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The first Dirichlet eigenvalue of birth–death process on trees ☆

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ABSTRACT

This paper investigates the birth–death ("B–D" for short) process on trees, emphasizing on estimating the principal eigenvalue (equivalently, the convergence rate) of the process with Dirichlet boundary at the unique root 0. Three kinds of variational formulas for the eigenvalue are presented. As an application, we obtain a criterion for positivity of the first eigenvalue for B–D processes on trees with one branch after some layer.

1. Introduction and main results

This paper deals with the first Dirichlet eigenvalue for B–D process on tree with the unique root 0 as absorbing boundary. One may refer to Miclo (2008) and Ma (2010) and the references therein for more related works. Our work is inspired by analogies research for B–D processes in Chen (2010) and Chen et al. (2013), in which the principal eigenvalues in dimension one with kinds of boundary conditions were studied. Let T be a tree of at least two vertexes with the edge set E (i.e., a connected graph without circle), such that the degree d_i for each $i \in T$ is finite. Let $|i|$ denote the layer of i, and $i \sim j$ if $(i, j) \in E$. We call $j \in T$ a son (correspondingly, the father) of vertex $i \in T$ if $i \sim j$ and $|j| = |i| + 1$ (correspondingly, $|j| = |i| - 1$). Consider a continuous time B–D process with Q-matrix such that $q_{ij} > 0$ if and only if $i \sim j$. Then the corresponding operator is

$$\Omega f(i) = \sum_{j\in J(i)} q_{ij}(f_j - f_i) + q_{ii^*}(f_{i^*} - f_i), \quad i \in T,$$

where $J(i)$ is the set of sons of i and i^* is the father of i. It is easy to obtain the unique symmetric measure μ on T:

$$\mu_0 = 1, \qquad \mu_k = \prod_{j\in \mathscr{P}(k)} \frac{q_{j^*j}}{q_{jj^*}}, \quad k \in T \setminus \{0\},$$

where $\mathscr{P}(i)$ is the set of all the vertexes (the root 0 is excluded) in the unique simple path from $i \in T \setminus \{0\}$ to the root. If (λ, g) with $g \neq 0$ is a solution to "eigenequation":

$$\Omega g(i) = -\lambda g(i), \quad i \in T \setminus \{0\}, \tag{1.1}$$

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then λ is called an "eigenvalue", and g is called an "eigenfunction" of the eigenvalue λ. Note that the "eigenvalue" and "eigenfunction" used in this paper in a generalized sense rather than the standard ones since we do not require $g \in L^2(\mu)$. In this paper, we focus on estimating the principal Dirichlet eigenvalue λ_0 (i.e., the corresponding eigenfunction satisfies boundary condition $g_0 = 0$), which has the following classical variational formula:

$$\lambda_0 = \inf\{D(f) : \mu(f^2) = 1, f_0 = 0\}, \tag{1.2}$$

where $\mu(f) = \sum_{k\in T\setminus\{0\}} \mu_k f_k$ and

$$D(f) = \sum_{i\in T\setminus\{0\}} \mu_i q_{ii^*}(f_i - f_{i^*})^2, \quad f \in \mathscr{D}(D)$$

with $\mathscr{D}(D) = \{f : D(f) < \infty, f_0 = 0\}$. Without loss of generality, we assume that the vertex 0 has only one son throughout this paper (i.e., $|J(0)| = 1$) and the layer counting begins from the son of the unique root 0. Denote by N ($N \leqslant \infty$) the maximal layer of tree T and T_i (i is included) a subtree of tree T with i as root. It is clear that $\lambda_0 > 0$ if $N < \infty$ (otherwise, $\Omega g(i) = 0$. Denoted by Λ_N the set of vertexes in the maximal layer N. By letting $i \in \Lambda_N$ in (1.1), we have $g_i = g_{i^*}$ for $i \in \Lambda_N$. By the induction, we have $g_i = g_0 = 0$ for $i \in T$, which is a contraction to $g \neq 0$). To state our results, we need some notations as follows. For $i \in T \setminus \{0\}$, define

$$I_i(f) = \frac{1}{\mu_i q_{ii^*}(f_i - f_{i^*})} \sum_{j\in T_i} \mu_j f_j \quad \text{(single summation form)},$$

$$II_i(f) = \frac{1}{f_i} \sum_{k\in \mathscr{P}(i)} \frac{1}{\mu_k q_{kk^*}} \sum_{j\in T_k} \mu_j f_j \quad \text{(double summation form)},$$

$$R_i(w) = q_{ii^*}(1 - w_i^{-1}) + \sum_{j\in J(i)} q_{ij}(1 - w_j) \quad \text{(difference form)}.$$

The forms of these operators defined above were initially introduced in Chen (1996, 2001, 2010) respectively for birth–death processes in dimension one. Shao and Mao in Shao and Mao (2007) extended the operator with single summation form from line to tree, and obtained the first operator defined above. The domains of the three operators are defined respectively as follows:

$$\mathscr{F}_I = \{f : f_0 = 0, f_i > f_{i^*} \text{ for } i \in T \setminus \{0\}\},$$
$$\mathscr{F}_{II} = \{f : f_0 = 0, f > 0 \text{ on } T \setminus \{0\}\},$$
$$\mathscr{W} = \{w : w > 1, w_0 = \infty\}.$$

These are used for the lower estimates of λ_0. For the upper bounds, some modifications are needed to avoid non-summable phenomenon, as shown below.

$$\widetilde{\mathscr{F}}_I = \{f > 0 : f_0 = 0, \exists 1 \leqslant n < N+1 \text{ such that } f_i > f_{i^*} \text{ for } |i| \leqslant n, \text{ and } f_i = f_{i^*} \text{ for } |i| \geqslant n+1\},$$
$$\widetilde{\mathscr{F}}_{II} = \{f > 0 : f_0 = 0, \exists 1 \leqslant n < N+1 \text{ such that } f_i = f_{i^*} \text{ for } |i| \geqslant n+1\},$$
$$\widetilde{\mathscr{W}} = \bigcup_{m:\, 1\leqslant m<N+1} \left\{ w : w_0 = \infty, w_i > 1 \text{ and } \sum_{j\in J(i)} q_{ij} w_j < q_{ii^*}(1 - w_i^{-1}) \right.$$
$$\left. + \sum_{j\in J(i)} q_{ij} \text{ for } |i| \leqslant m, \text{ and } w_i = 1 \text{ for } |i| \geqslant m+1 \right\}.$$

Define $\widetilde{R}$ acting on $\widetilde{\mathscr{W}}$ as a modified form of R by replacing q_{ii^*} with $\mu_i q_{ii^*} / \sum_{j\in T_i} \mu_j$ in $R_i(w)$ when $|i| = m$, where m is the same one in $\widetilde{\mathscr{W}}$. When using the approximating method, we also use $\widetilde{R}$ (at this time, q_{ii^*} is replaced with $\tilde{q}_{ii^*}$ for each $i \in T$, see the arguments before Lemma 2.1 and Step 4 in the proof of Theorem 1.1). Here and in what follows, we adopt the usual convention $1/0 = \infty$. The superscript "$\sim$" means modified.

