , and  $\bar{f}_{-1}$ , so that  $\bar{Q}|_E = Q, \bar{g}|_E = g$ , and  $\bar{f}|_E = f$ . In this way, one may rewrite  $\Omega g = f$  on  $E$  as  $\bar{Q}\bar{g} = \bar{f}$  on  $E \cup \{-1\}$ .

(3) To solve the Poisson equation, in view of (2.9), even for the simplest birth–death type, once  $c$  appears, it is necessary to go out to the larger class of single birth one, one cannot just stay within the class of birth–death processes. Actually, this observation is crucial to solve the Open Problem 9.13 in [7]. Refer to [8; Theorem 2.6].

For the remainder of this section, we consider only the processes on a finite state space  $\{0, 1, \dots, N\}$ . Note that here the rate  $q_{N,N+1}$  is not defined (or setting to be zero), but we allow  $c_N \neq 0$ . Hence,  $\tilde{F}_n^{(k)}$  is defined up to  $n = N - 1$  only. The next result is a localized version of Theorem 1.1.

**Proposition 2.6** *Given a single-birth  $Q$ -matrix  $(q_{ij})$  and a function  $c$  on the finite state space  $\{0, 1, \dots, N\}$  ( $N \geq 1$ ), the following assertions hold.*

(i) *The solution of the Poisson equation  $\Omega g = f$  has the following form:*

$$g_n = g_0 + \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \quad 0 \leq n \leq N, \tag{2.10}$$

*with boundary condition*

$$c_N g_0 = \sum_{k=0}^{N-1} \tilde{q}_N^{(k)} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}} + f_N.$$

(ii) *Let  $c \leq 0$ . Then the harmonic equation  $\Omega g = 0$  has only the trivial solution  $g_i \equiv 0$  if and only if there exists some  $c_i < 0$ .*

(iii) *The unique solution  $g$  to the equation  $\Omega g|_{\{0,1,\dots,N-1\}} = 0$  (locally harmonic) with  $g_0 = 1$  is as follows:*

$$g_n = 1 - \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)} c_j}{q_{j,j+1}}, \quad 0 \leq n \leq N, \tag{2.11}$$

*which is increasing once  $c \leq 0$ .*



*Proof* (a) The proof is nearly the same as the one of Theorem 1.1, except we have to take care for the boundary at  $N$ . By (2.9), for  $0 \leq i \leq N - 1$ , we have

$$(\Omega g)_i = q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)}(g_{k+1} - g_k) + c_i g_0.$$

Denote  $g_{k+1} - g_k$  by  $w_k$  for all  $0 \leq k < N$ . Then

$$\begin{aligned} (\Omega g)_i &= q_{i,i+1}w_i - \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)}w_k + c_i g_0, & 0 \leq i < N; \\ (\Omega g)_N &= - \sum_{k=0}^{N-1} \tilde{q}_N^{(k)}w_k + c_N g_0. \end{aligned}$$

Rewrite the Poisson equation as

$$w_i = \frac{1}{q_{i,i+1}} \left( \tilde{f}_i + \sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)}w_k \right), \quad 0 \leq i < N, \tag{2.12}$$

where  $\tilde{f}_i = f_i - c_i g_0$  for all  $0 \leq i \leq N$ . By Corollary 2.3, we get

$$w_i = \sum_{j=0}^i \frac{\tilde{F}_i^{(j)} \tilde{f}_j}{q_{j,j+1}}, \quad 0 \leq i < N. \tag{2.13}$$

So the solution of the Poisson equation satisfies

$$g_i = g_0 + \sum_{k=0}^{i-1} w_k = g_0 + \sum_{k=0}^{i-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)} \tilde{f}_j}{q_{j,j+1}}, \quad 1 \leq i \leq N.$$

Combining this with the boundary condition  $(\Omega g)_N = f_N$  and (2.13), we obtain the first assertion.

(b) We have just seen that the harmonic solution  $g$  satisfies

$$g_n = g_0 \left( 1 - \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad 1 \leq n \leq N, \tag{2.14}$$

and

$$g_0 \left( c_N + \sum_{k=0}^{N-1} \tilde{q}_N^{(k)} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right) = 0.$$

When  $c \leq 0$ , by irreducibility, we have not only  $\tilde{q}_N^{(N-1)} > 0$  but also  $\tilde{F}_{N-1}^{(j)} > 0$  for every  $j: 0 \leq j \leq N - 1$ . Hence, if there exists some  $c_i < 0$ , then we must have  $g_0 = 0$  by the last equation. Furthermore, by (2.14), we indeed have  $g \equiv 0$ .

Conversely, if  $c_i \equiv 0$ , then every constant function  $g \neq 0$  is a solution to the equation  $\Omega g = 0$ . Hence, the harmonic function  $g$  can be non-trivial.

(c) To prove the third assertion, based on the second one, we have to use a smaller space  $\{0, 1, \dots, N-1\}$  instead of the original  $\{0, 1, \dots, N\}$  to avoid the trivial solution. The assertion now follows from (2.14).  $\square$

The next result is exceptional of the paper. Instead of single birth, we consider single death processes on a finite state space. The result may be regarded as a dual of Proposition 2.6. It indicates that a large parts of the study in the paper is meaningful for the single death processes, but we will not go to the details here.

A matrix  $Q = (q_{ij})$  is called of single death if  $q_{i,i-j} > 0$  if and only if  $j = 1$  for  $i \geq 1$ .

**Proposition 2.7** *Given a single death  $Q$ -matrix  $Q = (q_{ij})$  and a function  $(c_i)$  on the finite state space  $\{0, 1, \dots, N\}$ , define  $\tilde{q}_n^{(k)} = \sum_{j=k}^N q_{nj} - c_n$  for  $k > n$  and*

$$\tilde{F}_i^{(i)} = 1, \quad \tilde{F}_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^i \tilde{q}_n^{(k)} \tilde{F}_k^{(i)}, \quad 1 \leq n < i.$$

Then

(i) *the solution  $g$  to the Poisson equation  $\Omega g = f$  has the following representation:*

$$g_n = g_N + \sum_{n+1 \leq k \leq N} \sum_{k \leq j \leq N} \frac{\tilde{F}_k^{(j)} (f_j - c_j g_N)}{q_{j,j-1}}, \quad 0 \leq n \leq N,$$

with boundary condition

$$c_0 g_N = \sum_{k=1}^N \tilde{q}_0^{(k)} \sum_{j=k}^N \frac{\tilde{F}_k^{(j)} (f_j - c_j g_N)}{q_{j,j-1}} + f_0;$$

(ii) *the unique solution with  $g_N = 1$  to equation  $Qg|_{\{1,2,\dots,N\}} = 0$  is as follows:*

$$g_n = 1 - \sum_{n+1 \leq k \leq N} \sum_{k \leq j \leq N} \frac{\tilde{F}_k^{(j)} c_j}{q_{j,j-1}}, \quad 0 \leq n \leq N,$$

which is decreasing in  $n$  once  $c \leq 0$ .

*Proof* For  $1 \leq i \leq N$ , we have

$$\begin{aligned} (\Omega g)_i &= q_{i,i-1}(g_{i-1} - g_i) + \sum_{i+1 \leq j \leq N} q_{ij} \sum_{k=i+1}^j (g_k - g_{k-1}) + c_i g_i \\ &= q_{i,i-1}(g_{i-1} - g_i) + \sum_{i+1 \leq k \leq N} \sum_{j=k}^N q_{ij} (g_k - g_{k-1}) + c_i g_i \\ &= q_{i,i-1}(g_{i-1} - g_i) - \sum_{i+1 \leq k \leq N} \tilde{q}_i^{(k)} (g_{k-1} - g_k) + c_i g_N. \end{aligned}$$

Denote  $g_{k-1} - g_k$  by  $w_k$  for all  $1 \leq k \leq N$ . Then

$$(\Omega g)_i = q_{i,i-1}w_i - \sum_{i+1 \leq j \leq N} \tilde{q}_i^{(k)} w_k + c_i g_N, \quad 1 \leq i \leq N;$$

$$(\Omega g)_0 = - \sum_{k=1}^N \tilde{q}_0^{(k)} w_k + c_0 g_N.$$

Now, we rewrite the Poisson equation as

$$w_i = \frac{1}{q_{i,i-1}} \left( \tilde{f}_i + \sum_{i+1 \leq j \leq N} q_i^{(k)} w_k \right), \quad 1 \leq i \leq N,$$

where  $\tilde{f}_i = f_i - c_i g_N$  for all  $0 \leq i \leq N$ . As an analogue of Corollary 2.3, by induction, we can verify that

$$w_i = \sum_{j=i}^N \frac{\tilde{F}_i^{(j)} \tilde{f}_j}{q_{j,j-1}}, \quad 1 \leq i \leq N.$$

From the argument above, it follows immediately that

$$g_i = g_N + \sum_{k=i+1}^N w_k = g_N + \sum_{i+1 \leq j \leq N} \sum_{k \leq j \leq N} \frac{\tilde{F}_k^{(j)} \tilde{f}_j}{q_{j,j-1}}, \quad 0 \leq i \leq N - 1.$$

Combining this with the boundary condition  $(\Omega g)_0 = f_0$ , we finish the proof of the first assertion. The second assertion is derived from the first one immediately.  $\square$

### 3 Uniqueness

Starting from this section, we handle with the problems for single birth processes, listed at the beginning of the paper. First, we study the uniqueness problem. To do so, we need a sequence  $(\tilde{m}_n)$  (to be used often subsequently):

$$\tilde{m}_0 = \frac{1}{q_{01}}, \quad \tilde{m}_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{m}_k \right), \quad n \geq 1. \tag{3.1}$$

By Corollary 2.3, we have

$$\tilde{m}_n = \sum_{k=0}^n \frac{\tilde{F}_n^{(k)}}{q_{k,k+1}}, \quad n \geq 0. \tag{3.2}$$

Again, we omit the superscript ‘ $\sim$ ’ everywhere in  $\tilde{m}$ ,  $\tilde{F}$ , and  $\tilde{q}$  once  $c_i \equiv 0$ . The following criterion is taken from [6,15,16].

