

Mixed principal eigenvalues in dimension one

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Abstract This is one of a series of papers exploring the stability speed of one-dimensional stochastic processes. The present paper emphasizes on the principal eigenvalues of elliptic operators. The eigenvalue is just the best constant in the L^2 -Poincaré inequality and describes the decay rate of the corresponding diffusion process. We present some variational formulas for the mixed principal eigenvalues of the operators. As applications of these formulas, we obtain case by case explicit estimates, a criterion for positivity, and an approximating procedure for the eigenvalue.

Keywords Eigenvalue, variational formula, explicit estimate, positivity criterion, approximating procedure

MSC 60J60, 34L15

1 Introduction

This paper is a continuation of [5] in which the stability speed was carefully studied in the discrete situation (birth–death processes) and partially in the continuous one (diffusions). For a large part of the study, the description of the problem is equivalent to that of the Poincaré-type inequalities or the principal eigenvalue. On the last two topics, there are a great number of publications (cf. [4,7] and references therein for the background and motivation of the study on these topics). However, to save the space here, most of the references are not repeated in this paper. Consider a finite interval $(0, D)$ for a moment. We are interested in some typical Sturm-Liouville eigenvalue problems. According to the Dirichlet (denoted by code ‘D’) and Neumann (denoted by code ‘N’) boundaries at the left- or right-endpoint, we have four cases of boundary condition: DD, ND, DN, and NN. In the diffusion context, the DD- and NN-cases are largely handled in [1–5,8,9]. The present paper is mainly devoted to the ND- and DN-cases. As will be seen in the next section, the classification

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for the boundaries is also meaningful when $D = \infty$.

This paper is organized as follows. In the next section, we focus on the ND-case. First, we introduce several variational formulas for the eigenvalue. As a consequence, we obtain the basic estimates, a criterion for positivity, an approximating procedure, and improved estimates for the eigenvalue. As far as we know, most of these results, except Theorem 2, have not yet appeared in the literature. The proofs of them are sketched in Section 3. From [5; Section 10], we know that the DN-case and the ND-case are dual to each other. Thus, as a dual to the ND-case, it is natural to study the DN-case, to which Section 4 is devoted, partial results come from the duality but some of them are not and need direct proofs. The main extension to the earlier study is that here we do not assume the uniqueness of the processes, instead of which we adopt the maximal extension of the Dirichlet form or the maximal process. Finally, some supplement to [2,3,9] in the NN-case (i.e., the ergodic case) is presented in Section 5. The complete proofs of the results presented in this paper are quite technical and long. However, a large part of them are parallel to [5] and so we omit mostly the ‘translation’ from the discrete situation to the continuous one. Instead, we emphasis on the difference between them (Lemmas 1–6, for instance), and illustrate a little of the translation for the reader’s reference. We may leave the details to our homepage or publish them elsewhere.

The basic estimates are also studied in [10] in terms of H -transform.

Some examples of the study are illustrated in [7; Section 5]. The most powerful application of the improved estimates presented in the paper is given by [6] where the lower and upper bounds are quite close to or almost coincide with each other.

Here, we discuss briefly about the problem on the whole line. First, we consider the ND-case. Then one may regard the whole line \mathbb{R} as a limit of $[M, \infty)$ as M decreases to $-\infty$. Thus, the mixed eigenvalue problem on line is known by what we are studying in the paper. Next, consider the DD-case, one may split \mathbb{R} into two parts: $(-\infty, 0)$ and $(0, \infty)$. The case with ND-boundaries on $(0, \infty)$ is studied in Sections 2 and 3. Besides, the case with DN-boundaries on $(-\infty, 0)$ is simply a reverse of the ND-case on $(0, \infty)$. Therefore, the behavior of the original operator on the whole line should be clear. However, there is an interesting point here. On $(0, \infty)$, we use the minimal Dirichlet form but on $(-\infty, 0)$ we adopt the maximal one. Thus, the domain of the original Dirichlet form on the whole line may be neither the maximal nor the minimal one. Therefore, it is essentially different from DD- or NN-cases on the whole line we have studied in [5,7,8].