In Theorem 1.1, "sup inf" are used for the lower bounds of λ_0, e.g., each test function $f \in \mathscr{F}_I$ produces a lower bound $\inf_{i\in T\setminus\{0\}} I_i(f)^{-1}$, so this part is called the variational formula for the lower estimate of λ_0. Dually, the "inf sup" are used for the upper estimates of λ_0. Among them, the ones expressed by operator R are easiest to compute in practice, and the ones expressed by II are hardest to compute but provide better estimates. Because of "inf sup", a localizing procedure is used for the test function to avoid $I(f) \equiv \infty$ for instance, which is removed out automatically for the "sup inf" part. Define another set

$$\widetilde{\mathscr{F}}'_{II} = \{f > 0 : fII(f) \in L^2(\mu)\}.$$

Then we present our main results.

Theorem 1.1. *The following variational formulas hold for λ_0 defined by (1.2).*

(1) *Single summation forms:*

$$\sup_{f\in\mathscr{F}_I}\inf_{i\in T\setminus\{0\}} I_i(f)^{-1}=\lambda_0=\inf_{f\in\widetilde{\mathscr{F}}_I}\sup_{i\in T\setminus\{0\}} I_i(f)^{-1},$$

(2) *Double summation forms:*

$$\sup_{f\in S(\mathscr{F})}\inf_{i\in T\setminus\{0\}} II_i(f)^{-1}=\lambda_0=\inf_{f\in S(\widetilde{\mathscr{F}})}\sup_{i\in T\setminus\{0\}} II_i(f)^{-1}$$

with $S(\mathscr{F})=\mathscr{F}_{II}$ *or* $\mathscr{F}_I$ *and* $S(\widetilde{\mathscr{F}})=\widetilde{\mathscr{F}}_{II}$, *or* $\widetilde{\mathscr{F}}_I$, *or* $\widetilde{\mathscr{F}}'_{II}\cup\widetilde{\mathscr{F}}_{II}$.

(3) *Difference forms:*

$$\sup_{w\in\mathscr{W}}\inf_{i\in T\setminus\{0\}} R_i(w)=\lambda_0=\inf_{w\in\widetilde{\mathscr{W}}}\sup_{i\in T\setminus\{0\}}\widetilde{R}_i(w).$$

We mention that the lower bounds of λ_0 in Theorem 1.1 (1) were known in Shao and Mao (2007) as an inequality. Liu et al. in Liu et al. (2013) extend the result in Shao and Mao (2007), obtaining lower estimates of λ_0 under some conditions. In view of the relation between the test functions of R, I and II (they are all the analogies of an eigenfunction, see arguments after Lemma 2.1 for details), it is not hard to check that these estimates in Theorem 1.1 can be sharp (Shao and Mao, 2007), which illustrated that the lower estimates with single summation form can be sharp.

Define $|A|$ = number of elements in the set A, $\mu(T_j):=\sum_{k\in T_j}\mu_k$, and

$$\varphi_j=\sum_{k\in\mathscr{P}(j)}\frac{1}{\mu_k q_{kk^*}},\quad j\in T\setminus\{0\}.$$

As applications of Theorem 1.1 (1) and (2), we have the following theorem.

Theorem 1.2. *Let* $\delta=\sup_{j\in T\setminus\{0\}}\mu\left(T_j\right)\varphi_j$. *Then*

$$\delta^{-1}\geqslant\lambda_0\geqslant\left[\left(2\sup_{i\in T\setminus\{0\}}C_i\right)\delta\right]^{-1},$$

where

$$C_i=1+|J(i)|+\sum_{s\in J(i)}\sum_{k\in T_s}\left(|J(k)|-1\right),\quad i\in T.$$

The theorem effectively presents to us the positive criterion of the first Dirichlet eigenvalue of a B–D process on trees with one branch after some layer. For the degenerated case of the tree (only one branch), the results reduce to that of the B–D process on a half line studied in Chen (2010) (the ratio of the upper and lower bounds for the estimates of λ_0 is no more than 4). It is worthy to point out that the B–D process on a tree with the root as a Dirichlet boundary can be a comparison with the B–D process on a line with bilateral reflecting boundaries. Let us have a look at the B–D process on a line with reflecting boundaries. From Chen (2010), we see that the eigenfunction of the first eigenvalue is strictly monotone with a unique zero. If we treat the unique zero of the eigenfunction as a root, then the B–D process on a line is just a B–D process on a tree with two branches and the unique "root" as a Dirichlet boundary (the intuition is pointed out by Professor Mao Y.H.). Concerning the B–D process on a line with reflecting boundaries, one may refer to Chen (2013).

2. Proofs of the main results

Define $\Lambda_m=\{i:|i|=m\}$, $T(n)=\cup_{m=0}^n\Lambda_m$ and

$$\tilde{\lambda}_0=\{D(f):\mu(f^2)=1,\exists 1\leqslant n<N+1\text{ such that }f_i=f_{i^*}\text{ for }|i|\geqslant n+1\}.$$

As will be seen in Lemma 2.1, $\lambda_0=\tilde{\lambda}_0$ once $\sum_{k\in T}\mu_k<\infty$. To this end, define

$$\lambda_0^{(m)}=\inf\left\{D(f):\mu(f^2)=1,f_i=f_{i^*}\text{ for }|i|\geqslant m+1\right\},\quad 1\leqslant m<N+1.$$

There is an explanation for $\lambda_0^{(m)}$ (see Chen, 2010, Section 4, p. 427): let

$$\tilde{\mu}_i=\mu_i,\qquad \tilde{q}_{ij}=q_{ij}\quad\text{for }|i|\leqslant m-1\text{ and }|j|\leqslant m-1;$$

$$\tilde{\mu}_i=\sum_{j\in T_i}\mu_j,\qquad \tilde{q}_{i^*i}=q_{i^*i},\qquad \tilde{q}_{ii^*}=\mu_i q_{ii^*}\Big/\sum_{j\in T_i}\mu_j\quad\text{for }|i|=m.$$

Noticing $\mu_i q_{ii^*}=\tilde{\mu}_i\tilde{q}_{ii^*}$, for f with $f_i=f_{i^*}$ for $|i|\geqslant m+1$, we have

$$D(f)=\sum_{i\in T(m)\setminus\{0\}}\tilde{\mu}_i\tilde{q}_{ii^*}(f_i-f_{i^*})^2=:\widetilde{D}(f),\qquad \mu(f^2)=\sum_{i\in T(m)}\tilde{\mu}_i f_i^2=:\tilde{\mu}(f^2).$$

So the Q-matrix $\widetilde{Q} = (\tilde{q}_{ij} : i, j \in T(m))$ is symmetric with respect to $\{\tilde{\mu}_i\}_{i\in T(m)}$ and $\lambda_0^{(m)}$ is the first Dirichlet eigenvalue of the local Dirichlet form $\big(\widetilde{D}, \mathscr{D}(\widetilde{D})\big)$ with state space $T(m)$.