**Proposition 3.1** *Corresponding to a given single birth  $Q$ -matrix  $Q = (q_{ij})$  (conservative), the process is unique (non-explosive) if and only if*

$$\sum_{n=0}^{\infty} m_n = \infty.$$

*Proof* By [6; Theorems 2.47 and 2.40], the single birth process is unique if and only if the solution  $(u_i)$  to the equation

$$(\lambda + q_i)u_i = \sum_{j \neq i} q_{ij}u_j, \quad i \geq 0; \quad u_0 = 1, \quad (3.3)$$

is unbounded for some (equivalently for all)  $\lambda > 0$ . Rewrite (3.3) as

$$\Omega u = Qu - \lambda u = 0; \quad u_0 = 1.$$

Applying Theorem 1.1 to  $c_i \equiv -\lambda$  and  $f_i \equiv 0$ , we obtain the unique solution:

$$u_n = 1 + \lambda \sum_{0 \leq k \leq n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} = 1 + \lambda \sum_{0 \leq k \leq n-1} \tilde{m}_k, \quad n \geq 0.$$

Clearly,  $u_n$  is increasing in  $n$  and then is unbounded if and only if  $\sum_n \tilde{m}_n = \infty$ . Thus, it remains to show that  $\sum_n \tilde{m}_n = \infty$  if and only if  $\sum_n m_n = \infty$ . Combining  $\tilde{m}_n$  with  $m_n$ , it is clear that

$$\tilde{m}_n = \sum_{j=0}^n \frac{\tilde{F}_n^{(j)}}{q_{j,j+1}} \downarrow \sum_{k=0}^n \frac{F_n^{(k)}}{q_{k,k+1}} = m_n \quad \text{as } \lambda \downarrow 0,$$

since

$$\tilde{q}_n^{(k)} = q_n^{(k)} + \lambda \downarrow q_n^{(k)} \quad \text{as } \lambda \downarrow 0.$$

This already shows that the condition  $\sum_n m_n = \infty$  is sufficient. It is nearly necessary since the conclusion does not depend on  $\lambda > 0$ , except there is a jump from  $\lambda > 0$  to  $\lambda = 0$ . Hopefully, we have thus seen some advantage of Theorem 1.1, even though there is still a distance to prove the necessity.

Actually, there are several ways to prove the equivalence

$$\sum_n \tilde{m}_n = \infty \text{ for a fixed } \lambda > 0 \iff \sum_n m_n = \infty.$$

From now on, for simplicity, assume that  $\lambda = 1$ .

(a) Observing that corresponding to the sequence  $(\tilde{m}_n)$ , the operator is  $\Omega = Q - I$ , which may be regarded as a bounded perturbation of the original operator  $Q$ . Since these two operators are zero-exit or not simultaneously, the equivalence above holds.

(b) In the original proof (cf. [6; Proof of Theorem 3.16]), it was proved that  $u_n$  is unbounded if and only if  $\sum_n m_n = \infty$ . Combining this with what proved above, we obtain the required equivalence.

(c) Here is a more direct proof. The idea comes from [20].

Assume that  $\sum_{k=0}^\infty \tilde{m}_k = \infty$ . If  $\sum_{k=0}^\infty m_k < \infty$ , then there exists  $N_0$  large enough such that for all  $n \geq N_0$ ,

$$\tilde{M}_n := \sum_{k=0}^n \tilde{m}_k > 1 \quad \text{and} \quad K := 2 \sum_{k=N_0+1}^\infty m_k < 1.$$

We now prove that for each  $n > N_0$ ,

$$\tilde{m}_k \leq 2m_k \tilde{M}_{n-1}, \quad 0 \leq k \leq n. \tag{3.4}$$

Since  $\tilde{m}_0 = m_0$  and  $\tilde{M}_{n-1} > 1$  (due to the fact that  $n - 1 \geq N_0$ ), (3.4) holds in the case of  $k = 0$ . Assume that (3.4) holds up to  $k = \ell - 1 < n$ . Then,

$$\begin{aligned} \tilde{m}_\ell &= \frac{1}{q_{\ell,\ell+1}} \left( 1 + \sum_{k=0}^{\ell-1} q_\ell^{(k)} \tilde{m}_k + \sum_{k=0}^{\ell-1} \tilde{m}_k \right) \quad (\text{since } \lambda = 1) \\ &\leq \frac{1}{q_{\ell,\ell+1}} \left( 1 + \sum_{k=0}^{\ell-1} q_\ell^{(k)} 2m_k \tilde{M}_{n-1} + \tilde{M}_{\ell-1} \right) \quad (\text{by assumption}) \\ &\leq \frac{1}{q_{\ell,\ell+1}} \left( 1 + \sum_{k=0}^{\ell-1} q_\ell^{(k)} m_k \right) 2\tilde{M}_{n-1} \\ &= 2m_\ell \tilde{M}_{n-1}. \end{aligned}$$

So (3.4) holds when  $k = \ell$ . By induction, we know that (3.4) holds for every  $k : 0 \leq k \leq n$ . Now, for each  $n > N_0$ , we have

$$\tilde{M}_n = \tilde{M}_{N_0} + \sum_{k=N_0+1}^n \tilde{m}_k \leq \tilde{M}_{N_0} + \sum_{k=N_0+1}^n 2m_k \tilde{M}_{n-1} \leq \tilde{M}_{N_0} + K \tilde{M}_{n-1}.$$

Furthermore, we have

$$\begin{aligned} \tilde{M}_n &\leq \tilde{M}_{N_0} (1 + K + \dots + K^{n-N_0-1}) + K^{n-N_0} \tilde{M}_{N_0} \\ &= \frac{\tilde{M}_{N_0} (1 - K^{n-N_0})}{1 - K} + K^{n-N_0} \tilde{M}_{N_0}. \end{aligned}$$

Thus, as  $n \rightarrow \infty$ , we would have  $\infty \leq \tilde{M}_{N_0} / (1 - K)$  which is a contradiction. Hence, once  $\sum_{k=0}^\infty \tilde{m}_k = \infty$ , we should also have  $\sum_{k=0}^\infty m_k = \infty$ .

We have therefore completed the proof of the equivalence mentioned above. □

To conclude this section, we mention that the uniqueness problem for the single birth  $Q$ -matrix with absorbing set  $H = \{0, 1, \dots, N\}$  ( $N < \infty$ ) can be dealt with by the same approach. Refer to [6; Theorem 3.16] and [14].

#### 4 Recurrence and extinction/return probability

For the recurrence, the following criterion is taken from [6; Theorem 4.52 (1)] and [15].

**Proposition 4.1** *Assume that the single birth  $Q$ -matrix  $Q = (q_{ij})$  is non-explosive and irreducible. Then the process is recurrent if and only if  $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$ , where  $(F_n^{(i)})$  was defined in (1.1) by setting  $c_i \equiv 0$ .*

*Proof* By [6; Lemma 4.51], we know that the single birth process is recurrent if and only if the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \quad 0 \leq x_i \leq 1, \quad i \geq 0, \quad (4.1)$$

has only zero solution, where  $\Pi_{ik} = (1 - \delta_{ik})q_{ik}/q_i$ . It is easily seen that equation (4.1) has a non-trivial solution if and only if the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \quad i \geq 0; \quad x_0 = 1,$$

has a nonnegative bounded solution. The following fact will be used several times below:

$$x_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i - \lambda} x_k + \frac{\gamma_i}{q_i - \lambda} \iff (Qx)_i + \lambda x_i = q_{ii_0}(1 - \delta_{ii_0})x_{i_0} - \gamma_i, \quad (4.2)$$

where  $\lambda \in \mathbb{R}$  satisfying some suitable condition. Certainly, here we preassume that  $x_i \in \mathbb{R}$  for every  $i \in E$ . By using this fact with  $\lambda = 0$  and  $i_0 = 0$ , we can rewrite the previous equation as

$$(Qx)_0 = 0, \quad (Qx)_i = q_{i0}, \quad i \geq 1; \quad x_0 = 1.$$

Applying Theorem 1.1 to  $c_i \equiv 0$  and  $f_i = q_{i0}(1 - \delta_{i0})$ , we obtain the unique solution as follows:

$$x_0 = 1, \quad x_n = 1 + \sum_{k=1}^{n-1} \sum_{j=1}^k \frac{F_k^{(j)} q_{j0}}{q_{j,j+1}} = 1 + \sum_{k=1}^{n-1} \sum_{j=1}^k \frac{F_k^{(j)} q_j^{(0)}}{q_{j,j+1}}, \quad n \geq 1.$$

By (2.7), it follows that

$$x_n = 1 + \sum_{k=1}^{n-1} F_k^{(0)} = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1.$$

Clearly,  $(x_n)$  is bounded if and only if  $\sum_{k=0}^\infty F_k^{(0)} < \infty$ . In other words, equation (4.1) has only a trivial solution if and only if  $\sum_{k=0}^\infty F_k^{(0)} = \infty$ . The assertion is now proven.  $\square$

**Extinction/return probability**

For the remainder of this section, we study the extinction probability. Here, the extinction time  $\tau_0$  is the first hitting time of the state 0. Thus, this topic is actually a refinement of what studied in the last proposition, in which we pay attention only on the result either  $\mathbb{P}_n[\tau_0 < \infty] = 1$  or  $< 1$  rather than its distribution. We will come back this point after the proof of the next proposition. For the extinction problem, the rates  $q_{0j}$  ( $j \neq 0$ ) play no role, so one may assume the state 0 to be an absorbing state. In other words, we may reduce the state space from  $E$  to  $E_1 := \{1, 2, \dots\}$ , and regard the rate  $q_{i0}$  ( $i \neq 0$ ) as a killing from  $i$ . Then we need to redefine the sequences  $(\tilde{q}_n^{(k)})$  and  $(\tilde{F}_n^{(k)})$  starting from 1 but not 0. However, for our convenience, we prefer to keep the notation  $E$ ,  $(\tilde{q}_n^{(k)})$ ,  $(\tilde{F}_n^{(k)})$ , and so on. For this, it is better to use the return time  $\sigma_0$  instead of the hitting time  $\tau_0$ . In the case that the state 0 is really an absorbing one, we can add a positive rate  $q_{01}$  and assume that the enlarged process becomes irreducible. Then, the solution of  $\mathbb{P}_n[\sigma_0 < \infty]$  restricted on  $E_1$  gives us the answer of  $\mathbb{P}_n[\tau_0 < \infty]$  on  $E_1$  (as a trivial application of the localization theorem [9; Theorem 3.4.1] or [6; Theorem 2.13]), so we can return to our original problem.

We remark that in the context of denumerable Markov processes, the topic of this section and much more problems were well studied in [9; Chapter IX]. In the present special case, for the single birth processes, the problem was studied in [1; Chapter 9] or [2], using a different technique.

**Proposition 4.2** *Let the single birth  $Q$ -matrix  $Q = (q_{ij})$  be non-explosive and irreducible. Then the return/extinction probability is as follows:*

$$\mathbb{P}_0(\sigma_0 < \infty) = \frac{\sum_{k=1}^\infty F_k^{(0)}}{\sum_{k=0}^\infty F_k^{(0)}}, \quad \mathbb{P}_n(\sigma_0 < \infty) = \frac{\sum_{k=n}^\infty F_k^{(0)}}{\sum_{k=0}^\infty F_k^{(0)}}, \quad n \geq 1.$$

Furthermore,  $\mathbb{P}_n(\sigma_0 < \infty) = 1$  for all  $n \geq 0$  if and only if  $\mathbb{P}_0(\sigma_0 < \infty) = 1$ , equivalently, if and only if  $\sum_{n=0}^\infty F_n^{(0)} = \infty$ .