To conclude this section, we mention that in a more general context, for the Poincaré-type inequalities, the DN-case was completed earlier (cf. [4; Chapter 6]), the basic estimates for the ND-case in the discrete situation was given by [5; Theorem 8.5], from which one can write down easily the continuous version.

2 ND-case

Define

$$\begin{aligned}\mathcal{C}[0, D] &= \{f: f \text{ is continuous on } [0, D]\}, \\ \mathcal{C}^k(0, D) &= \{f: f \text{ has continuous derivatives of order } k \text{ on } (0, D)\}, \quad k \geq 1.\end{aligned}$$

Here and in what follows, when $D = \infty$, the notation $\mathcal{C}[0, D]$ simply means $\mathcal{C}(0, D)$. The convention should be clear in other cases and we will not mention time by time. Let

$$L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$$

be an elliptic operator on an interval $(0, D)$ ($D \leq \infty$). Set

$$C(x) = \int_0^x \frac{b(u)}{a(u)} du.$$

Throughout this paper, we need the following hypothesis (which is trivial in the discrete situation):

The functions a, b are Borel measurable on $[0, D]$ and a is positive on $[0, D]$, b/a and e^C/a are locally integrable on $[0, D]$. (2.1)

Note that for continuous functions a and b , hypothesis (2.1) is reduced to the condition $a > 0$ only. In this section, we consider the ND-boundaries only. More precisely, as usual, the Dirichlet boundary condition at D means that $g(D) = 0$ when $D < \infty$. When $D = \infty$, it is natural to take ‘ $\lim_{x \rightarrow \infty} g(x) = 0$ ’ as a boundary condition. However, this is not pre-assumed but proved later (cf. Lemma 6 below). Therefore, the code ‘ND’ is still meaningful even if $D = \infty$.

Throughout this section, we work on the following mixed principal eigenvalue:

$$\lambda_0 = \inf\{D(f): \mu(f^2) = 1, f \in \mathcal{C}_K[0, D], f(D) = 0 \text{ if } D < \infty\}, \quad (2.2)$$

where

$$\begin{aligned}\mu(f) &:= \int_0^D f d\mu, \\ \mathcal{C}_K[0, D] &= \{f: f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], f \text{ has compact support}\}, \\ D(f) &= \int_0^D a f'^2 d\mu, \quad \mu(dx) = \frac{e^{C(x)}}{a(x)} dx.\end{aligned}$$

Besides μ , throughout the paper, we often use another measure:

$$\nu(dx) = e^{-C(x)} dx.$$

When $D < \infty$, λ_0 coincides with the minimal solution λ to the following eigenequation:

$$Lf = -\lambda f, \quad f'(0) = 0, \quad f(D) = 0 \text{ if } D < \infty.$$

To state our results, we need some notation. Define

$$\begin{aligned} I(f)(x) &= -\frac{e^{-C(x)}}{f'(x)} \int_0^x f d\mu \quad (\text{single integral form}), \\ II(f)(x) &= \frac{1}{f(x)} \int_{(x,D) \cap \text{supp}(f)} \nu(ds) \int_0^s f d\mu, \quad x \in \text{supp}(f) \\ &\quad (\text{double integral form}), \\ R(h)(x) &= -(ah^2 + bh + ah')(x) \quad (\text{differential form}). \end{aligned}$$

The domains of the three operators defined above are, respectively, as follows:

$$\begin{aligned} \mathcal{F}_I &= \{f: f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], f|_{(0,D)} > 0, f'|_{(0,D)} < 0\}, \\ \mathcal{F}_{II} &= \{f: f \in \mathcal{C}[0, D], f|_{(0,D)} > 0\}, \\ \mathcal{H} &= \{h: h \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], h(0) = 0, h|_{(0,D)} < (\text{resp. } \leq) 0 \\ &\quad \text{if } \nu(0, D) < (\text{resp. } =) \infty\}, \quad \nu(\alpha, \beta) := \int_\alpha^\beta d\nu. \end{aligned}$$

