A Generalization of Dobrushin Coefficient *

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Abstract

We generalize the well-known Dobrushin coefficient $\delta$ in total variation to weighted total variation $\delta_V$, which gives a criterion for the geometric ergodicity of discrete-time Markov chains.

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§1. Introduction and Main Results

In this paper, we generalize the classical Dobrushin coefficient, in order to give a criteria for geometric ergodicity of discrete-time Markov chains.

Let $X = (X_n)_{n \geq 0}$ be a discrete-time Markov chain taking values on a measurable space $(E, \mathcal{E})$. Denote by

$$P^n(x, A) := P[X_n \in A| X_0 = x], \quad x \in E, A \in \mathcal{E},$$

the $n$-step transition kernel. Set $P = P^1$ be the one-step kernel.

Throughout the paper, we assume that $X$ is $\psi$-irreducible and aperiodic, c.f. [1].

We are interested in the geometrical ergodicity that is, there is an invariant probability measure $\pi$ on $(E, \mathcal{E})$ and a constant $\rho \in [0, 1)$ and a function $C : E \to (0, \infty)$ such that

$$\|P^n(x, \cdot) - \pi\|_{\text{Var}} \leq C(x)\rho^n, \quad \text{for all } n > 0, \pi\text{-a.s. } x \in E, \quad (1.1)$$

where $\|\mu\|_{\text{Var}} := 2 \sup_{A \in \mathcal{E}} |\mu(A)| = \sup \left\{ \int_E f\,d\mu : |f| \leq 1 \right\}$ is the total variation for a signed measure $\mu$.

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From [1, 2], we know that (1.1) is equivalent to that there exist constants \( \rho \in [0, 1) \), \( C < \infty \) and a \( \pi \)-a.s. finite function \( V : E \to [1, \infty] \) such that

\[
\|P^n(x, \cdot) - \pi\|_V \leq CV(x)\rho^n;
\]

(1.2)

where \( \|\mu\|_V := \sup \left\{ \int_E f \, d\mu : \|f\|_V \leq 1 \right\} \) denotes the \( V \)-norm on signed measures. Recall that for a kernel \( K(x, A) \) \( (x \in E, A \in \mathcal{E}) \),

\[
\|K\|_V := \sup_{x \in E} \frac{\|K(x, \cdot)\|_V}{V(x)},
\]

(1.3)

and for a function \( f \),

\[
\|f\|_V := \sup_{x \in E} \frac{|f(x)|}{V(x)},
\]

(1.4)

C.f. [1, Chapter 14].

Specially, if in (1.1) \( C(x) \leq C < \infty \) for \( \pi \)-a.s. \( x \in E \), (1.1) is strengthened to the so-called strong ergodicity. Equivalently in (1.2), we can choose \( V : E \to [1, \infty) \) to be (upper) bounded. Recall that \( X \) is said to be strongly ergodic if there exist constants \( \rho \in [0, 1) \) and \( C < \infty \) such that

\[
\text{ess}_{\pi} \sup_{x \in E} \|P^n(x, \cdot) - \pi\|_{\text{Var}} \leq C\rho^n, \quad \text{for all } n > 0.
\]

(1.5)

Dobrushin (1956) gave an elegant criterion for strong ergodicity in (1.5). Let

\[
\delta(P) = \frac{1}{2} \text{ess}_{\pi} \sup_{x, y \in E} \|P(x, \cdot) - P(y, \cdot)\|_{\text{Var}}.
\]

Then, \( X \) is strongly ergodic if and only if there exists \( n \), such that \( \delta(P^n) < 1 \). \( \delta(P) \) is now in term of Dobrushin coefficient.

In the context of \( E \) countable, an elegant description of \( \delta(P) \) and strong ergodicity can be found in [4, §6.1, §6.3]. Although [4, §6.3] only considered continuous-time Markov chains, these arguments remain valid for a general Markov process. Or this can be viewed as a special case where we will do in the following (by taking \( V(x) \equiv 1 \)).