*Proof* By [6; Lemma 4.46] with  $H = \{0\}$ ,  $(\mathbb{P}_i(\sigma_0 < \infty) : i \in E)$  is the minimal nonnegative solution to the equation

$$x_i = \sum_{k \neq 0, i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i} (1 - \delta_{i0}), \quad i \in E.$$

The study on recurrence usually starts from here, the lemma [6; Lemma 4.51] used in the last proof simplifies our study on the recurrence problem, as we have just seen above. By (4.2), the last equation is equivalent to

$$(Qx)_i = q_{i0}(1 - \delta_{i0})(x_0 - 1), \quad i \geq 0.$$

Applying Theorem 1.1 to  $c_i \equiv 0$  and  $f_i = q_{i0}(1 - \delta_{i0})(x_0 - 1)$ , we obtain the solution to the last equation:

$$\begin{aligned} x_n &= x_0 + \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{F_k^{(j)}}{q_{j,j+1}} q_{j0}(1 - \delta_{j0})(x_0 - 1) \\ &= x_0 \left\{ 1 + \sum_{1 \leq k \leq n-1} \sum_{1 \leq j \leq k} \frac{F_k^{(j)}}{q_{j,j+1}} q_j^{(0)} \right\} - \sum_{1 \leq k \leq n-1} \sum_{1 \leq j \leq k} \frac{F_k^{(j)}}{q_{j,j+1}} q_j^{(0)} \\ &= x_0 \left( 1 + \sum_{1 \leq k \leq n-1} F_k^{(0)} \right) - \sum_{1 \leq k \leq n-1} F_k^{(0)}, \quad n \geq 0 \quad (\text{by (2.7)}). \end{aligned}$$

Because  $x_n > 0$ , it follows that

$$x_0 \geq \sup_{n \geq 1} \frac{\sum_{k=1}^{n-1} F_k^{(0)}}{\sum_{k=0}^{n-1} F_k^{(0)}} = \sup_{n \geq 1} \frac{\sum_{k=0}^{n-1} F_k^{(0)} - 1}{\sum_{k=0}^{n-1} F_k^{(0)}} = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}.$$

From here, we obtain the minimal nonnegative solution:

$$x_0^* = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad x_n^* = 1 - \frac{\sum_{k=0}^{n-1} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad n \geq 1.$$

We have thus proved the first assertion. The second one is obvious.  $\square$

Rewrite the solution just obtained as follows:

$$1 - x_0^* = \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad 1 - x_n^* = \frac{\sum_{k=0}^{n-1} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad n \geq 1.$$

Renormalize them so that the initial value becomes 1:

$$x_0 = 1, \quad x_n = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1,$$

which is what we obtained in the last proof. We have thus seen the relation between the last two propositions.

The study on the Laplace transform of extinction/return time is delayed to Section 7 (Proposition 7.3 which is based on Lemma 7.1).

## 5 Ergodicity, strong ergodicity, and the first moment of return time

Let  $E = \mathbb{Z}_+$  and  $H \subset E$ ,  $H \neq \emptyset, E$ . Define  $\sigma_H = \inf\{t \geq \eta_1 : X(t) \in H\}$ , where  $\eta_1$  is the first jump of the process. When  $H$  is a singleton,  $H = \{0\}$ , for instance, denote  $\sigma_{\{0\}}$  by  $\sigma_0$  for simplicity. We now consider the first moment of



the return time  $\sigma_0$ . To do so, we introduce the following lemma (cf. [9; Lemma 9.4.1]).

**Lemma 5.1** *Let  $(q_{ij})$  be irreducible and assume that its  $Q$ -process is recurrent. Then  $(x_i^* := \mathbb{E}_i \sigma_H : i \in E)$  is the minimal nonnegative solution (may be infinite) to the equation*

$$x_i = \frac{1}{q_i} \sum_{k \notin H \cup \{i\}} q_{ik} x_k + \frac{1}{q_i}, \quad i \in E,$$

where  $1 \cdot \infty = \infty$  and  $0 \cdot \infty = 0$  by convention.

*Proof* Let  $(y_i^* : i \in E)$  be the minimal nonnegative solution to the equation

$$y_i = \frac{1}{q_i} \sum_{k \notin H \cup \{i\}} q_{ik} y_k + \frac{1}{q_i}, \quad i \in E.$$

By assumption and [6; Lemma 4.46], the quantity  $f_{iH}$  defined there is equal to 1 for every  $i \in E$ . Then,  $(y_i^* : i \in E)$  coincides with  $(e_{iH}(0) : i \in E)$  used in [6; Lemma 4.48]. Note that

$$e_{iH}(0) = \int_0^\infty \mathbb{P}_i(\sigma_H > t) dt = \mathbb{E}_i \sigma_H.$$

The assertion now follows immediately. □

In what follows, we use often another sequence  $(\tilde{d}_n)$  similar to  $(\tilde{m}_n)$  having different initial value:

$$\tilde{d}_0 = 0, \quad \tilde{d}_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{d}_k \right), \quad n \geq 1, \tag{5.1}$$

where  $\tilde{q}_n^{(k)}$  is defined in (1.2). By Corollary 2.3, we have

$$\tilde{d}_n = \sum_{1 \leq j \leq n} \frac{\tilde{F}_n^{(j)}}{q_{j,j+1}}, \quad n \geq 0, \tag{5.2}$$

which is very much the same as (3.2). Again, we omit the superscript ‘ $\sim$ ’ everywhere in  $(\tilde{d}_n)$  once  $c_i \equiv 0$ . Note that if we rewrite

$$\begin{aligned} \tilde{d}_n &= \frac{1}{q_{n,n+1}} \left( 1 + \sum_{1 \leq k \leq n-1} \tilde{q}_n^{(k)} \tilde{d}_k \right), \quad n \geq 1, \\ \tilde{F}_n^{(0)} &= \frac{1}{q_{n,n+1}} \left( \tilde{q}_n^{(0)} + \sum_{1 \leq k \leq n-1} \tilde{q}_n^{(k)} \tilde{F}_k^{(0)} \right), \quad n \geq 1, \end{aligned}$$

then it is clear that the sequences  $(\tilde{d}_n)_{n \geq 1}$  and  $(\tilde{F}_n^{(0)})_{n \geq 1}$  are also quite close each other.

The main result in this section is as follows. Refer to [6; Theorem 4.52 (2)], [1; Proposition 2.4], and [15,17,18].

**Proposition 5.2** *Assume that the single birth  $Q$ -matrix  $Q = (q_{ij})$  is irreducible and corresponding process is recurrent. Then*

$$\mathbb{E}_0\sigma_0 = \frac{1}{q_{01}} + d, \quad \mathbb{E}_n\sigma_0 = \sum_{k=0}^{n-1} (F_k^{(0)}d - d_k), \quad n \geq 1,$$

where

$$d = \overline{\lim}_{k \rightarrow \infty} \frac{\sum_{n=0}^k d_n}{\sum_{n=0}^k F_n^{(0)}} = \lim_{n \rightarrow \infty} \frac{d_n}{F_n^{(0)}} \text{ if the limit exists.}$$

Furthermore, the process is ergodic (i.e., positive recurrent) if and only if  $d < \infty$ ; and it is strongly ergodic if and only if

$$\sup_{k \in E} \sum_{n=0}^k (F_n^{(0)}d - d_n) < \infty.$$

Actually, for the last conclusion, the recurrence assumption can be replaced by the uniqueness one.

*Proof* Let  $H = \{0\}$ . By Lemma 5.1,  $(\mathbb{E}_i\sigma_0 : i \in E)$  is the minimal nonnegative solution  $(x_i^*)$  to the equation

$$x_i = \frac{1}{q_i} \sum_{k \notin \{0,i\}} q_{ik}x_k + \frac{1}{q_i}, \quad i \in E. \quad (5.3)$$

Suppose for a moment that  $x_i^* < \infty$  first for some  $i \in E$  and then for all  $i$  by irreducibility. Next, let  $(x_i)$  be a (finite) solution to (5.3). Then, by (4.2), we have

$$(Qx)_i = q_{i0}x_0 - 1, \quad i \geq 1; \quad (Qx)_0 = -1.$$

Applying Theorem 1.1 to  $c = 0$  and  $f_i = q_{i0}(1 - \delta_{i0})x_0 - 1$  ( $i \geq 0$ ), we obtain the solution to the last equation:

$$x_n = x_0 + \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{F_k^{(j)}f_j}{q_{j,j+1}} = x_0 \left( 1 + \sum_{k=1}^{n-1} \sum_{j=1}^k \frac{F_k^{(j)}q_{j0}}{q_{j,j+1}} \right) - \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{F_k^{(j)}}{q_{j,j+1}}, \quad n \geq 1.$$

By (2.7) and (5.2), we obtain

$$x_n = x_0 \sum_{k=0}^{n-1} F_k^{(0)} - \sum_{k=0}^{n-1} \left( \frac{F_k^{(0)}}{q_{01}} + d_k \right) = \sum_{k=0}^{n-1} \left[ F_k^{(0)} \left( x_0 - \frac{1}{q_{01}} \right) - d_k \right], \quad n \geq 1.$$

Since  $x_n > 0$ , it follows that

$$x_0 \sum_{k=0}^{n-1} F_k^{(0)} > \sum_{k=0}^{n-1} \left( \frac{F_k^{(0)}}{q_{01}} + d_k \right), \quad n \geq 1.$$

This gives us

$$x_0 \geq \sup_{n \geq 1} \frac{\sum_{k=0}^{n-1} (\frac{F_k^{(0)}}{q_{01}} + d_k)}{\sum_{k=0}^{n-1} F_k^{(0)}} = \frac{1}{q_{01}} + \sup_{n \geq 1} \frac{\sum_{k=0}^{n-1} d_k}{\sum_{k=0}^{n-1} F_k^{(0)}}.$$