These sets are used for the lower estimates of λ_0 . For the upper bounds, some modifications are needed to avoid the non-integrability problem, as shown below:

$$\begin{aligned} \widetilde{\mathcal{F}}_I &= \{f: f \in \mathcal{C}^1(x_0, x_1) \cap \mathcal{C}[x_0, x_1], f'|_{(x_0, x_1)} < 0 \text{ for some} \\ &\quad x_0, x_1 \in [0, D) \text{ with } x_0 < x_1, \text{ and } f = f(\cdot \vee x_0) \mathbb{1}_{[0, x_1]}\}, \\ \widetilde{\mathcal{F}}_{II} &= \{f: \exists x_0 \in (0, D) \text{ such that } f = f \mathbb{1}_{[0, x_0]} \text{ and } f \in \mathcal{C}[0, x_0]\}, \\ \widetilde{\mathcal{H}} &= \{h: \exists x_0 \in (0, D) \text{ such that } h \in \mathcal{C}^1(0, x_0) \cap \mathcal{C}[0, x_0], h|_{(0, x_0)} < 0, \\ &\quad h|_{[x_0, D]} = 0, \text{ and } h(0) = 0, \sup_{(0, x_0)}(ah^2 + bh + ah') < 0\}. \end{aligned}$$

Here and in what follows, we adopt the usual convention $1/0 = \infty$. The superscript ‘ $\widetilde{}$ ’ means modified. In the formulas of Theorem 1 below, ‘sup inf’ is used for lower bounds of λ_0 , each test function f produces a lower bound $\inf_x I(f)(x)^{-1}$, and so this part is called variational formula for the lower estimate of λ_0 . Dually, the ‘inf sup’ is used for upper estimates of λ_0 . Among them, the ones expressed by the operator R are easiest to compute in practice, and the ones expressed by II are hardest to compute but provide better estimates. Because of ‘inf sup’, a localizing procedure is used for the test functions to avoid $I(f) \equiv \infty$ for instance, which is removed out automatically for the ‘sup inf’ part. Each part of Theorem 1 below plays a role in our

study. Parts (1) and (2) are applied to Theorems 2 and 3, respectively. Part (3) is a comparison with Proposition 2, which is then used as a dual form of Theorem 4 (3).

Theorem 1 *Under hypothesis (2.1), the following variational formulas hold for λ_0 defined by (2.2).*

(1) *Single integral forms:*

$$\inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0, D)} I(f)(x)^{-1} = \lambda_0 = \sup_{f \in \tilde{\mathcal{F}}_I} \inf_{x \in (0, D)} I(f)(x)^{-1},$$

(2) *Double integral forms:*

$$\inf_{f \in \tilde{\mathcal{F}}_{II}} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} = \lambda_0 = \sup_{f \in \tilde{\mathcal{F}}_{II}} \inf_{x \in (0, D)} II(f)(x)^{-1}.$$

Moreover, if $a, b \in \mathcal{C}[0, D]$, then we have additionally

(3) *differential forms:*

$$\inf_{h \in \tilde{\mathcal{H}}} \sup_{x \in (0, D)} R(h)(x) = \lambda_0 = \sup_{h \in \tilde{\mathcal{H}}} \inf_{x \in (0, D)} R(h)(x).$$

Furthermore, the supremum on the right-hand side of the above three formulas can be attained.

The next result, similar to the discrete case, either extends the domain of λ_0 , or adds some additional sets of test functions for operators I and II , respectively. Besides, as an application of the lower variational formula (Theorem 1 (2)), we obtain the vanishing property of the eigenfunction (Lemma 6) which leads to the crucial part (1) of the proposition below. The vanishing property is the meaning of the Dirichlet boundary at $D = \infty$ as we expected. A more common description of λ_0 is given by Lemma 2 below.