For simplicity, we use "iff" to denote "if and only if" and $\uparrow$ (resp. $\downarrow$) to denote increasing and decreasing throughout the paper.

Lemma 2.1. *Assume that $\sum_{k\in T} \mu_k < \infty$. We have $\lambda_0 = \tilde{\lambda}_0$ and $\lambda_0^{(n)} \downarrow \lambda_0$ as $n \to N$.*

Proof. By the definition of λ_0, for any $\varepsilon > 0$, there exists f such that $D(f)/\mu(f^2) \leqslant \lambda_0 + \varepsilon$. Construct $f^{(n)}$ such that $f_i^{(n)} = f_i$ for $|i| \leqslant n$ and $f_i^{(n)} = f_{i^*}$ for $|i| \geqslant n + 1$. Since $\sum_{k\in T} \mu_k < \infty$, by symmetry, we have

$$D(f^{(n)}) = \sum_{i\in T\setminus\{0\}} \mu_i q_{ii^*} (f_i - f_{i^*})^2 = \sum_{i\in T(n)\setminus\{0\}} \mu_i q_{ii^*} (f_i - f_{i^*})^2 \uparrow D(f)$$

$$\mu(f^{(n)2}) = \sum_{i\in T(n)\setminus\{0\}} \mu_i f_i^2 + \sum_{i\in \Lambda_{n+1}} \mu(T_i) f_{i^*}^2 \to \mu(f^2).$$

By definitions of λ_0, $\tilde{\lambda}_0$ and $\lambda_0^{(n)}$, the required assertion holds. $\square$

This lemma presents us with an approximating procedure, making it sometimes possible that we only need to show that some assertion or property holds for finite trees even if $N = \infty$ (see Step 6 (b) and Step 8 in proofs of Theorem 1.1). The following lemma known in Shao and Mao (2007), gives us an important property of eigenfunction g. The property provides us with the basis for the choices of those test function sets of operators I, II and R.

Lemma 2.2 (*Shao and Mao, 2007, Proposition 2.4*). *For a B–D process on tree T (may have infinite vertexes). If (λ_0, g) is a solution to (1.1) with boundary condition $g_0 = 0$ and $g \in L^2(\mu)$ holds, then $g_i > g_{i^*}$ for each $i \in T \setminus \{0\}$.*

Obviously, for a B–D process on finite tree T (a tree with maximal layer $N < \infty$), the eigenfunction g of the first Dirichlet eigenvalue satisfies $g_i > g_{i^*}$ for every $i \in T$. Before moving further, we introduce a general equation and discuss the origin of operators. Consider

$$\text{Poisson equation}: \ \Omega g(i) = -f_i, \quad i \in T \setminus \{0\}.$$

By multiplying μ_i on both sides of the equation and making a summation with respect to $i \in T_k \cap T(n)$ for some $k \in T \setminus \{0\}$ with $|k| \leqslant n$, it is easy to check that

$$\sum_{j\in \Lambda_{n+1}\cap T_k} \mu_j q_{jj^*}(g_{j^*} - g_j) + \mu_k q_{kk^*}(g_k - g_{k^*}) = \sum_{j\in T_k\cap T(n)} \mu_j f_j, \quad |k| \leqslant n. \tag{2.1}$$

If $\lim_{n\to N} \sum_{j\in \Lambda_{n+1}\cap T_k} \mu_j q_{jj^*}(g_{j^*} - g_j) = 0$ (which is obvious for $N < \infty$), then we obtain the form of the operator I by letting $n \to N$ and $f = \lambda g$ in (2.1). Moreover, if $g_0 = 0$ (which is clear for the eigenfunction of Dirichlet eigenvalue λ_0), then

$$g_i = \sum_{k\in \mathscr{P}(i)} \frac{1}{\mu_k q_{kk^*}} \sum_{j\in T_k} \mu_j f_j.$$

This explains where the operator II comes from. Similarly, from the eigenequation (1.1), we obtain the operator R by letting $w_i = g_i/g_{i^*}$. The eigenequation is a "bridge" among these operators. Based on Chen (2010), Chen et al. (2013) and taking full advantage of these relations, we present the proofs of the main results.

Proof of Theorem 1.1. We introduce the following circle arguments for lower bounds of λ_0.

$$\lambda_0 \geqslant \sup_{f\in \mathscr{F}_{II}} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1} = \sup_{f\in \mathscr{F}_I} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1} = \sup_{f\in \mathscr{F}_I} \inf_{i\in T\setminus\{0\}} I_i(f)^{-1} \geqslant \sup_{w\in \mathscr{W}} \inf_{i\in T\setminus\{0\}} R_i(w) \geqslant \lambda_0.$$

Step 1 Prove that $\lambda_0 \geqslant \sup_{f\in \mathscr{F}_{II}} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1}$.

For positive sequence $\{h_i\}_{i\in T\setminus\{0\}}$ and g with $g_0 = 0$, $\mu(g^2) = 1$, we have

$$\begin{aligned}
1 &= \sum_{k\in T\setminus\{0\}} \mu_k g_k^2 = \sum_{k\in T\setminus\{0\}} \mu_k \left(\sum_{i\in \mathscr{P}(k)} (g_i - g_{i^*}) \right)^2 \quad (\text{since } g_0 = 0) \\
&\leqslant \sum_{k\in T\setminus\{0\}} \mu_k \sum_{j\in \mathscr{P}(k)} \frac{\mu_j q_{jj^*}}{h_j} \left(g_j - g_{j^*}\right)^2 \sum_{i\in \mathscr{P}(k)} \frac{h_i}{\mu_i q_{ii^*}} \quad (\text{by Cauchy's ineq.}) \\
&= \sum_{j\in T\setminus\{0\}} \mu_j q_{jj^*} \left(g_j - g_{j^*}\right)^2 \frac{1}{h_j} \sum_{k\in T_j} \mu_k \sum_{i\in \mathscr{P}(k)} \frac{h_i}{\mu_i q_{ii^*}}
\end{aligned}$$

(by exchanging the order of sums, and $j \in \mathscr{P}(k)$ iff $k \in T_j$).