Now, the minimal property implies that

$$x_0^* = \frac{1}{q_{01}} + \sup_{n \geq 1} \frac{\sum_{k=0}^{n-1} d_k}{\sum_{k=0}^{n-1} F_k^{(0)}}$$

and then

$$x_n^* = \sum_{k=0}^{n-1} \left( F_k^{(0)} \sup_{n \geq 1} \frac{\sum_{j=0}^{n-1} d_j}{\sum_{j=0}^{n-1} F_j^{(0)}} - d_k \right), \quad n \geq 1,$$

gives us the solution  $(\mathbb{E}_i \sigma_0 : i \in E)$ . We claim that the supremum in the last line has to be achieved at infinity. Otherwise, if it is achieved at some finite  $n_0$ :

$$\frac{\sum_{j=0}^{n_0-1} d_j}{\sum_{j=0}^{n_0-1} F_j^{(0)}} = \sup_{n \geq 1} \frac{\sum_{j=0}^{n-1} d_j}{\sum_{j=0}^{n-1} F_j^{(0)}},$$

then

$$x_0^* = \frac{1}{q_{01}} + \frac{\sum_{j=0}^{n_0-1} d_j}{\sum_{j=0}^{n_0-1} F_j^{(0)}}$$

and furthermore,  $x_{n_0}^* = 0$ , which is a contradiction with  $x_i^* = \mathbb{E}_i \sigma_0 > 0$ . Therefore,

$$\sup_{n \geq 1} \frac{\sum_{j=0}^{n-1} d_j}{\sum_{j=0}^{n-1} F_j^{(0)}} = \overline{\lim}_{n \rightarrow \infty} \frac{\sum_{j=0}^n d_j}{\sum_{j=0}^n F_j^{(0)}} =: d,$$

as required. The next limit in the expression of  $d$  is an application of Stolz's Theorem. Now,  $d < \infty$  since  $x_0^* < \infty$  by assumption. To remove the finiteness assumption of  $(x_i^*)$ , we claim that the expressions in the first assertion for  $\mathbb{E}_n \sigma_0 (= x_n^*)$  still hold even  $x_i^* = \infty$ , since then we must have  $d = \infty$ . If otherwise,  $d < \infty$ , then by the last assertion of Theorem 1.1 and (4.2), we would obtain a finite solution to (5.3), which deduces a contradiction to the assumption  $x_i^* = \infty$  by the comparison theorem for the nonnegative solutions (cf. [6; Theorem 2.6]). We have thus proved the first assertion.

Let us remark that the trick used above replacing  $\sup_{n \geq 1}$  by  $\overline{\lim}_{n \rightarrow \infty}$  was missed in the previous publications. This trick and the one assuming the finiteness of  $(x_i^*)$ , will be used several times below but we may not mention it time by time.

Finally, by [6; Theorem 4.44], the single process is ergodic if and only if  $\mathbb{E}_0 \sigma_0 < \infty$  which is now equivalent to  $d < \infty$ . By the same cited theorem, the process is strongly ergodic if and only if  $\sup_{i \in E} \mathbb{E}_i \sigma_0 < \infty$ , equivalently,

$$\sup_{n \in E} \sum_{k=0}^n (F_k^{(0)} d - d_k) < \infty,$$

which follows from the first assertion. As mentioned in the proof of the cited book, for ergodicity, the uniqueness assumption is enough instead of the recurrence one. The proof is now finished.  $\square$

## 6 Polynomial moments of hitting time and life time

### 6.1 Polynomial moments of hitting time

We have just studied the first moment of the time of first hitting/return 0 in the last section. Now, we study the higher-order moments of the first hitting time.

Fix  $i_0 \geq 0$ . Recall that  $\sigma_{i_0}$  is the time of first return to  $i_0$  after the first jump. For its higher-moments, we have the following result (cf. [19,21]).

**Proposition 6.1** *Assume that the single birth  $Q$ -matrix  $Q = (q_{ij})$  is irreducible and the corresponding process is  $(\ell-1)$ -ergodic ( $\ell \geq 1$ ), i.e.,  $\mathbb{E}_i \sigma_{i_0}^{\ell-1} < \infty$  for every  $i \geq 0$ . When  $\ell = 1$ , assume additionally that the process is unique. Then we have*

$$\mathbb{E}_n \sigma_{i_0}^\ell = \begin{cases} \ell \sum_{n \leq k \leq i_0-1} v_k^{(\ell)} + \left[ 1 - \sum_{n \leq k \leq i_0-1} u_k \right] \mathbb{E}_{i_0} \sigma_{i_0}^\ell, & 0 \leq n \leq i_0; \\ -\ell \sum_{i_0 \leq k \leq n-1} v_k^{(\ell)} + \left[ 1 + \sum_{i_0 \leq k \leq n-1} u_k \right] \mathbb{E}_{i_0} \sigma_{i_0}^\ell, & n > i_0, \end{cases}$$

where

$$u_k = \sum_{j=i_0-1}^k q_{j,j+1}^{-1} F_k^{(j)} q_{ji_0} (1 - \delta_{ji_0}), \quad k \geq i_0,$$

$$u_{i_0-1} = 1, \quad u_k = 0, \quad 0 \leq k \leq i_0 - 2,$$

$$v_k^{(\ell)} = \sum_{j=0}^k \frac{F_k^{(j)}}{q_{j,j+1}} \mathbb{E}_j \sigma_{i_0}^{\ell-1}, \quad k \geq 0,$$

$$\begin{aligned} \mathbb{E}_{i_0} \sigma_{i_0}^\ell &= \ell \overline{\lim}_{n \rightarrow \infty} \left( \sum_{i_0 \leq k \leq n} v_k^{(\ell)} \right) \left[ 1 + \sum_{i_0 \leq k \leq n} u_k \right]^{-1} \\ &= \ell \lim_{n \rightarrow \infty} \frac{v_n^{(\ell)}}{u_n} \text{ if the limit exists.} \end{aligned}$$

*Proof* By [9; Theorem 9.3.3] (cf. [6; Proposition 4.56], or [10; Theorem 3.1]),  $(y_i^* := \mathbb{E}_i \sigma_{i_0}^\ell : i \in E)$  is the the minimal nonnegative solution to the following equation:

$$y_i = \sum_{k \neq i, i_0} \frac{1}{q_i} q_{ik} y_k + \frac{\ell}{q_i} \mathbb{E}_i \sigma_{i_0}^{\ell-1}, \quad i \in E.$$

As remarked in the last section, we may assume that  $y_i^* < \infty$  for every  $i \in E$ . Then, by (4.2), we obtain the Poisson equation:

$$(Qy)_i = q_{ii_0}(1 - \delta_{ii_0})y_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}, \quad i \in E.$$

Applying Theorem 1.1 to  $c = 0$  and  $f_i = q_{ii_0}(1 - \delta_{ii_0})y_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}$ , it follows that the solution to the last equation is as follows:

$$y_n = y_0 + \sum_{0 \leq k \leq n-1} \sum_{j=0}^k \frac{F_k^{(j)} f_j}{q_{j,j+1}} = y_0 + y_{i_0} \sum_{0 \leq k \leq n-1} u_k - \ell \sum_{0 \leq k \leq n-1} v_k^{(\ell)}, \quad n \geq 0.$$

Here, in the summation of  $u_k$ , we have used the character of single birth:  $q_{ji_0}(1 - \delta_{ji_0}) > 0$  only if either  $j = i_0 - 1$  or  $j \geq i_0 + 1$ . In particular, by setting  $n = i_0$ , it follows that

$$y_0 = \ell \sum_{0 \leq k \leq i_0-1} v_k^{(\ell)} + y_{i_0} \left( 1 - \sum_{0 \leq k \leq i_0-1} u_k \right).$$

Return to the original  $y_n$ , we get

$$\begin{aligned} y_n &= \ell \left[ \sum_{0 \leq k \leq i_0-1} v_k^{(\ell)} - \sum_{0 \leq k \leq n-1} v_k^{(\ell)} \right] + y_{i_0} \left[ 1 - \sum_{0 \leq k \leq i_0-1} u_k + \sum_{0 \leq k \leq n-1} u_k \right] \\ &= \begin{cases} -\ell \sum_{i_0 \leq k \leq n-1} v_k^{(\ell)} + y_{i_0} \left[ 1 + \sum_{i_0 \leq k \leq n-1} u_k \right], & n \geq i_0 + 1, \\ \ell \sum_{n \leq k \leq i_0-1} v_k^{(\ell)} + y_{i_0} \left[ 1 - \sum_{n \leq k \leq i_0-1} u_k \right], & n \leq i_0. \end{cases} \end{aligned} \tag{6.1}$$

When  $n \leq i_0$ , since  $\sum_{k \leq i_0-1} u_k \leq 1$  by definition of  $(u_k)$ , it is clear that  $y_n > 0$ . When  $n \geq i_0 + 1$ , for  $y_n > 0$ , one requires the condition

$$y_{i_0} > \frac{\ell \sum_{i_0 \leq k \leq n-1} v_k^{(\ell)}}{1 + \sum_{i_0 \leq k \leq n-1} u_k},$$

and then

$$y_{i_0} \geq \sup_{n \geq i_0+1} \frac{\ell \sum_{i_0 \leq k \leq n-1} v_k^{(\ell)}}{1 + \sum_{i_0 \leq k \leq n-1} u_k}.$$

By a reason explained in the last section, this leads to

$$y_{i_0}^* = \ell \overline{\lim}_{n \rightarrow \infty} \frac{\sum_{i_0 \leq k \leq n} v_k^{(\ell)}}{1 + \sum_{i_0 \leq k \leq n} u_k},$$

which gives us  $\mathbb{E}_{i_0} \sigma_{i_0}^\ell$ . Combining it with (6.1), we obtain the required assertion. The limit in  $\mathbb{E}_{i_0} \sigma_{i_0}^\ell$  is again an application of Stolz's Theorem since  $\sum_k u_k = \infty$

by the recurrence of the process. To see the last assertion, define a single birth process on  $\{i_0, i_0 + 1, \dots\}$  (regarding the set  $\{0, 1, \dots, i_0\}$  as a single state) with rates

$$\bar{q}_{ij} = \begin{cases} q_{ij} & \text{if } j \geq i_0 + 1, \\ \sum_{k \leq i_0} q_{ik} & \text{if } j = i_0, i \geq i_0. \end{cases}$$

Then  $(\bar{q}_{ij})$  is irreducible and recurrent because so is  $(q_{ij})$ . Next, as in (1.1), we can define a sequence  $(\bar{F}_k^{(j)})$  on  $\{i_0, i_0 + 1, \dots\}$ . By induction, it is easy to check that  $\bar{F}_k^{(j)} = \tilde{F}_k^{(j)}$  for every  $k \geq j \geq i_0$ . Hence, we have

$$\sum_k \bar{F}_k^{(i_0)} = \sum_k \tilde{F}_k^{(i_0)} = \infty$$

by Proposition 4.1. It should be now easy to see that  $\sum_k u_k = \infty$  as claimed. □

### 6.2 Polynomial moments of life time

Recall that  $\tau_n$  is the time of first hitting the state  $n$ . If we start from  $i \leq n - 1$ , then  $\tau_n$  coincides with the time of first hitting the set  $\{n, n + 1, \dots\}$ . For the remainder of this section, we are going to study the time  $\tau_\infty := \lim_{n \rightarrow \infty} \tau_n$ . Next, because  $\tau_\infty$  is actually equal to the life time  $\eta := \lim_{n \rightarrow \infty} \eta_n$  almost everywhere, where  $\{\eta_n\}$  are the successive jumping times:

$$\eta_0 \equiv 0, \quad \eta_n = \inf\{t \geq \eta_{n-1} : X(t) \neq X(\eta_{n-1})\}, \quad n \geq 1,$$

therefore,  $\tau_\infty = \infty$  a.e. if the single birth  $Q$ -matrix is non-explosive. Thus, the study on the moments of  $\tau_\infty$  is meaningful only for explosive single birth  $Q$ -matrix. The next result is taken from [21].