Proposition 1 *Let hypothesis (2.1) hold. Then*

(1) *we have*

$$\lambda_0 = \inf\{D(f) : \mu(f^2) = 1, f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D] \text{ and } f(D) = 0\} =: \tilde{\lambda}_0,$$

where $f(D) = \lim_{x \rightarrow D} f(x)$ in the case of $D = \infty$.

(2) *Moreover, we have*

$$\inf_{f \in \tilde{\mathcal{F}}_I'} \sup_{x \in (0, D)} I(f)(x)^{-1} = \lambda_0 = \sup_{f \in \tilde{\mathcal{F}}_I} \inf_{x \in (0, D)} II(f)(x)^{-1}, \tag{2.3}$$

$$\inf_{f \in \tilde{\mathcal{F}}_{II} \cup \tilde{\mathcal{F}}_{II}'} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} = \lambda_0 = \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1}, \tag{2.4}$$

where

$$\tilde{\mathcal{F}}_I' = \{f : \exists x_0 \in (0, D), f = f \mathbf{1}_{[0, x_0]}, f \in \mathcal{C}^1(0, x_0) \cap \mathcal{C}[0, x_0], f'|_{(0, x_0)} < 0\},$$

$$\tilde{\mathcal{F}}_{II}' = \{f : f > 0, f \in \mathcal{C}[0, D], f II(f) \in L^2(\mu)\}.$$

Besides, the supremum over $\{f \in \mathcal{F}_I\}$ in (2.3) can be attained.

The operator \bar{R} defined below was first introduced in [9; Theorem 2.1] based on a probabilistic (coupling) technique. Different from R , it is a ‘bridge’ in proving the duality of the ND- and DN-cases. It also leads to a different variational formula for λ_0 as follows.

Proposition 2 *Suppose that $a, b \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]$ and $a > 0$ on $(0, D)$. Set*

$$\bar{\mathcal{H}} = \{h: h(0) = 0, h \in \mathcal{C}^2(0, D) \cap \mathcal{C}[0, D], h|_{(0, D)} < 0\}$$

and define

$$\bar{R}(h)(x) = -\frac{(ah' + bh)'(x)}{h(x)}.$$

Then

(1) we have

$$\sup_{h \in \bar{\mathcal{H}}} \inf_{x \in (0, D)} \bar{R}(h)(x) \geq \lambda_0$$

and the equality holds once $\mu(0, D) = \infty$.

(2) In general, we have

$$\lambda_0 = \sup_{h \in \mathcal{H}_*} \inf_{x \in (0, D)} \bar{R}(h)(x), \quad (2.5)$$

where

$$\mathcal{H}_* = \{h \in \mathcal{C}^2(0, D) \cap \mathcal{C}[0, D]: h(0) = 0, \text{ and } h < 0, h' < -a^{-1}bh \text{ on } (0, D)\}.$$

Moreover, the supremum in (2.5) can be attained.

Remark 1 (Comparison of R and \bar{R}) With $h = g'/g$, we have

$$-\frac{Lg}{g} = -(ah^2 + bh + ah') = R(h).$$

Next, with $h = g'$, we have

$$-\frac{(Lg)'}{g'} = -\frac{(ah' + bh)'}{h} = \bar{R}(h).$$

As an application of Theorem 1 (1) to the test function $\nu(x, D)^\gamma$ with $\gamma = 1/2$ or 1, we obtain the basic estimates and furthermore a criterion as follows.