For every f with $fII(f) < \infty$, let $h_i = \sum_{k\in T_i} \mu_k f_k$. By the proportional property, we get

$$\mu(g^2) \leqslant D(g) \sup_{j\in T\setminus\{0\}} \left(\sum_{k\in T_j} \mu_k \sum_{i\in \mathscr{P}(k)} \frac{h_i}{\mu_i q_{ii^*}} \right) \Big/ \sum_{k\in T_j} \mu_k f_k \leqslant D(g) \sup_{j\in T\setminus\{0\}} II_j(f).$$

By (1.2), we have

$$\lambda_0 \geqslant \inf_{j\in T\setminus\{0\}} II_j(f)^{-1}, \quad f \in \mathscr{F}_{II},$$

and the required assertion follows by making the supremum with respect to $f \in \mathscr{F}_{II}$.

Step 2 Prove that $\sup_{f\in\mathscr{F}_{II}} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1} = \sup_{f\in\mathscr{F}_I} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1} = \sup_{f\in\mathscr{F}_I} \inf_{i\in T\setminus\{0\}} I_i(f)^{-1}$.

(a) Prove that the direction $\geqslant$. The first inequality is clear since $\mathscr{F}_I \subset \mathscr{F}_{II}$. Replacing f in the denominator of $II_j(f)$ with $\sum_{k\in\mathscr{P}(j)}(f_k - f_{k^*})$. Using the proportional property, for $f \in \mathscr{F}_I$, we have

$$\sup_{j\in T\setminus\{0\}} II_j(f) = \sup_{j\in T\setminus\{0\}} \left[\left(\sum_{k\in\mathscr{P}(j)} \frac{1}{\mu_k q_{kk^*}} \sum_{i\in T_k} \mu_i f_i \right) \Big/ \sum_{k\in\mathscr{P}(j)} (f_k - f_{k^*}) \right] \leqslant \sup_{k\in T\setminus\{0\}} I_k(f).$$

So the required assertion holds.

(b) To prove the equality, it suffices to show that

$$\sup_{f\in\mathscr{F}_I} \inf_{i\in T\setminus\{0\}} I_i(f)^{-1} \geqslant \sup_{f\in\mathscr{F}_{II}} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1}.$$

For $f \in \mathscr{F}_{II}$, without loss of generality, assume that $II(f) < \infty$. Let $g = fII(f)$. Then $g \in \mathscr{F}_I$,

$$g_i - g_{i^*} = \frac{1}{\mu_i q_{ii^*}} \sum_{j\in T_i} \mu_j f_j \geqslant \sum_{j\in T_i} \mu_j g_j \inf_{k\in T\setminus\{0\}} \frac{f_k}{g_k}, \quad i \in T\setminus\{0\},$$

and then the required assertion follows immediately since $f \in \mathscr{F}_{II}$ is arbitrary.

There is another choice to show the equality. By Lemma 2.2, we see that the eigenfunction g satisfies that $g_i > g_{i^*}$ for $i \in T\setminus\{0\}$ provided $N < \infty$. So $g \in \mathscr{F}_I$ and $\lambda_0 = I_i(g)^{-1}$ for $i \in T\setminus\{0\}$ (Shao and Mao, 2007, Lemma 2.3). By making the infimum with respect to $i \in T\setminus\{0\}$ first and then the supremum with respect to $f \in \mathscr{F}_I$, we have $\lambda_0 \geqslant \sup_{f\in\mathscr{F}_I} \inf_{i\in T\setminus\{0\}} I_i(f)^{-1}$. There is a small gap in the proof since the eigenfunction g may not belong to L^2 when $N = \infty$. However, one may avoid this by a standard approximating procedure (according to the approximating idea used in Step 4 below). Combining this with Step 1 above, the required assertion holds.

Step 3 Prove that $\sup_{f\in\mathscr{F}_{II}} \inf_{i\in T\setminus\{0\}} II_i(f)^{-1} \geqslant \sup_{w\in\mathscr{W}} \inf_{i\in T\setminus\{0\}} R_i(w)$.

We first change the form of $R_i(w)$. For $w \in \mathscr{W}$, let u with $u_0 = 0$ be a positive function on $T\setminus\{0\}$ such that $w_i = u_i/u_{i^*}$ for $i \in T\setminus\{0\}$, i.e.,

$$u_i = \prod_{j\in\mathscr{P}(i)} w_j \quad \text{for } i \in T\setminus\{0\}, \qquad u_0 = 0.$$

Then $u_i > u_{i^*}$ for $i \in T\setminus\{0\}$ and

$$R_i(w) = \frac{1}{u_i}\left[\sum_{j\in J(i)} q_{ij}(u_i - u_j) + q_{ii^*}(u_i - u_{i^*}) \right] = -\frac{\Omega u(i)}{u_i}. \tag{2.2}$$

Now we turn to the main text. For any fixed $w \in \mathscr{W}$, without loss of generality, assume that $R(w) > 0$. Let u be a function mentioned above such that $w_i = u_i/u_{i^*}$ and $f = uR(w) > 0$. Then $f \in \mathscr{F}_{II}$ and $\Omega u(i) = -f_i$. Since $u_i > u_{i^*}$, by (2.1), we have

$$\sum_{j\in T_k\cap T(n)} \mu_j f_j \leqslant \mu_k q_{kk^*}(u_k - u_{k^*}) < \infty, \qquad |k| \leqslant n.$$

So $f \in L^1(\mu)$ and

$$u_k - u_{k^*} \geqslant \frac{1}{\mu_k q_{kk^*}} \sum_{j\in T_k} \mu_j f_j$$

by letting $n \to N$. Moreover,

$$u_i \geqslant \sum_{k\in\mathscr{P}(i)} \frac{1}{\mu_k q_{kk^*}} \sum_{j\in T_k} \mu_j f_j.$$

Hence,

$$\inf_{i\in T\setminus\{0\}} R_i(w) = \inf_{i\in T\setminus\{0\}} \frac{f_i}{u_i} \leqslant \inf_{i\in T\setminus\{0\}} II_i(f)^{-1}, \quad i \in T \setminus \{0\},$$

and the assertion follows by making the supremum over $f \in \mathscr{F}_I$ first and then over $w \in \mathscr{W}$.