**Proposition 6.2** *Let the single birth  $Q$ -matrix  $Q = (q_{ij})$  be irreducible and explosive (i.e.,  $\sum_n m_n < \infty$  by Proposition 3.1). Assume that the minimal process has finite  $(\ell - 1)$ -th moments of  $\tau_\infty$  for some integer  $\ell \geq 1$  (i.e.,  $E_i \tau_\infty^{\ell-1} < \infty$  for all  $i \geq 0$ ). Then*

$$E_n \tau_\infty^\ell = \ell \sum_{k \geq n} \bar{m}_k^{(\ell)}, \quad n \geq 0,$$

where

$$\bar{m}_n^{(\ell)} = \frac{1}{q_{n,n+1}} \left[ E_n \tau_\infty^{\ell-1} + \sum_{0 \leq k \leq n-1} q_n^{(k)} \bar{m}_k^{(\ell)} \right] = \sum_{j=0}^n \frac{F_n^{(j)} E_j \tau_\infty^{\ell-1}}{q_{j,j+1}}, \quad n \geq 0.$$

*Proof* The last equality of  $\bar{m}_n^{(\ell)}$  comes from Corollary 2.3. By [6; Proposition 4.56] or [11], we know that  $(E_i \tau_\infty^\ell : i \in E)$  is the minimal nonnegative solution  $(y_i^* : i \in E)$  to the following equation:

$$y_i = \sum_{k \neq i} \frac{1}{q_i} q_{ik} y_k + \frac{\ell}{q_i} E_i \tau_\infty^{\ell-1}, \quad i \in E.$$

That is,

$$(Qy)_i = -\ell \mathbb{E}_i \tau_\infty^{\ell-1}, \quad i \in E.$$

Applying Theorem 1.1 to  $c = 0$  and  $f_i = -\ell \mathbb{E}_i \tau_\infty^{\ell-1}$  ( $i \geq 0$ ), it follows that the solution to the last equation can be expressed as

$$y_n = y_0 - \ell \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{F_k^{(j)} \mathbb{E}_j \tau_\infty^{\ell-1}}{q_{j,j+1}}, \quad n \geq 1.$$

Hence,

$$y_n = y_0 - \ell \sum_{k=0}^{n-1} \bar{m}_k^{(\ell)}, \quad n \geq 1.$$

By the nonnegative and minimal properties, it follows that

$$y_0^* = \sup_{n \geq 1} \left( \ell \sum_{k=0}^{n-1} \bar{m}_k^{(\ell)} \right) = \ell \sum_{k=0}^{\infty} \bar{m}_k^{(\ell)}, \quad y_n^* = \ell \sum_{k=n}^{\infty} \bar{m}_k^{(\ell)}, \quad n \geq 1.$$

Hence, we obtain

$$\mathbb{E}_n \tau_\infty^\ell = \ell \sum_{k \geq n} \bar{m}_k^{(\ell)}, \quad n \geq 0,$$

which is the required assertion. □

## 7 Exponential ergodicity and Laplace transform of return time

### 7.1 Exponential moments of return time and exponential ergodicity

In this section, we consider the exponential moments of return time. At first, we introduce the following lemma for general  $Q$ -matrices.

**Lemma 7.1** *Let  $(q_{ij})$  be irreducible and assume that its  $Q$ -process is recurrent. Next, let  $\lambda \in \mathbb{R}$ ,  $\lambda < q_i$  for every  $i \in E$ . Then for fixed  $H \subset E$ ,  $H \neq \emptyset, E$ ,  $(\mathbb{E}_i \exp(\lambda \sigma_H) : i \in E)$  is the minimal solution to the equation*

$$x_i = \frac{1}{q_i - \lambda} \sum_{k \notin H \cup \{i\}} q_{ik} x_k + \frac{1}{q_i - \lambda} \sum_{k \in H \setminus \{i\}} q_{ik}, \quad i \in E. \quad (7.1)$$

*Proof* Let  $(y_i^* : i \in E)$  be the minimal nonnegative solution to the equation

$$y_i = \frac{1}{q_i - \lambda} \sum_{k \notin H \cup \{i\}} q_{ik} y_k + \frac{1}{q_i - \lambda}, \quad i \in E.$$

By the recurrent assumption and [6; Lemma 4.46], the quantity  $f_{iH}$  defined there is equal to 1 for every  $i \in E$ . Then,  $(y_i^* : i \in E)$  coincides with  $(e_{iH}(\lambda) : i \in E)$  used in [6; Lemma 4.48]. Moreover, by the proof given on [6; p. 148], we

have  $\mathbb{E}_i \exp(\lambda \sigma_H) = 1 + \lambda y_i^*$  for every  $i \in E$ . Besides, it can be checked that  $(1 + \lambda y_i^* : i \in E)$  is a nonnegative solution to equation (7.1). Hence,  $\mathbb{E}_i \exp(\lambda \sigma_H) = 1 + \lambda y_i^* \geq x_i^*$  for every  $i \in E$ , where  $(x_i^* : i \in E)$  is the minimal nonnegative solution to equation (7.1). We are now going to prove that  $\mathbb{E}_i \exp(\lambda \sigma_H) = x_i^*$  for all  $i \in E$ . The proof is split into two parts: either  $\lambda \geq 0$  or  $\lambda < 0$ .

First, let  $\lambda \geq 0$ . It is easily seen that  $(x_i^* - 1 : i \in E)$  is a nonnegative solution to the equation

$$y_i = \frac{1}{q_i - \lambda} \sum_{k \notin H \cup \{i\}} q_{ik} y_k + \frac{\lambda}{q_i - \lambda}, \quad i \in E.$$

Hence,  $x_i^* - 1 \geq \lambda y_i^*$  since  $(\lambda y_i^*)$  is the minimal nonnegative solution to the equation above, by the linear combination theorem [6; Theorem 2.12 (1)]. That is,  $x_i^* \geq 1 + \lambda y_i^*$ . Combining what we have proved in the last paragraph, it follows that  $x_i^* = \mathbb{E}_i \exp(\lambda \sigma_H)$  for all  $i \in E$ .

Next, let  $\lambda < 0$ . Denote by  $(\bar{y}_i : i \in E)$  the minimal nonnegative solution to the equation

$$y_i = \frac{1}{q_i - \lambda} \sum_{k \notin H \cup \{i\}} q_{ik} y_k + \left[ 1 - \frac{1}{q_i - \lambda} \sum_{k \notin H \cup \{i\}} q_{ik} \right], \quad i \in E. \quad (7.2)$$

Clearly, we have  $\bar{y}_i \leq 1$  since  $y_i \equiv 1$  is a solution to the equation. We claim that  $\bar{y}_i \equiv 1$ . To see this, note that  $(1 - \bar{y}_i : i \in E)$  is the maximal solution to the equation

$$y_i = \frac{1}{q_i - \lambda} \sum_{k \notin H \cup \{i\}} q_{ik} y_k, \quad 0 \leq y_i \leq 1, \quad i \in E. \quad (7.3)$$

By a comparison lemma [6; Lemma 3.14], it suffices to show that the equation

$$y_i = \frac{1}{q_i} \sum_{k \notin H \cup \{i\}} q_{ik} y_k, \quad 0 \leq y_i \leq 1, \quad i \in E,$$

has only trivial (i.e., zero-) solution. Then this follows by the recurrence assumption and [6; Lemma 4.46]. We remark that there is an alternative way to prove that  $\bar{y}_i \equiv 1$ , using the uniqueness rather than the recurrence assumption. Actually, equation (7.3) is an exit equation for a modified  $Q$ -matrix (any local modification of a  $Q$ -matrix does not interfere the uniqueness). The exit solution to (7.3) should be zero by uniqueness assumption.

We now return to our main proof. By the linear combination theorem [6; Theorem 2.12 (1)],  $(x_i^* - \lambda y_i^* : i \in E)$  is the minimal nonnegative solution to equation (7.2). Hence,  $x_i^* - \lambda y_i^* = \bar{y}_i \equiv 1$  as we have just proved in the last paragraph. Therefore, we conclude that  $x_i^* = 1 + \lambda y_i^* = \mathbb{E}_i \exp(\lambda \sigma_H)$  for all  $i \in E$ . We have thus completed the proof of the lemma.  $\square$

Now, we present our results about the exponential moments of the return time  $\sigma_0$ , which can be referred in [18].



**Proposition 7.2** *Let the single birth  $Q$ -matrix  $(q_{ij})$  be irreducible. Assume that its process is ergodic. Define  $(\tilde{F}_k^{(i)})$  and  $(\tilde{d}_k)$  by setting  $c_i \equiv \lambda > 0$ . Then for small  $\lambda$ ,*

$$\mathbb{E}_0 e^{\lambda \sigma_0} = \frac{q_{01}(1 + \lambda \tilde{d})}{q_{01} - \lambda} < \infty, \quad \mathbb{E}_n e^{\lambda \sigma_0} = 1 + \lambda \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} \tilde{d} - \tilde{d}_k) < \infty, \quad n \geq 1,$$

if and only if

$$\tilde{d} := \overline{\lim}_{n \rightarrow \infty} \mathbb{1}_{\{\sum_{k=0}^n \tilde{F}_k^{(0)} > 0\}} \frac{\sum_{k=0}^n \tilde{d}_k}{\sum_{k=0}^n \tilde{F}_k^{(0)}} < \infty$$

and

$$\tilde{d} \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} > \sum_{k=0}^{n-1} \tilde{d}_k \quad \text{whenever} \quad \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} \leq 0 \quad \text{for } n \geq 2. \quad (7.4)$$

Furthermore, once  $\tilde{F}_n^{(0)} > 0$  for large enough  $n$  and  $\sum_n \tilde{F}_n^{(0)} = \infty$ , we have

$$\tilde{d} = \lim_{n \rightarrow \infty} \frac{\tilde{d}_n}{\tilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

Finally, the process is exponentially ergodic if and only if both  $\tilde{d} < \infty$  and (7.4) holds.