Theorem 2 (Criterion and basic estimates) *Let hypothesis (2.1) hold. Then $\lambda_0 > 0$ if and only if*

$$\delta := \sup_{x \in (0, D)} \mu(0, x) \nu(x, D) < \infty, \quad \mu(\alpha, \beta) := \int_\alpha^\beta d\mu.$$

More precisely, we have

$$(4\delta)^{-1} \leq \lambda_0 \leq \delta^{-1}.$$

In particular, when $D = \infty$, we have $\lambda_0 = 0$ if $\nu(0, D) = \infty$, and $\lambda_0 > 0$ if

$$\int_0^\infty \mu(0, x)\nu(dx) < \infty.$$

The next result is an application of Theorem 1 (2), repeated with $f = f_n$, starting from the initial function f_1 , the test function just mentioned before Theorem 2. The result provides us a way to improve the basic estimates step by step. In view of the last criterion, for any improvement, one may assume that $\delta < \infty$.

Theorem 3 (Approximating procedure) *Let hypothesis (2.1) hold and assume that $\delta < \infty$. Set $\varphi(x) = \nu(x, D)$.*

(1) *Let*

$$f_1 = \sqrt{\varphi}, \quad f_n = f_{n-1}H(f_{n-1}), \quad \delta_n = \sup_{x \in (0, D)} H(f_n)(x), \quad n \geq 1.$$

Then δ_n is decreasing in n and

$$\lambda_0 \geq \delta_n^{-1} \geq (4\delta)^{-1}, \quad n \geq 1.$$

(2) *For fixed $x_0, x_1 \in [0, D]$ with $x_0 < x_1$, define*

$$f_1^{x_0, x_1} = \nu(\cdot \vee x_0, x_1)\mathbb{1}_{[0, x_1]}, \quad f_n^{x_0, x_1} = (f_{n-1}^{x_0, x_1} H(f_{n-1}^{x_0, x_1}))(\cdot \vee x_0)\mathbb{1}_{[0, x_1]}, \quad n \geq 1,$$

and let

$$\delta'_n = \sup_{x_0, x_1: x_0 < x_1} \inf_{x < x_1} H(f_n^{x_0, x_1})(x).$$

Then $\delta^{-1} \geq \delta'_n{}^{-1} \geq \lambda_0$ for $n \geq 1$.

(3) *Define*

$$\bar{\delta}_n = \sup_{x_0, x_1: x_0 < x_1} \frac{\|f_n^{x_0, x_1}\|}{D(f_n^{x_0, x_1})}, \quad n \geq 1.$$

Then $\bar{\delta}_n{}^{-1} \geq \lambda_0$, $\bar{\delta}_{n+1} \geq \delta'_n$ for $n \geq 1$, and $\bar{\delta}_1 = \delta'_1$.

The next result comes from the first step of the approximation above.

Corollary 1 (Improved estimates) *Let hypothesis (2.1) hold. For λ_0 , we have*

$$\delta^{-1} \geq \delta'_1{}^{-1} \geq \lambda_0 \geq \bar{\delta}_1{}^{-1} \geq (4\delta)^{-1},$$

where

$$\begin{aligned} \delta_1 &= \sup_{x \in (0, D)} \frac{1}{\sqrt{\varphi(x)}} \int_0^D \sqrt{\varphi} \varphi(\cdot \vee x) d\mu \\ &= \sup_{x \in (0, D)} \left(\sqrt{\varphi(x)} \int_0^x \sqrt{\varphi} d\mu + \frac{1}{\sqrt{\varphi(x)}} \int_x^D \varphi^{3/2} d\mu \right), \end{aligned} \tag{2.6}$$

$$\delta'_1 = \sup_{x \in (0, D)} \left(\mu(0, x) \varphi(x) + \frac{1}{\varphi(x)} \int_x^D \varphi^2 d\mu \right) \in [\delta, 2\delta]. \quad (2.7)$$

3 Partial proofs of results in Section 2

Some preparations are needed to prove our main results. The first six lemmas below, except Lemma 2, are mainly devoted to describe the eigenfunction of λ_0 . These lemmas are essential in our study. Note that their proofs are very different from the discrete situation. The first one below is taken from [11; Theorems 1.2.1 and 2.2.1].