Now, we will generalize \( \delta(P) \) to \( \delta_V(P) \), which gives a criteria for geometric ergodicity in (1.1) or (1.2).

**Definition 1.1** For a transition kernel \( P \), and a \( \pi \)-a.s. finite function \( V : E \to [1, \infty] \), we define the generalized Dobrushin coefficient \( \delta_V(P) \):

\[
\delta_V(P) := \text{ess}_{\pi} \sup_{x, y \in E} \frac{1}{V(x) + V(y)} \|P(x, \cdot) - P(y, \cdot)\|_V.
\]

(1.6)
Here is the main result.

**Theorem 1.1** A $\psi$-irreducible and aperiodic Markov chain $X = (X_n)_{n \geq 0}$ is geometrically ergodic if and only if $X$ is ergodic with stationary distribution $\pi$, and there is a $\pi$-a.s. finite function $V : E \to [1, \infty]$ such that $\pi(V) < \infty$ and $\delta_V(P^n) < 1$ for $n$ large enough.

Note that when $V(x) \equiv 1$, $\delta_V(P) = \delta(P)$ and hence Theorem 1.1 generalizes the classical Dobrushin coefficient.

The key ingredient for the proof of Theorem 1.1 comes from an excellent observation. By defining a metric on $E$ as in [5], we can transfer $\delta_V(P)$ to be a Wasserstein distance on space of probability measures.

### §2. Proof of Theorem

For simplicity, in the following, $\sup$ is referred to $\text{ess}_\pi \sup$.

Following [5], we introduce a metric on $E$. Let $V : E \to [1, \infty)$. For $x, y \in E$,

$$d_V(x, y) = \begin{cases} 0, & x = y; \\ V(x) + V(y), & x \neq y. \end{cases}$$

Then $(E, d_V)$ is a complete metric space. Let $\varphi$ be a function on $(E, d_V)$, the Lipschitz norm of $\varphi$ is

$$\|\varphi\|_{\text{Lip}(d_V)} := \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d_V(x, y)} = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{V(x) + V(y)}.$$

For two (probability) measures $\mu_1, \mu_2$ on $(E, \mathcal{E})$, Wasserstein metric $W_{d_V}$ is defined by

$$W_{d_V}(\mu_1, \mu_2) = \sup_{\varphi : \|\varphi\|_{\text{Lip}(d_V)} \leq 1} \int E \varphi d(\mu_1 - \mu_2).$$

The following was proved in [5, Lemma 2.1].

**Lemma 2.1** For two probability measures $\mu_1, \mu_2$ on $(E, \mathcal{E})$,

$$\|\mu_1 - \mu_2\|_V = W_{d_V}(\mu_1, \mu_2). \quad (2.1)$$

**Remark 1** (1) For $V(x) \equiv 1$, both sides of (2.1) are reduced to the total variation of $\mu_1 - \mu_2$. For the left side, when $V(x) \equiv 1$,

$$\|\mu_1 - \mu_2\|_V = \sup \left\{ \int_E f d(\mu_1 - \mu_2) : |f| \leq 1 \right\} = \|\mu_1 - \mu_2\|_{\text{Var}}.$$
For the right side, $d_V$ is nothing but discrete metric. Hence by [6, Theorem 5.7], $W_{d_V}(\mu_1, \mu_2) = \|\mu_1 - \mu_2\|_{\text{var}}$.

(2) As pointed out in [5], (2.1) is also true for measures $\mu_1, \mu_2$ with equal mass.

The following properties are essential to the proof of Theorem 1.1.

**Proposition 2.1** Let $P$ and $\tilde{P}$ be probability transition kernels on $(E, \mathcal{E})$. Then

(i) $\delta_V(P\tilde{P}) \leq \delta_V(P)\delta_V(\tilde{P})$,

(ii) $|\delta_V(P) - \delta_V(\tilde{P})| \leq \|P - \tilde{P}\|_V$,

(iii) Let $R$ be a transition kernel on $(E, \mathcal{E})$ such that $R(x, E) = 0$, for all $x \in E$ and $\|R\|_V < \infty$. Then $\|RP\|_V \leq \|R\|_V \delta_V(P)$.