Step 4 Prove that $\sup_{w\in\mathscr{W}} \inf_{i\in T\setminus\{0\}} R_i(w) \geqslant \lambda_0$.

We first prove that $\sup_{w\in\mathscr{W}} \inf_{i\in T\setminus\{0\}} R_i(w) \geqslant 0$. Let $f \in L^1(\mu)$ be a positive function on $T \setminus \{0\}$ and $h = fII(f)$ on $T \setminus \{0\}$, $h_0 = 0$. Then

$$h_i - h_{i^*} = \frac{1}{\mu_i q_{ii^*}} \sum_{k\in T_i} \mu_k f_k.$$

Put $\bar{w}_i = h_i/h_{i^*}$ for $i \in T \setminus \{0\}$. By symmetry, we have

$$\begin{aligned} -\Omega h(i) &= q_{ii^*}(h_i - h_{i^*}) - \sum_{j\in J(i)} q_{jj^*}(h_j - h_{j^*}) \quad (\text{by } j \in J(i) \text{ iff } i = j^*) \\ &= \frac{1}{\mu_i} \sum_{k\in T_i} \mu_k f_k - \sum_{j\in J(i)} \frac{q_{j^*j}}{\mu_j q_{jj^*}} \sum_{k\in T_j} \mu_k f_k \\ &= \frac{1}{\mu_i} \left[\sum_{k\in T_i} \mu_k f_k - \sum_{j\in J(i)} \sum_{k\in T_j} \mu_k f_k \right] = f_i. \end{aligned}$$

So

$$R_i(\bar{w}) = -\frac{\Omega h(i)}{h_i} = \frac{f_i}{h_i} > 0, \quad i \in T \setminus \{0\},$$

and the required assertion then follows by making the infimum with respect to $i \in T \setminus \{0\}$ first and then the supremum with respect to $w \in \mathscr{W}$.

By Lemma 2.2, if $\lambda_0 > 0$ and the maximal layer of the tree $N < \infty$, then the eigenfunction g satisfies $g_i > g_{i^*}$ for every $i \in T \setminus \{0\}$. Let $\bar{w}_i = g_i/g_{i^*}$. Then $\bar{w} \in \mathscr{W}$ and

$$R_i(\bar{w}) = \frac{-\Omega g(i)}{g_i} = \lambda_0, \quad i \in T \setminus \{0\}.$$

So the assertion holds for $N < \infty$. If $N = \infty$, then an approximating procedure is used. Let $m \in \mathbb{N}^+$ and $1 \leqslant m < N + 1$. Then $\lambda_0^{(m)} \downarrow \lambda_0$ as $m \uparrow N$ by Lemma 2.1. Noticing the explanation of $\lambda_0^{(m)}$ at the beginning of this section and the assertion we have just shown for $N < \infty$, we have

$$\lambda_0^{(m)} = \sup_{w\in\mathscr{W}(m)} \inf_{i\in T(m)\setminus\{0\}} \widetilde{R}_i(w),$$

where $\mathscr{W}(m) = \{w : w_i > 1, i \in T(m), w_0 = \infty\}$, $\widetilde{R}$ is a modified form of R by replacing q_{ii^*} with $\tilde{q}_{ii^*}$. So for any $\varepsilon > 0$, there exists $\bar{w} \in \mathscr{W}(m)$ such that

$$\lambda_0^{(m)} < \inf_{i\in T(m)\setminus\{0\}} \widetilde{R}_i(\bar{w}) + \varepsilon \leqslant \inf_{i\in T(m-1)\setminus\{0\}} \widetilde{R}_i(\bar{w}) + \varepsilon.$$

Extend $\bar{w}$ to T by setting $\bar{w}_i = \bar{w}_{i^*}$ for $|i| > m$. Noticing $q_{ii^*} = \tilde{q}_{ii^*}$ for $|i| < m$, we have $\widetilde{R}_i(\bar{w}) = R_i(\bar{w})$ for $|i| < m$. Since $\inf_{i\in T(m-1)\setminus\{0\}} R_i(\bar{w}) \to \inf_{i\in T\setminus\{0\}} R_i(\bar{w})$ as $m \to \infty$, the required assertion follows by letting $m \to \infty$.

We adopt the following circle to prove the upper bounds of λ_0.

$$\begin{aligned} \lambda_0 &\leqslant \inf_{f\in\mathscr{F}'_{II}\cup\mathscr{F}_{II}} \sup_{i\in T\setminus\{0\}} II_i(f)^{-1} \\ &\leqslant \inf_{f\in\widetilde{\mathscr{F}}_{II}} \sup_{i\in T\setminus\{0\}} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I}} \sup_{i\in T\setminus\{0\}} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I}} \sup_{i\in T\setminus\{0\}} I_i(f)^{-1} \\ &\leqslant \inf_{w\in\widetilde{\mathscr{W}}} \sup_{i\in T\setminus\{0\}} \widetilde{R}_i(w) \leqslant \lambda_0. \end{aligned}$$

The second inequality above is clear and we only need to prove the remainders.

Step 5 Prove that $\lambda_0 \leqslant \inf_{f\in\widetilde{\mathscr{F}}'_{II}\cup\widetilde{\mathscr{F}}_{II}} \sup_{i\in T\setminus\{0\}} II_i(f)^{-1}$.

For $f \in \widetilde{\mathscr{F}}_{II}$, there exists $n \in E$ such that $f_i = f_{i^*}$ for $|i| \geqslant n+1$. Let $g_i = f_i II_i(f)$ for $|i| \leqslant n$ and $g_i = g_{i^*}$ for $|i| \geqslant n+1$. Then $g \in L^2(\mu)$ and

$$g_i - g_{i^*} = \frac{1}{\mu_i q_{ii^*}} \sum_{j\in T_i} \mu_j f_j \mathbf{1}_{\{i:|i|\leqslant n\}}.$$

Inserting this term into $D(g)$, we have

$$\begin{aligned} D(g) &= \sum_{j\in T\setminus\{0\}} (g_j - g_{j^*}) \sum_{k\in T_j} \mu_k f_k \mathbf{1}_{\{j:|j|\leqslant n\}} \\ &= \sum_{k\in T\setminus\{0\}} \mu_k f_k \sum_{j\in\mathscr{P}(k)} \mathbf{1}_{\{j:|j|\leqslant n\}} (g_j - g_{j^*}) \quad (\text{since } k \in T_j \text{ iff } j \in \mathscr{P}(k)) \\ &= \sum_{k\in T\setminus\{0\}} \mu_k f_k g_k \quad (\text{since } g_i = g_{i^*} \text{ for } |i| \geqslant n+1). \end{aligned}$$

Since $g \in L^2(\mu)$, we further obtain

$$D(g) \leqslant \sum_{k\in T\setminus\{0\}} \mu_k g_k^2 \sup_{k\in T\setminus\{0\}} \frac{f_k}{g_k} \leqslant \mu(g^2) \sup_{k\in T\setminus\{0\}} II_k(f)^{-1}.$$

Hence,

$$\lambda_0 \leqslant \frac{D(g)}{\mu(g^2)} \leqslant \sup_{k\in T\setminus\{0\}} II_k(f)^{-1}.$$

This inequality also holds for $f \in \widetilde{\mathscr{F}}'_{II}$ since the key point in its proof is $g = fII(f) \in L^2(\mu)$, which holds also for $f \in \widetilde{\mathscr{F}}'_{II}$. So the required assertion holds.