Lemma 1 (1) *Let hypothesis (2.1) hold. Then, whenever g and g' are initially not vanished simultaneously, there exists uniquely a non-zero function $g \in \mathcal{C}^1[0, D]$ such that g' is absolutely continuous on each compact subinterval of $[0, D)$ and the eigenequation $Lg = -\lambda g$ holds almost everywhere.*

(2) *Suppose that additionally a and b are continuous on $[0, D]$. Then $g \in \mathcal{C}^2[0, D]$ and the eigenequation holds everywhere on $[0, D]$.*

In what follows, we call the function g given in Lemma 1 (1) *a.e. eigenfunction of λ* . Remember we need ‘a.e.’ only in the case where g'' is used. Of course, we remove ‘a.e.’ if the eigenequation holds everywhere.

The next result enables us to return to a more common description of the eigenvalue.

Lemma 2 *Let $\mathcal{A}[\alpha, \beta]$ be the set of all absolutely continuous functions on $[\alpha, \beta]$. Define*

$$\lambda_* = \inf \{ D(f) : f \in \mathcal{A}[0, D], \|f\| = 1, f \text{ has compact support and } f(D) = 0 \text{ if } D < \infty \}$$

Then $\lambda_0 = \lambda_$.*

Proof It is obvious that $\lambda_* \leq \lambda_0$. Next, let g be the a.e. eigenfunction of λ_* . Then, $g \in \mathcal{C}^1[0, D]$ by Lemma 1 (1). By making inner product with g on the both sides of $Lg = -\lambda_* g$ with respect to μ , it follows that

$$-(e^C gg')|_0^D + D(g) = \lambda_* \|g\|^2.$$

Since $g'(0) = 0$ and $(gg')(D) \leq 0$, we have $\lambda_* \geq D(g)/\|g\|^2$. Because $g \in \mathcal{C}^1[0, D]$, it is clear that $D(g)/\|g\|^2 \geq \lambda_0$. We have thus obtained that

$$\lambda_0 \leq \lambda_* \leq \lambda_0,$$

and so $\lambda_0 = \lambda_*$. There is a small gap in the proof above since in the case of $D = \infty$, the a.e. eigenfunction g may not belong to $L^2(\mu)$ and we have not yet proved that $(gg')(D) \leq 0$. However, one may avoid this by a standard

approximating procedure, using $[0, p_n]$ instead of $[0, D]$ with $p_n \uparrow D$ provided $D = \infty$:

$$\begin{aligned} \lim_{n \rightarrow \infty} \lambda_0^{(0, p_n)} &= \lim_{n \rightarrow \infty} \inf \{ D(f) : f \in \mathcal{C}[0, p_n] \cap \mathcal{C}^1(0, p_n), \mu(f^2) = 1, f|_{[p_n, D]} = 0 \} \\ &= \inf \{ D(f) : \mu(f^2) = 1, f \in \mathcal{C}_K[0, D], f(D) = 0 \text{ if } D < \infty \} \\ &= \lambda_0. \end{aligned} \quad \square$$

Clearly, because of hypothesis (2.1), we have $\lambda_0 > 0$ once $D < \infty$. The next result is a simple comparison. For given α, β ($\alpha < \beta$), denote by $\lambda_0^{(\alpha, \beta)}$ and $\lambda_1^{(\alpha, \beta)}$, respectively, the principal ND- and NN-eigenvalues (the latter is also called the first nontrivial eigenvalue or the spectral gap in the ergodic case). For simplicity, we use \downarrow (resp. $\downarrow\downarrow, \uparrow, \uparrow\uparrow$) to denote decreasing (resp. strictly decreasing, increasing, strictly increasing).

Lemma 3 (1) For $p, q \in (0, D)$ with $p < q$, we have $\lambda_0^{(0, p)} > \lambda_0^{(0, q)}$. Furthermore, $\lambda_0^{(0, p_n)} \downarrow\downarrow \lambda_0$ as $p_n \uparrow\uparrow D$.