*Proof* Let  $\lambda \in (0, q_i)$  for every  $i \in E$  and set  $H = \{0\}$ . Then by Lemma 7.1,  $(\mathbb{E}_i e^{\lambda \sigma_0} : i \in E)$  is the minimal solution  $(x_i^*)$  of the following equation:

$$x_i = \frac{1}{q_i - \lambda} \sum_{k \notin \{0, i\}} q_{ik} x_k + \frac{q_{i0}(1 - \delta_{i0})}{q_i - \lambda}, \quad x_i \geq 1, \quad i \in E.$$

Assume that  $x_i^* < \infty$  for every  $i \in E$  for a moment, and let  $(x_i)$  be a finite nonnegative solution to the last equation. Then, by (4.2), we have

$$(Qx)_i + \lambda x_i = q_{i0}(x_0 - 1), \quad i \geq 1; \quad (Qx)_0 + \lambda x_0 = 0. \quad (7.5)$$

Applying Theorem 1.1 to  $c_i \equiv \lambda$  and  $f_i = q_{i0}(1 - \delta_{i0})(x_0 - 1)$  for all  $i \geq 0$ , we obtain

$$\begin{aligned} x_n &= x_0 \left( 1 - \lambda \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} \right) + (x_0 - 1) \sum_{k=1}^{n-1} \sum_{j=1}^k \frac{\tilde{F}_k^{(j)} q_{j0}}{q_{j,j+1}} \\ &= x_0 \left( 1 - \lambda \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} \right) + (x_0 - 1) \sum_{k=1}^{n-1} \sum_{j=1}^k \frac{\tilde{F}_k^{(j)} (\tilde{q}_j^{(0)} + \lambda)}{q_{j,j+1}}, \quad n \geq 1. \end{aligned}$$

Due to the explicit representation of  $\tilde{F}_n^{(k)}$ ,  $\tilde{m}_n$ , and  $\tilde{d}_n$ , given in (2.7), (3.2), and (5.2), respectively, we have not only

$$\tilde{m}_n = \sum_{0 \leq j \leq n} \frac{\tilde{F}_n^{(j)}}{q_{j,j+1}} = \frac{1}{q_{01}} \tilde{F}_n^{(0)} + \tilde{d}_n, \quad n \geq 0, \quad (7.6)$$

but also that

$$\begin{aligned} x_n &= x_0 \left( 1 - \lambda \sum_{k=0}^{n-1} \tilde{m}_k \right) + (x_0 - 1) \sum_{k=1}^{n-1} (\tilde{F}_k^{(0)} + \lambda \tilde{d}_k) \\ &= x_0 \left( 1 - \frac{\lambda}{q_{01}} \right) \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} - \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} + \lambda \tilde{d}_k) + 1, \quad n \geq 1. \end{aligned} \quad (7.7)$$

Since  $x_n > 1$ , we get

$$x_0 \left( 1 - \frac{\lambda}{q_{01}} \right) \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} > \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} + \lambda \tilde{d}_k), \quad n \geq 1.$$

That is,

$$\left[ x_0 \left( \frac{1}{\lambda} - \frac{1}{q_{01}} \right) - \frac{1}{\lambda} \right] \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} > \sum_{k=0}^{n-1} \tilde{d}_k, \quad n \geq 1. \quad (7.8)$$

Note that on one hand, if  $x_0^* = x_0^*(\lambda_0) < \infty$ , then  $x_0^* = x_0^*(\lambda) < \infty$  for every  $\lambda \in (0, \lambda_0)$ , by the comparison theorem (cf. [6; Theorem 2.6]). On the other hand, when  $\lambda = 0$ , we have

$$\sum_{k=0}^n \tilde{F}_k^{(0)} = \sum_{k=0}^n F_k^{(0)} > 0 \quad \text{and} \quad \sum_{k=0}^n \tilde{d}_k = \sum_{k=0}^n d_k > 0, \quad n \geq 1.$$

For each fixed  $n$ ,  $\sum_{k=0}^n \tilde{F}_k^{(0)}$  and  $\sum_{k=0}^n \tilde{d}_k$  are analytic in  $\lambda$ , and so should be positive for sufficient small  $\lambda$ , say  $\lambda \leq \lambda_1$  for some  $\lambda_1 \leq \lambda_0$ . Then by (7.8), we should have

$$x_0 \left( \frac{1}{\lambda} - \frac{1}{q_{01}} \right) - \frac{1}{\lambda} > 0, \quad \lambda \in (0, \lambda_1),$$

independent of  $n$ . Therefore, by the minimal property, we have

$$x_0^* \left( \frac{1}{\lambda} - \frac{1}{q_{01}} \right) - \frac{1}{\lambda} = \overline{\lim}_{n \rightarrow \infty} \mathbb{1}_{\{\sum_{k=0}^n \tilde{F}_k^{(0)} > 0\}} \left[ \sum_{k=0}^n \tilde{d}_k \right] \left[ \sum_{k=0}^n \tilde{F}_k^{(0)} \right]^{-1} = \tilde{d},$$

i.e.,

$$\mathbb{E}_0 e^{\lambda \sigma_0} = x_0^* = \frac{q_{01}(1 + \lambda \tilde{d})}{q_{01} - \lambda}. \quad (7.9)$$

Since  $x_0^*$  satisfies (7.8), we obtain condition (7.4). Then

$$\mathbb{E}_n e^{\lambda \sigma_0} = 1 + \lambda \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} \tilde{d} - \tilde{d}_k), \quad n \geq 1.$$

Conversely, if  $\tilde{d} < \infty$  and (7.4) holds, then starting from  $x_0 = x_0^*$  given in (7.9) and defining  $x_n$  by (7.7), we obtain a solution ( $x_i > 1: i \in E$ ) to (7.5).

By (4.2), we obtain a finite nonnegative solution to the original equation for  $(\mathbb{E}_i e^{\lambda\sigma_0} : i \in E)$ , and hence the minimal solution  $(x_i^* = \mathbb{E}_i e^{\lambda\sigma_0} : i \in E)$  should be finite.

Finally, by [6; Theorem 4.44], the process is exponentially ergodic if and only if  $\mathbb{E}_0 e^{\lambda\sigma_0} < \infty$ , equivalently,  $\tilde{d} < \infty$  and (7.4) holds. The last assertion of the proposition then follows.  $\square$

In contract to the ergodic case, one may study the exponential decay (in the transient case) for which the Poisson equation becomes

$$Qg + \lambda g = 0, \quad g > 0.$$

With  $c_i \equiv \lambda$ , by Theorem 1.1, the solution is

$$g_n = g_0 \left[ 1 - \lambda \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} \right] = g_0 \left[ 1 - \lambda \sum_{0 \leq k \leq n-1} \tilde{m}_k \right], \quad n \geq 0.$$

This is somehow simpler than the previous one. However, these two exponential cases are actually much harder than the others, for instance, we do not know at the moment how to remove condition (7.4). That is showing for some  $\lambda > 0$ , small enough,  $\sum_{k=0}^n \tilde{F}_k^{(0)} > 0$  for all  $n$  (or equivalently,  $\underline{\lim}_{n \rightarrow \infty} \sum_{k=0}^n \tilde{F}_k^{(0)} > 0$ ). This seems necessary for the exponential ergodicity since  $\sum_{k=0}^{\infty} \tilde{F}_k^{(0)} = \infty$  when  $\lambda = 0$  by the recurrence (which is much weaker than exponential ergodicity) and  $\lambda$  is allowed to be very small. Actually, to figure out a criterion, one needs much more work using different approaches, refer to [6; Chapter 9] and [7] for some details.

**7.2 Laplace transform of return/extinction time**

Note that for negative  $\lambda$ ,  $\mathbb{E}_i e^{\lambda\sigma_0}$  is the Laplace transform of  $\sigma_0$ . The proof of Proposition 7.2 is still available. So we get the following result.

**Proposition 7.3** *Define  $(\tilde{F}_k^{(i)})$  and  $(\tilde{d}_k)$  by (1.1) and (5.1), respectively, with  $c_i \equiv -\lambda < 0$ . Let the single birth process be recurrent. Then the Laplace transform of  $\sigma_0$  is given by*

$$\mathbb{E}_0 e^{-\lambda\sigma_0} = \frac{q_{01}(1 - \lambda\tilde{d})}{q_{01} + \lambda}, \quad \mathbb{E}_n e^{-\lambda\sigma_0} = 1 - \lambda \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} \tilde{d} - \tilde{d}_k), \quad n \geq 1,$$

where

$$\tilde{d} = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{n-1} \tilde{d}_k}{\sum_{k=0}^{n-1} \tilde{F}_k^{(0)}} = \lim_{n \rightarrow \infty} \frac{\tilde{d}_n}{\tilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

*Proof* Following the proof of Proposition 7.2, replacing  $\lambda$  by  $-\lambda$ , we arrive at

$$\begin{aligned} x_n &= x_0 \left( 1 + \frac{\lambda}{q_{01}} \right) \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} - \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} - \lambda\tilde{d}_k) + 1, \\ &=: x_0 \alpha_{n-1} - \beta_{n-1}, \quad n \geq 1. \end{aligned}$$

By the minimal nonnegative property,  $x_0^* = \sup_{n \geq 1} \beta_n / \alpha_n$ , and then we indeed have

$$x_0^* = \overline{\lim}_{n \rightarrow \infty} \frac{\beta_n}{\alpha_n}.$$

We now show that we can replace  $\overline{\lim}_{n \rightarrow \infty}$  by  $\lim_{n \rightarrow \infty}$ . Noting that on one hand, since  $x_n \in (0, 1]$ , we have

$$\frac{\beta_n}{\alpha_n} < x_0 \leq \frac{\beta_n + 1}{\alpha_n}, \quad n \geq 1.$$

On the other hand, following the proof for

$$\sum_k \tilde{m}_k = \infty \iff \sum_k m_k = \infty$$

given in Section 3, we can prove that  $\sum_k \tilde{F}_k^{(0)} = \infty$  since  $\sum_k F_k^{(0)} = \infty$  by the recurrent assumption (i.e.,  $\gamma_j \equiv 1$ ). Hence, we can rewrite  $\overline{\lim}_{n \rightarrow \infty} \beta_n / \alpha_n$  as  $\lim_{n \rightarrow \infty} \beta_n / \alpha_n$ . Therefore, we have

$$\begin{aligned} x_0^* &= \lim_{n \rightarrow \infty} \left[ \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} - \lambda \tilde{d}_k) \right] \left\{ \left[ 1 + \frac{\lambda}{q_{01}} \right] \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} \right\}^{-1} \\ &= \frac{q_{01}}{q_{01} + \lambda} \lim_{n \rightarrow \infty} \left[ 1 - \lambda \frac{\sum_{k=0}^{n-1} \tilde{d}_k}{\sum_{k=0}^{n-1} \tilde{F}_k^{(0)}} \right] \\ &= \frac{q_{01}}{q_{01} + \lambda} [1 - \lambda \tilde{d}]. \end{aligned}$$

Furthermore,

$$x_n^* = (1 - \lambda \tilde{d}) \sum_{k=0}^{n-1} \tilde{F}_k^{(0)} - \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} - \lambda \tilde{d}_k) + 1 = 1 - \lambda \sum_{k=0}^{n-1} (\tilde{F}_k^{(0)} \tilde{d} - \tilde{d}_k), \quad n \geq 1.$$

The last limit in  $\tilde{d}$  is an application of Stolz's Theorem. □

### 7.3 Exponential moments and Laplace transform of life time

Now we return to  $\tau_\infty$ .