Step 6 Prove that $\inf_{f\in\widetilde{\mathscr{F}}_{II}} \sup_{i\in T\setminus\{0\}} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{i\in T\setminus\{0\}} II_i(f)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{i\in T\setminus\{0\}} I_i(f)^{-1}$.

(a) We first prove the direction "$\leqslant$". Since $\widetilde{\mathscr{F}}_I \subset \widetilde{\mathscr{F}}_{II}$, the first inequality is clear. For $f \in \widetilde{\mathscr{F}}_I$, there exists $1 \leqslant n < N+1$ such that $f_i = f_{i^*}$ for $|i| \geqslant n+1$ and $f_i > f_{i^*}$ for $|i| \leqslant n$. Since $f_i = \sum_{k\in\mathscr{P}(i)} (f_k - f_{k^*})$ for $|i| \leqslant n$, inserting this term into the denominator of $II(f)$ and using the proportional property, we have

$$\inf_{i\in T\setminus\{0\}} II(f) = \inf_{i\in T(n)\setminus\{0\}} II(f) \geqslant \inf_{i\in T\setminus\{0\}} I_i(f)$$

and the required assertion holds since $f \in \widetilde{\mathscr{F}}_I$ is arbitrary.

(b) Prove the equality.

For $f \in \widetilde{\mathscr{F}}_{II}$, $\exists\, n \in [1, N+1)$ such that $f_i = f_{i^*}$ for $|i| \geqslant n+1$ and $f > 0$. Let $g_i = f_i II_i(f)$ for $0 < |i| \leqslant n$, $g_0 = 0$ and $g_i = g_{i^*}$ for $|i| \geqslant n+1$. Then $g \in \widetilde{\mathscr{F}}_I$ and

$$g_i - g_{i^*} = \frac{1}{\mu_i q_{ii^*}} \sum_{j\in T_i} \mu_j f_j, \qquad |i| \leqslant n.$$

Moreover,

$$\mu_i q_{ii^*} (g_i - g_{i^*}) \leqslant \sum_{j\in T_i} \mu_j g_j \sup_{j\in T_i} \frac{f_j}{g_j} = \sum_{j\in T_i} \mu_j g_j \sup_{j\in T_i} II_i(f)^{-1}, \quad i \in T\setminus\{0\}.$$

Hence,

$$\sup_{k\in T\setminus\{0\}} I_k(g)^{-1} \leqslant \sup_{k\in T\setminus\{0\}} II_k(f)^{-1}.$$

Then the assertion follows by making the infimum over $\widetilde{\mathscr{F}}_I$ first and then the infimum over $\widetilde{\mathscr{F}}_{II}$.

Alternatively, there is another method to prove the equality. Combining with the arguments in Step 5 and Step 6 (a), it suffices to show that

$$\inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{k\in T\setminus\{0\}} I_k(f)^{-1} \leqslant \lambda_0.$$

To see this, assume that g is an eigenfunction corresponding to $\lambda_0^{(m)}$. Then $g_i > g_{i^*}$ for $i \in T(m)$. Extend g to the whole space by letting $g_i = g_{i^*}$ for $|i| \geqslant m+1$. Then $g \in \widetilde{\mathscr{F}}_I$ and

$$\lambda_0^{(m)} = \sup_{k\in T(m)\setminus\{0\}} I_k(g)^{-1} = \sup_{k\in T\setminus\{0\}} I_k(g)^{-1} \geqslant \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{k\in T\setminus\{0\}} I_k(f)^{-1}.$$

Noticing Lemma 2.1, the required assertion then holds by letting $m \to \infty$.

Step 7 Prove that $\inf_{f\in\widetilde{\mathscr{F}}_{II}}\sup_{i\in T\setminus\{0\}} II_i(f)^{-1}\leqslant\inf_{w\in\widetilde{\mathscr{W}}}\sup_{i\in T\setminus\{0\}}\widetilde{R}_i(w)$.

First, we change the form of $\widetilde{R}$. For $w\in\widetilde{\mathscr{W}}$ with $w_i=1$ for $|i|\geqslant m+1$, let g be a positive function on $T\setminus\{0\}$ with $g_0=0$ such that $w_i=g_i/g_{i^*}$. Then $g_i>g_{i^*}$ for $|i|\leqslant m$ and $g_i=g_{i^*}$ for $|i|\geqslant m+1$. Since

$$\sum_{j\in J(i)} q_{ij}w_j<q_{ii^*}(1-w_i^{-1})+\sum_{j\in J(i)}q_{ij}\quad\text{for }|i|\leqslant m,$$

we have $\widetilde{R}_i(w)=-\widetilde{\Omega}g(i)/g_i>0$ for $|i|\leqslant m$ and $\widetilde{R}_i(w)=0$ for $|i|\geqslant m+1$, where $\widetilde{\Omega}$ is defined on $T(m)$, corresponding to Q-matrix $(\tilde{q}_{ij})$ (see the arguments before Lemma 2.1).