**Proof** (i) By the definition of $\delta_V(P)$ and Lemma 2.1, we have

$$W_{d_V}(P(x, \cdot), P(y, \cdot)) = \|P(x, \cdot) - P(y, \cdot)\|_V \leq \delta_V(P)(V(x) + V(y)), \quad \pi\text{-a.s. } x, y \in E.$$ 

Hence by the definition of $W_{d_V}$, we get

$$|P\varphi(x) - P\varphi(y)| \leq \delta_V(P)\|\varphi\|_{\text{Lip}(d_V)}(V(x) + V(y))$$

or

$$\frac{1}{V(x) + V(y)}|P\varphi(x) - P\varphi(y)| \leq \delta_V(P)\|\varphi\|_{\text{Lip}(d_V)},$$

and

$$\|P\varphi\|_{\text{Lip}(d_V)} \leq \delta_V(P)\|\varphi\|_{\text{Lip}(d_V)}.$$  \hspace{1cm} (2.2)

Since $\int(\tilde{P}P)(x, dz)\varphi(z) = \int(P\varphi)(z)\tilde{P}(x, dz)$, we get from (2.2) that

$$W_{d_V}((\tilde{P}P)(x, \cdot), (\tilde{P}P)(y, \cdot)) = \sup_{\|\varphi\|_{\text{Lip}(d_V)} \leq 1} \int(P\varphi)(z)(\tilde{P}(x, dz) - \tilde{P}(y, dz))$$

$$\leq \delta_V(P) \sup_{\|\psi\|_{\text{Lip}(d_V)} \leq 1} \int\psi(z)(\tilde{P}(x, dz) - \tilde{P}(y, dz))$$

$$= \delta_V(P)W_{d_V}(\tilde{P}(x, \cdot), \tilde{P}(y, \cdot)).$$

Therefore by Lemma 2.1 again, we have

$$\delta_V(\tilde{P}P) = \sup_{x, y \in E} \frac{1}{V(x) + V(y)}\|(\tilde{P}P)(x, \cdot) - (\tilde{P}P)(y, \cdot)\|_V$$

$$= \sup_{x, y \in E} \frac{1}{V(x) + V(y)}W_{d_V}((\tilde{P}P)(x, \cdot), (\tilde{P}P)(y, \cdot))$$

$$\leq \delta_V(P) \sup_{x, y \in E} \frac{1}{V(x) + V(y)}W_{d_V}(\tilde{P}(x, \cdot), \tilde{P}(y, \cdot))$$

$$= \delta_V(P) \delta_V(\tilde{P}).$$
(ii) Using the fact that $\left| \sup_x |u(x)| - \sup_x |v(x)| \right| \leq \sup_x |u(x) - v(x)|$, we get

$$\left| \delta_V(P) - \delta_V(\tilde{P}) \right|$$

$$= \left| \sup_{x,y \in E} \frac{1}{V(x) + V(y)} \|P(x, \cdot) - P(y, \cdot)\|_V - \sup_{x,y \in E} \frac{1}{V(x) + V(y)} \|\tilde{P}(x, \cdot) - \tilde{P}(y, \cdot)\|_V \right|$$

$$\leq \sup_{x,y \in E} \frac{1}{V(x) + V(y)} \sup_{|f| \leq V} \left| (Pf(x) - Pf(y)) - (\tilde{P}f(x) - \tilde{P}f(y)) \right|$$

$$\leq \sup_{x,y \in E} \frac{1}{V(x) + V(y)} \sup_{|f| \leq V} \left| Pf(x) - \tilde{P}f(x) \right| + \left| Pf(y) - \tilde{P}f(y) \right|$$