**Proposition 7.4** *Assume that the single birth  $Q$ -matrix  $Q = (q_{ij})$  is explosive and irreducible. Define  $(\tilde{m}_k)$  by (3.1) with  $c_i \equiv \lambda$ . For the corresponding minimal process,*

- (i) *if there exists a  $\lambda > 0$  such that  $\lambda \sum_{k=0}^{n-1} \tilde{m}_k < 1$  for every  $n > 1$ , then*

$$\mathbb{E}_n e^{\lambda \tau_\infty} = 1 + \lambda \left[ \bar{c} \left( 1 - \lambda \sum_{k=0}^{n-1} \tilde{m}_k \right) - \sum_{k=0}^{n-1} \tilde{m}_k \right], \quad n \geq 0,$$

where

$$\bar{c} = \overline{\lim}_{n \rightarrow \infty} \frac{\sum_{k=0}^n \tilde{m}_k}{1 - \lambda \sum_{k=0}^n \tilde{m}_k}.$$

Furthermore, the process decays exponentially fast provided  $\bar{c} < \infty$ .

(ii) For  $\lambda > 0$ , the Laplace transform of  $\tau_\infty$  is given by

$$\mathbb{E}_n e^{-\lambda \tau_\infty} = \frac{1 + \lambda \sum_{0 \leq k \leq n-1} \tilde{m}_k}{1 + \lambda \sum_{k \geq 0} \tilde{m}_k}, \quad n \geq 0.$$

*Proof* Define

$$e_{i\infty}(\lambda) = \int_0^\infty e^{\lambda t} \mathbb{P}_i(\tau_\infty > t) dt$$

with  $\lambda < q_i$  for all  $i \geq 0$ . Note that the process is explosive and

$$\mathbb{E}_i e^{\lambda \tau_\infty} = 1 + \lambda e_{i\infty}(\lambda).$$

Because  $\mathbb{P}_m(\tau_n < \eta) = 1$  for every pair  $m < n$ , we have  $\mathbb{P}_m(\tau_n < \infty) = 1$  and furthermore  $\mathbb{P}_m(\tau_\infty < \infty) = 1$  for every  $m$ , as  $n$  goes to  $\infty$ . Then by [6; Lemma 4.48],  $(e_{i\infty}(\lambda))$  is the minimal solution to the equation

$$x_i = \frac{q_i}{q_i - \lambda} \sum_k \Pi_{ik} x_k + \frac{1}{q_i - \lambda}, \quad i \geq 0.$$

By (4.2), we can rewrite the equation as

$$(Qx)_i + \lambda x_i = -1, \quad i \geq 0.$$

Applying Theorem 1.1 to  $c_i \equiv \lambda$  and  $f_i \equiv -1$ , the solution of the equation has the form:

$$\begin{aligned} x_n &= x_0 \left( 1 - \lambda \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} \right) - \sum_{k=0}^{n-1} \sum_{j=0}^k \frac{\tilde{F}_k^{(j)}}{q_{j,j+1}} \\ &= x_0 \left( 1 - \lambda \sum_{k=0}^{n-1} \tilde{m}_k \right) - \sum_{k=0}^{n-1} \tilde{m}_k, \quad n \geq 1. \end{aligned}$$

Note that  $\lambda < q_0 = q_{01}$  and  $\lambda \tilde{m}_0 < 1$ . If there exists a positive  $\lambda$  small enough so that  $\lambda \sum_{k=0}^{n-1} \tilde{m}_k < 1$  for every  $n > 1$ , then by the argument above and the minimal property of the solution, one gets

$$e_{0\infty}(\lambda) = \sup_{n \geq 1} \frac{\sum_{k=0}^{n-1} \tilde{m}_k}{1 - \lambda \sum_{k=0}^{n-1} \tilde{m}_k} = \overline{\lim}_{n \rightarrow \infty} \frac{\sum_{k=0}^n \tilde{m}_k}{1 - \lambda \sum_{k=0}^n \tilde{m}_k} =: \bar{c}$$

and

$$e_{n\infty}(\lambda) = \bar{c} \left( 1 - \lambda \sum_{k=0}^{n-1} \tilde{m}_k \right) - \sum_{k=0}^{n-1} \tilde{m}_k, \quad n \geq 1.$$

Then the first assertion follows.

For the Laplace transform of  $\tau_\infty$ , the argument above still works because now we deal with the case of  $-\lambda < 0$ . By the explosive property, we know that  $\sum_{k=0}^\infty \tilde{m}_k < \infty$ . Hence, we have

$$e_{0\infty}(-\lambda) = \bar{c} = \frac{\sum_{k=0}^\infty \tilde{m}_k}{1 + \lambda \sum_{k=0}^\infty \tilde{m}_k}$$

and

$$e_{n\infty}(-\lambda) = \bar{c} \left( 1 + \lambda \sum_{k=0}^{n-1} \tilde{m}_k \right) - \sum_{k=0}^{n-1} \tilde{m}_k = \frac{\sum_{k=n}^\infty \tilde{m}_k}{1 + \lambda \sum_{k=0}^\infty \tilde{m}_k}, \quad n \geq 1.$$

Finally, we have

$$\mathbb{E}_n e^{-\lambda \tau_\infty} = 1 - \frac{\lambda \sum_{k=n}^\infty \tilde{m}_k}{1 + \lambda \sum_{k=0}^\infty \tilde{m}_k} = \frac{1 + \lambda \sum_{0 \leq k \leq n-1} \tilde{m}_k}{1 + \lambda \sum_{k \geq 0} \tilde{m}_k}, \quad n \geq 0.$$

The proof for the second assertion is now finished. □

A more careful study on part (i) of Proposition 7.4, refer to Proposition 7.2.

### 8 Examples

In the special case of birth–death processes, the problems studied here have rather complete solutions, see, for instance, [6; Theorem 4.55]. As mentioned in the introduction of the paper, much more models have been studied in the past years. Here, we make a little addition. The following example is taken from [3].

**Example 8.1** (uniform catastrophes) Let

$$q_{i,i+1} = bi, \quad i \geq 0; \quad q_{ij} = a, \quad j = 0, 1, \dots, i - 1;$$

and  $q_{ij} = 0$  for other  $j > i + 1$ , where  $a$  and  $b$  are positive constants. Then the extinction of the process has an exponential distribution

$$\mathbb{E}_n e^{-\lambda \tau_0} = \frac{a}{a + \lambda}, \quad \lambda > 0, \quad n \geq 1.$$

It is surprising that the distribution is independent of  $b$  and the starting point  $n$ . Redefine  $q_{01} = 1$ . Then the irreducible process is indeed strongly ergodic.

*Proof* We need to consider the case that  $q_{01} > 0$  only. With  $c_i \equiv -\lambda \in \mathbb{R}$  and then  $\tilde{q}_n^{(k)} = (k + 1)a + \lambda$  for  $k \leq n - 1$ , by using (1.1), (5.1), and induction, one may check that

$$\begin{aligned} \tilde{F}_n^{(0)} &= \frac{a + \lambda}{nb} \prod_{1 \leq k \leq n-1} \left( 1 + \frac{(k + 1)a + \lambda}{kb} \right), & \prod_{\emptyset} &= 1, \\ \tilde{d}_n &= \frac{1}{nb} \prod_{1 \leq k \leq n-1} \left( 1 + \frac{(k + 1)a + \lambda}{kb} \right), & n &\geq 1. \end{aligned}$$

Since for each fixed  $\lambda \in \mathbb{R}$ ,

$$\log \left( 1 + \frac{(n+1)a + \lambda}{nb} \right) \rightarrow \log \left( 1 + \frac{a}{b} \right) > 0 \quad \text{as } n \rightarrow \infty,$$

we have  $\lim_{n \rightarrow \infty} \tilde{F}_n^{(0)} = \infty$  and so  $\sum_n \tilde{F}_n^{(0)} = \infty$ . As an application of this fact with  $\lambda = 0$ , it follows that the process is recurrent (Proposition 4.1) and then should be non-explosive ((7.6) and Proposition 3.1).

Next, because

$$\sum_n \tilde{F}_n^{(0)} = \infty, \quad \tilde{F}_n^{(0)} = (a + \lambda)\tilde{d}_n, \quad n \geq 1,$$

it follows that

$$\tilde{d} = \lim_{n \rightarrow \infty} \frac{\tilde{d}_n}{\tilde{F}_n^{(0)}} = \frac{1}{a + \lambda}.$$

Hence, we have

$$\tilde{F}_n^{(0)} \tilde{d} = \tilde{d}_n, \quad n \geq 1,$$

From here, when  $\lambda = 0$  in particular, we obtain

$$\sup_k \sum_{n=0}^k (F_n^{(0)} d - d_n) = d = a^{-1} < \infty.$$

Hence, the process is strongly ergodic by Proposition 5.2.

By using Proposition 7.3, we obtain

$$\begin{aligned} \mathbb{E}_0 e^{-\lambda \sigma_0} &= \frac{aq_{01}}{(a + \lambda)(q_{01} + \lambda)}, \\ \mathbb{E}_n e^{-\lambda \sigma_0} &= 1 - \lambda \tilde{d} = \frac{a}{a + \lambda} = \mathbb{E}_n e^{-\lambda \tau_0}, \quad n \geq 1. \end{aligned}$$

Therefore, we have proved the first assertion.