Now, we come back to the main assertion. For $w\in\widetilde{\mathscr{W}}$ with $w_i=1$ for $|i|\geqslant m+1$, let g be the function mentioned above and

$$f_i=\begin{cases}\sum_{j\in J(i)}q_{ij}(g_i-g_j)+q_{ii^*}(g_i-g_{i^*}), & |i|\leqslant m-1,\\ \tilde{q}_{ii^*}(g_i-g_{i^*}), & |i|=m,\\ f_{i^*}, & |i|\geqslant m+1.\end{cases}$$

Then $f_i=-\widetilde{\Omega}g(i)>0$ for $|i|\leqslant m$. By (2.1), we have

$$\sum_{j\in\Lambda_m\cap T_k}\mu_jq_{jj^*}(g_{j^*}-g_j)+\mu_kq_{kk^*}(g_k-g_{k^*})=\sum_{j\in T_k\cap T(m-1)}\mu_jf_j,\qquad |k|\leqslant m-1.$$

Since

$$\mu_iq_{ii^*}(g_i-g_{i^*})=\sum_{j\in T_i}\mu_jf_i=\sum_{j\in T_i}\mu_jf_j,\qquad |i|=m,$$

we have

$$\sum_{j\in\Lambda_m\cap T_k}\mu_jq_{jj^*}(g_{j^*}-g_j)=\sum_{j\in\Lambda_m\cap T_k}\sum_{i\in T_j}\mu_if_i=\sum_{j\in(T\setminus T(m-1))\cap T_k}\mu_jf_j,\qquad |k|\leqslant m.$$

Hence, for $0<|k|\leqslant m$, we obtain

$$\mu_kq_{kk^*}(g_k-g_{k^*})=\sum_{j\in T_k\cap T(m-1)}\mu_jf_j+\sum_{j\in\Lambda_m\cap T_k}\mu_jq_{jj^*}(g_j-g_{j^*})=\sum_{j\in T_k}\mu_jf_j.$$

Moreover,

$$g_i=\sum_{k\in\mathscr{P}(i)}\frac{1}{\mu_kq_{kk^*}}\sum_{j\in T_k}\mu_jf_j,\quad 0<|i|\leqslant m,$$

and $\widetilde{R}_i(w)=f_i/g_i=II_i(f)^{-1}$ for $0<|i|\leqslant m$. Since $\widetilde{R}_i(w)=0$ and $f_i=f_{i^*}$ for $|i|\geqslant m+1$, we obtain

$$\sup_{i\in T\setminus\{0\}}\widetilde{R}_i(w)=\sup_{i\in T\setminus\{0\}}II_i(f)^{-1}\geqslant\inf_{f\in\widetilde{\mathscr{F}}_{II}}\sup_{i\in T\setminus\{0\}}II_i(f)^{-1},\quad w\in\widetilde{\mathscr{W}},$$

and the required assertion holds.

Step 8 Prove that $\inf_{w\in\widetilde{\mathscr{W}}}\sup_{i\in T\setminus\{0\}}\widetilde{R}_i(w)\leqslant\lambda_0$.

Let g with $g_0=0$ be an eigenfunction of local first eigenvalue $\lambda_0^{(m)}$ and extend g to $T\setminus\{0\}$ by setting $g_i=g_{i^*}$ for $|i|\geqslant m+1$. Put $w_i=g_i/g_{i^*}$ for $i\in T\setminus\{0\}$. Then $w\in\widetilde{\mathscr{W}}$. Since $m<\infty$, we have $\widetilde{R}_i(w)=\lambda_0^{(m)}>0$ for $i\in T(m)\setminus\{0\}$, and $\widetilde{R}_i(w)=0$ for $T\setminus T(m)$. Therefore,

$$\begin{aligned}\lambda_0^{(m)}&=\max_{i\in T\setminus\{0\}}\widetilde{R}_i(w)\\ &\geqslant\inf_{w\in\widetilde{\mathscr{W}}:w_i=1\text{ for }|i|\geqslant m+1}\ \max_{i\in T(m)\setminus\{0\}}\widetilde{R}_i(w)\\ &\geqslant\inf_{w\in\widetilde{\mathscr{W}}:\exists n\geqslant1\text{ such that }w_i=1\text{ for }|i|\geqslant n+1}\ \max_{i\in T\setminus\{0\}}\widetilde{R}_i(w)\\ &\geqslant\inf_{w\in\widetilde{\mathscr{W}}}\max_{i\in T\setminus\{0\}}\widetilde{R}_i(w).\end{aligned}$$

The assertion then follows by letting $m\to N$. □

Define $T_{i,j}=T_i\cup T_j$. Then $T_{J(i)}=\{k:s\in J(i)\text{ and }k\in T_s\}$. Similarly, we have $J(T_i)=\{k:s\in T_i\text{ and }k\in J(s)\}$. It is obvious that $J(T_i)=T_{J(i)}$. Without loss of generality, we adopt convention that $\mu(T_k)=0$ if $T_k=\phi$. The proof of Theorem 1.2, which is an application of Theorem 1.1, is presented as follows.

Proof of Theorem 1.2. First, we prove that $\lambda_0^{-1} \leqslant \left(2\sup_{i\in T\setminus\{0\}} C_i\right)\delta$. It is easy to see that

$$\begin{aligned}\sum_{j\in T_i}\mu_j f_j &= \sum_{j\in T_i} f_j\left[\mu(T_j) - \sum_{k\in J(j)}\mu(T_k)\right]\\ &= \sum_{j\in T_i}\mu(T_j)f_j - \sum_{k\in T_{J(i)}}\mu(T_k)f_{k^*} \quad (\text{since } J(T_i) = T_{J(i)} \text{ and } k\in J(j) \text{ iff } j = k^*)\\ &= \mu(T_i)f_i + \sum_{k\in T_{J(i)}}\mu(T_k)\left(f_k - f_{k^*}\right) \quad \left(\text{since } T_i = \{i\}\cup T_{J(i)}\right).\end{aligned}$$

Put $f_j = \sqrt{\varphi_j}$ for $j \in T$. Then

$$\sum_{j\in T_i}\mu_j\sqrt{\varphi_j} = \mu(T_i)\sqrt{\varphi_i} + \sum_{k\in T_{J(i)}}\mu(T_k)\left(\sqrt{\varphi_k}-\sqrt{\varphi_{k^*}}\right) \leqslant \delta\left[{\varphi_i}^{-1/2} + \sum_{k\in T_{J(i)}}\frac{1}{\varphi_k}\left(\sqrt{\varphi_k}-\sqrt{\varphi_{k^*}}\right)\right].$$

Since $\varphi_k \geqslant \varphi_{k^*}$, we obtain

$$\sum_{k\in T_{J(i)}}\frac{1}{\varphi_k}\left(\sqrt{\varphi_k}-\sqrt{\varphi_{k^*}}\right) \leqslant \sum_{k\in T_{J(i)}}\left({\varphi_{k^*}}^{-1/2} - {\varphi_k}^{-1/2}\right),$$

Noticing that $T_{J(i)} = J(T_i)$ and $k \in J(j)$ if and only if $k^* = j$, we have

$$\sum_{k\in T_{J(i)}}{\varphi_{k^*}}^{-1/2} = \sum_{k\in J(T_i)}{\varphi_{k^*}}^{-1/2} = \sum_{j\in T_i}\sum_{k\in J(j)}{\varphi_j}^{-1/2} = \sum_{j\in T_i}|J(j)|{\varphi_j}^{-1/2}.$$