(2) For $p \in (0, D)$, we have $\lambda_1^{(0, p)} > \lambda_0^{(0, p)}$.

Proof (a) Let $g (\neq 0)$ be an a.e. eigenfunction of $\lambda_0^{(0, p)}$. Then $g'(0) = 0$, $g(p) = 0$, and $Lg = -\lambda_0^{(0, p)}g$ a.e. on $(0, p)$ by Lemma 1 (1). Moreover,

$$\lambda_0^{(0, p)} = \frac{D_{0, p}(g)}{\|g\|_{L^2(0, p; \mu)}^2}, \quad D_{\alpha, \beta}(f) = \int_{\alpha}^{\beta} a f'^2 d\mu.$$

By Lemma 2, the proof of the first assertion in part (1) will be done once we choose a function $\tilde{g} \in \mathcal{A}[0, q]$ such that $\tilde{g}'(0) = 0$, $\tilde{g}(q) = 0$, and

$$\frac{D_{0, p}(g)}{\|g\|_{L^2(0, p; \mu)}^2} > \frac{D_{0, q}(\tilde{g})}{\|\tilde{g}\|_{L^2(0, q; \mu)}^2} \quad (\geq \lambda_0^{(0, q)}). \tag{3.1}$$

To do so, without loss of generality, assume that $g|_{(0, p)} > 0$ (this is a well-known property as a reverse of the DN-case for finite intervals, cf. [4; Theorem 3.7]). Then the required assertion follows for

$$\tilde{g}(x) = (g + \varepsilon)\mathbb{1}_{[0, p)}(x) + \frac{\varepsilon(x - q)}{p - q} \mathbb{1}_{[p, q]}(x), \quad x \in [0, q],$$

once ε is sufficiently small. Actually, by simple calculation, we have

$$\begin{aligned} D_{0, q}(\tilde{g}) &= D_{0, p}(g) + \frac{\varepsilon^2}{(p - q)^2} \int_p^q e^{C(x)} dx, \\ \|\tilde{g}\|_{L^2(0, q; \mu)}^2 &= \|g\|_{L^2(0, p; \mu)}^2 + \varepsilon \int_0^p (2g + \varepsilon) d\mu + \frac{\varepsilon^2}{(p - q)^2} \int_p^q (x - q)^2 d\mu. \end{aligned}$$

Thus, (3.1) holds if and only if

$$\begin{aligned} & \frac{\varepsilon}{(p-q)^2} \int_p^q e^C dx \|g\|_{L^2(0,p;\mu)}^2 \\ & < \left(\int_0^p (2g + \varepsilon) d\mu + \frac{\varepsilon}{(p-q)^2} \int_p^q (x-q)^2 d\mu \right) D_{0,p}(g). \end{aligned}$$

Since $\lambda_0^{(0,p)} = D_{0,p}(g) / \|g\|_{L^2(0,p;\mu)}^2$, it suffices to show that

$$\frac{\varepsilon}{(p-q)^2} \int_p^q e^C dx < \lambda_0^{(0,p)} \left(2 \int_0^p g d\mu \right),$$

which is obvious for sufficiently small ε .

The second assertion in part (1) has just been proved at the end of the last proof.

(b) Part (2) of the lemma strengthens in the present situation a general result that $\lambda_1 \geq \lambda_0$ proved in [2; Proposition 3.2]. Let $g \neq \text{constant}$ be an a.e. eigenfunction of $\lambda_1^{(0,p)}$. Then $g'(0) = 0$, $g'(p) = 0$, and $Lg = -\lambda_1^{(0,p)}g$ a.e. on $(0, p)$ by Lemma 1 (1). Moreover,

$$\lambda_1^{(0,p)} = \frac{D_{0,p}(g)}{\text{Var}_{(0,p)}(g)}, \quad \text{Var}_{(\alpha,\beta)}(f) = \int_\alpha^\beta f^2 d\mu - \frac{\mu_{\alpha,\beta}(f)^2}{\mu(\alpha,\beta)}.$$