Even though it is now automatic that the process is exponentially ergodic, implied by the strongly ergodicity, we would like to check the effectiveness of Proposition 7.2 for this model. To do so, reset  $c_i \equiv \lambda > 0$ . Then

$$\begin{aligned} \tilde{F}_n^{(0)} &= \frac{a - \lambda}{nb} \prod_{1 \leq k \leq n-1} \left( 1 + \frac{(k+1)a - \lambda}{kb} \right), \\ \tilde{d}_n &= \frac{1}{nb} \prod_{1 \leq k \leq n-1} \left( 1 + \frac{(k+1)a - \lambda}{kb} \right), \end{aligned} \quad n \geq 1.$$

Clearly,  $\tilde{F}_n^{(0)} > 0$  and so does  $\tilde{d}_n$  for every  $\lambda \in (0, a)$ . As we have proved above

$$\sum_n \tilde{F}_n^{(0)} = \infty, \quad \tilde{d} = \lim_{n \rightarrow \infty} \frac{\tilde{d}_n}{\tilde{F}_n^{(0)}} = \frac{1}{a - \lambda} < \infty,$$

and hence the process is exponentially ergodic by Proposition 7.2. Actually, we have

$$\begin{aligned}\mathbb{E}_0 e^{\lambda \sigma_0} &= \frac{aq_{01}}{(a-\lambda)(q_{01}-\lambda)}, \\ \mathbb{E}_n e^{\lambda \sigma_0} &= \frac{a}{a-\lambda}, \quad n \geq 1,\end{aligned}\quad \lambda \in (0, a \wedge q_{01}). \quad \square$$

**Example 8.2** Consider the single birth  $Q$ -matrix  $(q_{ij})$  with

$$q_{i0}, q_{i,i+1} > 0, \quad q_{ij} = 0 \text{ for all other } j \neq i.$$

Let  $c_i \in \mathbb{R}$ . Then

(1) we have

$$\begin{aligned}\tilde{F}_i^{(i)} &= 1, \\ \tilde{F}_n^{(i)} &= \frac{q_{n0} - c_n}{q_{n,n+1}} \prod_{i+1 \leq k \leq n-1} \left[ 1 + \frac{q_{k0} - c_k}{q_{k,k+1}} \right], \quad n > i \geq 0,\end{aligned}\quad (8.1)$$

where  $\prod_{\emptyset} =: 1$ , and then  $(\tilde{m}_n)$  and  $(\tilde{d}_n)$  are given by (3.2) and (5.2), respectively.

(2) In particular, if  $q_{n0} - c_n \equiv q_{10} - c_1$  for every  $n \geq 1$ , then

$$\begin{aligned}\tilde{F}_i^{(i)} &= 1, \quad \tilde{F}_n^{(i)} = \frac{q_{10} - c_1}{q_{n,n+1}} \prod_{k=i+1}^{n-1} \left[ 1 + \frac{q_{10} - c_1}{q_{k,k+1}} \right], \quad n > i \geq 0, \quad \prod_{\emptyset} =: 1, \\ \tilde{m}_0 &= \frac{1}{q_{01}}, \quad \tilde{m}_n = \frac{1}{q_{n,n+1}} \prod_{k=0}^{n-1} \left[ 1 + \frac{q_{10} - c_1}{q_{k,k+1}} \right], \quad n \geq 1, \\ \tilde{d}_0 &= 0, \quad \tilde{d}_n = \frac{1}{q_{n,n+1}} \prod_{1 \leq k \leq n-1} \left[ 1 + \frac{q_{10} - c_1}{q_{k,k+1}} \right], \quad n \geq 1.\end{aligned}$$

Furthermore, the process is explosive if

$$\kappa' := \lim_{n \rightarrow \infty} \frac{n(q_{n+1,n+2} - q_{n,n+1} - q_{10})}{q_{n,n+1} + q_{10}} > 1$$

( $q_{n,n+1} = (n+1)^\gamma$  for  $\gamma > 1$  for example). Otherwise, if  $\kappa' < 1$  ( $q_{n,n+1} = (n+1)^\gamma$  for some  $\gamma \leq 1$  for instance), then the process is unique. If so, the process is indeed strongly ergodic.

*Proof* (a) By assumption, we have  $\tilde{q}_n^{(k)} = q_{n0} - c_n$  for every  $k < n$ . Hence, by (1.1), we obtain

$$\tilde{F}_n^{(i)} = \frac{\tilde{q}_n^{(0)}}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{F}_k^{(i)}. \quad (8.2)$$

Thus, to prove (8.1), it suffices to show that



$$\sum_{k=i}^{n-1} \tilde{F}_k^{(i)} = \prod_{i+1 \leq k \leq n-1} \left[ 1 + \frac{\tilde{q}_k^{(0)}}{q_{k,k+1}} \right], \quad n > i \geq 0.$$

This clearly holds when  $n = i + 1$ . Suppose that it holds when  $n = \ell$ . Then

$$\begin{aligned} \sum_{k=i}^{\ell} \tilde{F}_k^{(i)} &= \sum_{k=i}^{\ell-1} \tilde{F}_k^{(i)} + \tilde{F}_{\ell}^{(i)} \\ &= \sum_{k=i}^{\ell-1} \tilde{F}_k^{(i)} + \frac{\tilde{q}_{\ell}^{(0)}}{q_{\ell,\ell+1}} \sum_{k=i}^{\ell-1} \tilde{F}_k^{(i)} \quad (\text{by (8.2)}) \\ &= \left[ 1 + \frac{\tilde{q}_{\ell}^{(0)}}{q_{\ell,\ell+1}} \right] \sum_{k=i}^{\ell-1} \tilde{F}_k^{(i)} \\ &= \prod_{i+1 \leq k \leq \ell} \left[ 1 + \frac{\tilde{q}_k^{(0)}}{q_{k,k+1}} \right] \quad (\text{by inductive assumption}). \end{aligned}$$

Therefore, the required assertion holds for  $n = \ell$  and it then holds for all  $n > i$  by induction. We have thus proved the first assertion.

(b) By assumption, we have  $\tilde{q}_n^{(k)} = q_{10} - c_1$  for every  $k < n$ . Hence, by (3.1) and (5.1), we obtain

$$\tilde{m}_n = \frac{1}{q_{n,n+1}} \left( 1 + \tilde{q}_1^{(0)} \sum_{k=0}^{n-1} \tilde{m}_k \right), \quad \tilde{d}_n = \frac{1}{q_{n,n+1}} \left( 1 + \tilde{q}_1^{(0)} \sum_{k=0}^{n-1} \tilde{d}_k \right), \quad n \geq 1.$$

As in the last proof, by using induction, we obtain the explicit expressions of  $(\tilde{m}_n)$  and  $(\tilde{d}_n)$ .

To study the divergence of  $\sum_n m_n$ , we adopt the following result.

**Kummer Test** *Let  $(u_n)$  and  $(v_n)$  be two sequences of positive numbers. Suppose that  $\sum_0^{\infty} \frac{1}{v_n} = \infty$  and the limit  $\kappa := \lim_{n \rightarrow \infty} \kappa_n$  exists, where*

$$\kappa_n = v_n \cdot \frac{u_n}{u_{n+1}} - v_{n+1}.$$

*Then, the series  $\sum u_n$  converges or diverges according to  $\kappa > 0$  or  $\kappa < 0$ , respectively.*

Set  $v_n \equiv n$  and  $u_n = m_n$ :

$$m_n = \frac{1}{q_{n,n+1}} \prod_{0 \leq k \leq n-1} \left[ 1 + \frac{q_{10}}{q_{k,k+1}} \right], \quad n \geq 0.$$

Then

$$v_n \frac{u_n}{u_{n+1}} - v_{n+1} = \frac{n(q_{n+1,n+2} - q_{n,n+1} - q_{10})}{q_{n,n+1} + q_{10}} - 1.$$

Hence,  $\sum_n u_n < \infty$  if  $\kappa' > 1$  (resp.  $\sum_n u_n = \infty$  once  $\kappa' < 1$ ). Clearly,  $\sum_n m_n = \infty$  implies  $\sum_n F_n^{(0)} = \infty$ . Hence,

$$d = \lim_{n \rightarrow \infty} \frac{d_n}{F_n^{(0)}} = \frac{1}{q_{01}}.$$

Furthermore,

$$\sup_{k \in E} \sum_{n=0}^k (F_n^{(0)} d - d_n) = F_0^{(0)} d = d < \infty.$$

This gives us the strong ergodicity by Proposition 5.2.

We mention that Proposition 7.2 (with  $0 < c_i \equiv \lambda < q_{10}$ ) is also available for this model.  $\square$

**Remark 8.3** For exponential ergodicity, the sufficient condition

$$M := \sup_{n \geq 1} \left[ \sum_{k=1}^{n-1} F_k^{(0)} \right] \left[ \sum_{j=n}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} \right] < \infty, \quad (8.3)$$

introduced in [12], is sufficient for Example 8.1 but is not for Example 8.2.

*Proof* It is obvious that  $M < \infty$  if and only if

$$\overline{\lim}_{n \rightarrow \infty} \left[ \sum_{k=1}^{n-1} F_k^{(0)} \right] \left[ \sum_{j=n}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} \right] < \infty. \quad (8.4)$$

For Example 8.1, because  $q_{j,j+1} F_j^{(0)}$  is growing exponentially fast and so it is easy to check that  $M < \infty$ . For Example 8.2, it suffices to consider  $q_{n,n+1} = b(n+1)$  for some  $b > 0$ . By Kummer test, one may show that

$$\sum_{j=n}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} = \infty$$

for suitable  $b > 0$  and then  $M = \infty$ .  $\square$

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**Appendix Key formulas used in proofs**

(A) Solution to the Poisson equation  $\Omega g = Qg + cg$ :

$$g_n = g_0 + \sum_{0 \leq k \leq n-1} \sum_{0 \leq j \leq k} \frac{\tilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \quad n \geq 0.$$

(B) Three sequences.

(a)  $\tilde{F}$ -sequence:

$$\tilde{F}_i^{(i)} = 1, \quad \tilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \tilde{F}_k^{(i)}, \quad n > i \geq 0, \quad (1.1)$$

where

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^k q_{nj} - c_n, \quad 0 \leq k < n. \quad (1.2)$$

(b)  $\tilde{m}$ -sequence:

$$\tilde{m}_0 = \frac{1}{q_{01}}, \quad \tilde{m}_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{m}_k \right), \quad n \geq 1. \quad (3.1)$$

(c)  $\tilde{d}$ -sequence:

$$\tilde{d}_0 = 0, \quad \tilde{d}_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} \tilde{q}_n^{(k)} \tilde{d}_k \right), \quad n \geq 1. \quad (5.1)$$

Representation of the three sequences:

$$\tilde{F}_i^{(i)} = 1, \quad \tilde{F}_n^{(i)} = \sum_{k=i+1}^n \frac{\tilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, \quad n \geq i+1, \quad (2.7)$$

$$\tilde{d}_n = \sum_{1 \leq k \leq n} \frac{\tilde{F}_n^{(k)}}{q_{k,k+1}}, \quad (5.2)$$

$$\tilde{m}_n = \sum_{k=0}^n \frac{\tilde{F}_n^{(k)}}{q_{k,k+1}}, \quad n \geq 0. \quad (3.2)$$

Relation of the three sequences:

$$\tilde{m}_n = \frac{1}{q_{01}} \tilde{F}_n^{(0)} + \tilde{d}_n, \quad n \geq 0. \quad (7.6)$$

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