Inserting this term to the inequality above, it holds[1]

$$\sum_{k\in T_{J(i)}}\frac{1}{\varphi_k}\left(\sqrt{\varphi_k}-\sqrt{\varphi_{k^*}}\right) \leqslant \left[|J(i)| + \sum_{k\in T_{J(i)}}\left(|J(k)|-1\right)\right]{\varphi_i}^{-1/2}.$$

Hence,

$$\sum_{j\in T_i}\mu_j\sqrt{\varphi_j} \leqslant \left[1 + |J(i)| + \sum_{s\in J(i)}\sum_{k\in T_s}\left(|J(k)|-1\right)\right]\delta{\varphi_i}^{-1/2} = C_i\delta{\varphi_i}^{-1/2}.$$

Since

$$\frac{1}{\sqrt{\varphi_i}-\sqrt{\varphi_{i^*}}} = \frac{1}{\varphi_i - \varphi_{i^*}}\left(\sqrt{\varphi_i}+\sqrt{\varphi_{i^*}}\right) = \mu_i q_{ii^*}\left(\sqrt{\varphi_i}+\sqrt{\varphi_i}\right),$$

we obtain

$$I_i(\sqrt{\varphi}) = \frac{1}{\mu_i q_{ii^*}\left(\sqrt{\varphi_i}-\sqrt{\varphi_{i^*}}\right)}\sum_{j\in T_i}\mu_j\sqrt{\varphi_j} \leqslant C_i\delta{\varphi_i}^{-1/2}\left(\sqrt{\varphi_i}+\sqrt{\varphi_{i^*}}\right) \leqslant 2C_i\delta.$$

It is clear that $\sqrt{\varphi} \in \mathscr{F}_I$, by Theorem 1.1 (1), we have

$$\lambda_0^{-1} \leqslant \inf_{f\in\mathscr{F}_I}\sup_{i\in T\setminus\{0\}} I_i(f) \leqslant \sup_{i\in T\setminus\{0\}} I_i(\sqrt{\varphi}) \leqslant \left(2\sup_{i\in T\setminus\{0\}} C_i\right)\delta.$$

Now, we prove that $\lambda_0 \leqslant \delta^{-1}$. For $i_0 \in T\setminus\{0\}$, let f be a function such that

$$f_i = \begin{cases}\varphi_i & \text{if } i\in\mathscr{P}(i_0),\\ \varphi_{i_0} & \text{if } i\in T_{i_0},\\ 0 & \text{Others.}\end{cases}$$

Then

$$\sum_{j\in T_i}\mu_j f_j = \sum_{j\in T_i\cap T_{i_0}}\mu_j\varphi_{i_0} + \sum_{k\in T_i\cap(\mathscr{P}(i_0)\setminus\{0\})}\sum_{j\in T_k}\mu_j\varphi_k.$$

[1] More details are presented in the Appendix.

Since $f_i - f_{i^*} = 1/(\mu_i q_{ii^*})$ for $i \in \mathscr{P}(i_0)$ and $f_i - f_{i^*} = 0$ for $i \in T \setminus \mathscr{P}(i_0)$. We have

$$\begin{aligned}\lambda_0^{-1} &= \sup_{g\in\widetilde{\mathscr{F}}_I} \inf_{i\in T\setminus\{0\}} I_i(g) \geqslant \inf_{i\in T\setminus\{0\}} I_i(f)\\ &= \inf_{i\in\mathscr{P}(i_0)} \left(\sum_{j\in T_i\cap T_{i_0}} \mu_j\varphi_{i_0} + \sum_{k\in T_i\cap(\mathscr{P}(i_0)\setminus\{0\})} \sum_{j\in T_k} \mu_j\varphi_k \right)\\ &= \mu(T_{i_0})\varphi_{i_0}, \quad i_0 \in T\setminus\{0\}.\end{aligned}$$

Then the assertion follows by taking the supremum over $T \setminus \{0\}$. □

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Appendix

It is not certain whether $\sum_{j\in T_{J(i)}} \varphi_j^{-1/2} < \infty$ or not, so we adopt the following methods to show the required assertion. It is easy to see

$$\sum_{s\in J(i)} \sum_{k\in T_s} \frac{1}{\varphi_k}\left(\sqrt{\varphi_k} - \sqrt{\varphi_{k^*}}\right) \leqslant \sum_{s\in J(i)} \sum_{k\in T_s} \left(\varphi_{k^*}^{-1/2} - \varphi_k^{-1/2}\right) = \lim_{n\to N} \sum_{m=|i|+1}^{n} \sum_{k\in\Lambda_m\cap T_i} \left(\varphi_{k^*}^{-1/2} - \varphi_k^{-1/2}\right).$$

Since

$$\begin{aligned}\sum_{m=|i|+1}^{n} \sum_{k\in\Lambda_m\cap T_i} \varphi_{k^*}^{-1/2} &= \sum_{m=|i|+1}^{n} \sum_{k\in\Lambda_{m-1}\cap T_i} |J(k)|\varphi_k^{-1/2} = \sum_{m=|i|}^{n-1} \sum_{k\in\Lambda_m\cap T_i} |J(k)|\varphi_k^{-1/2}\\ &= |J(i)|\varphi_i^{-1/2} + \sum_{m=|i|+1}^{n-1} \sum_{k\in\Lambda_m\cap T_i} |J(k)|\varphi_k^{-1/2},\end{aligned}$$

we have

$$\begin{aligned}\sum_{s\in J(i)} \sum_{k\in T_s} \frac{1}{\varphi_k}\left(\sqrt{\varphi_k} - \sqrt{\varphi_{k^*}}\right) &\leqslant \lim_{n\to N} \sum_{m=|i|+1}^{n} \sum_{k\in\Lambda_m\cap T_i} \left(\varphi_{k^*}^{-1/2} - \varphi_k^{-1/2}\right)\\ &= |J(i)|\varphi_i^{-1/2} + \lim_{n\to N} \left(\sum_{m=|i|+1}^{n-1} \sum_{k\in\Lambda_m\cap T_i} \left(|J(k)| - 1\right)\varphi_k^{-1/2} - \sum_{k\in\Lambda_n\cap T_i} |J(k)|\varphi_k^{-1/2} \right)\\ &\leqslant |J(i)|\varphi_i^{-1/2} + \sum_{m=|i|+1}^{N} \sum_{k\in\Lambda_m\cap T_i} \left(|J(k)| - 1\right)\varphi_k^{-1/2}\\ &= |J(i)|\varphi_i^{-1/2} + \sum_{k\in T_{J(i)}} \left(|J(k)| - 1\right)\varphi_k^{-1/2},\end{aligned}$$

and the required assertion holds. □

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