$$\leq \sup_{x \in E} \frac{\|P(x, \cdot) - \tilde{P}(x, \cdot)\|_V + \|P(y, \cdot) - \tilde{P}(y, \cdot)\|_V}{V(x) + V(y)}$$

$$\leq \sup_{x \in E} \frac{\|P(x, \cdot) - \tilde{P}(x, \cdot)\|_V}{V(x)} = \|P - \tilde{P}\|_V,$$

where in the last inequality, we use the element that

$$\frac{u(x) + u(y)}{v(x) + v(y)} \leq \max \left\{ \frac{u(x)}{v(x)}, \frac{u(y)}{v(y)} \right\}.$$

(iii) Since $R(x, E) = 0$, for all $x \in E$, we have Jordan-Hahn decomposition $R = R^+ - R^-$, hence

$$\|RP\|_V = \sup_{x \in E} \frac{1}{V(x)} \|(R^+P)(x, \cdot) - (R^-P)(x, \cdot)\|_V$$

$$= \sup_{x \in E} \frac{1}{V(x)} W_{d_V}((R^+P)(x, \cdot), (R^-P)(x, \cdot)).$$

It follows from (2.2) and Lemma 2.1 that,

$$W_{d_V}((R^+P)(x, \cdot), (R^-P)(x, \cdot)) = \sup_{\varphi, \|\varphi\|_{Lip(\mathcal{V})} \leq 1} \int \varphi(y)((R^+P)(x, dy) - (R^-P)(x, dy))$$

$$= \sup_{\varphi, \|\varphi\|_{Lip(\mathcal{V})} \leq 1} \int (P\varphi)(y)[R^+(x, dy) - R^-(x, dy)]$$

$$\leq \delta_V(P) \sup_{\varphi, \|\varphi\|_{Lip(\mathcal{V})} \leq 1} \int \phi(y)[R^+(x, dy) - R^-(x, dy)]$$

$$= \delta_V(P) W_{d_V}(R^+(x, \cdot), R^-(x, \cdot))$$

$$= \delta_V(P)\|R^+(x, \cdot) - R^-(x, \cdot)\|_V.$$

Finally we obtain

$$\|RP\|_V \leq \delta_V(P) \sup_{x \in E} \frac{1}{V(x)} \|R^+(x, \cdot) - R^-(x, \cdot)\|_V = \delta_V(P)\|R\|_V. \quad \square$$
Proof of Theorem 1.1  Set $\Pi(x, \cdot) = \pi$, for all $x \in E$. As noted in [1], the function $V$ in (1.6) can be chosen to be such that $\pi(V) < \infty$. Since $\delta_V(\Pi) = 0$, we obtain
\[
\delta_V(P^n) = |\delta_V(P^n) - \delta_V(\Pi)| \leq \|P^n - \Pi\|_V = \sup_{x \in E} \frac{1}{V(x)} \|P^n(x, \cdot) - \pi\|_V \to 0.
\]

Conversely, we have $\|I - \Pi\|_V \leq \sup_{x \in E} \{V(x) + \pi(V)\}/V(x) = 1 + \pi(V) < \infty$ and $(I - \Pi)(x, E) = 0$ for all $x \in E$. Then
\[
\|P^n - \Pi\|_V = \|P^n - \Pi P^n\|_V = \|(I - \Pi)P^n\|_V \leq \|I - \Pi\|_V \delta_V(P^n),
\]
and
\[
\|P^n(x, \cdot) - \pi\|_V \leq \|P^n - \Pi\|_V V(x) \leq \|I - \Pi\|_V V(x) \delta_V(P^n).
\]

References


Dobrushin系数的推广

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本文将一般的全变差距离下的Dobrushin系数$\delta$推广到加权的全变差下的$\delta_V$，并利用$\delta_V$系数得到了离散时间马氏链的几何遍历的判定准则。

关键词：$V$范数，$\delta_V$系数，几何遍历。

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