Without loss of generality, assume that g is strictly increasing (cf. [5; Proposition 6.4]). Then we have

$$\tilde{g}(x) := g(p) - g(x) > 0 \quad \text{on } (0, p).$$

Thus, $\tilde{g}'(0) = 0$, $\tilde{g}(p) = 0$, and moreover,

$$\lambda_1^{(0,p)} = \frac{D_{0,p}(\tilde{g})}{\text{Var}_{(0,p)}(\tilde{g})} = \frac{D_{0,p}(\tilde{g})}{\|\tilde{g}\|_{L^2(0,p;\mu)}^2 - \frac{\mu_{0,p}(\tilde{g})^2}{\mu(0,p)}} > \frac{D_{0,p}(\tilde{g})}{\|\tilde{g}\|_{L^2(0,p;\mu)}^2} \geq \lambda_0^{(0,p)}. \quad \square$$

Before moving further, let us mention a nice expression of L :

$$L = \frac{d}{d\mu} \frac{d}{d\nu},$$

which can be checked by a simple computation. Next, a large part of the results in the last section is related to the Poisson equation $Lg = -f$, a.e., from which we obtain

$$\frac{d}{d\nu} g(\beta) - \frac{d}{d\nu} g(\alpha) = - \int_\alpha^\beta f d\mu, \quad \alpha, \beta \in [0, D], \quad \alpha < \beta. \quad (3.2)$$

Furthermore, if $g'(\alpha) = 0$, then we have

$$g(q) - g(p) = - \int_p^q \nu(d\beta) \int_\alpha^\beta f d\mu, \quad p, q \in [0, D], \quad p < q. \quad (3.3)$$

Especially, because

$$\frac{d}{d\nu} g(0) = e^{C(0)} g'(0) = 0,$$

and (3.2), with $f = \lambda_0 g$, it follows that

$$\frac{d}{d\nu} g(s) = -\lambda_0 \int_0^s g d\mu, \quad s \in (0, D). \quad (3.4)$$

Lemmas 4–6 given below consist of the basis of the test functions used in the definitions of $\mathcal{F}_\#$ and \mathcal{H} .

Lemma 4 *Let g be a non-zero a.e. eigenfunction of $\lambda_0 > 0$. Then g is strictly monotone.*

Proof Because $\lambda_0 > 0$, g cannot be a constant. We need only to prove that $g' \neq 0$ on $(0, D)$. Suppose that there is a $p \in (0, D)$ such that $g'(p) = 0$. Then, by the eigenequation restricted to $(0, p)$, we would have $\lambda_0 \geq \lambda_1^{(0,p)}$, where $\lambda_1^{(0,p)}$ is the minimal eigenvalue with Neumann boundaries at 0 and p . To see this, by (3.4), we have $\mu_{0,p}(g) = 0$ since $g'(0) = 0$ and $g'(p) = 0$. Here, it is quite standard to prove the required assertion. By making inner product with g on the both sides of the eigenequation with respect to $\mu_{0,p}$, it follows that

$$-(e^C g g')|_0^p + D_{0,p}(g) = \lambda_0 \mu_{0,p}(g^2).$$

Again, because of $g'(0) = g'(p) = 0$, we obtain $\lambda_0 = D_{0,p}(g)/\mu_{0,p}(g^2)$. Hence,

$$\begin{aligned} \lambda_0 &= \frac{D_{0,p}(g)}{\mu_{0,p}(g^2)} \\ &= \frac{D_{0,p}(g)}{\text{Var}_{(0,p)}(g)} \quad (\text{since } \mu_{0,p}(g) = 0) \\ &\geq \inf \left\{ \frac{D_{0,p}(f)}{\text{Var}_{(0,p)}(f)} : f \in \mathcal{C}^1(0,p) \cap \mathcal{C}[0,p], f \in L^2(0,p;\mu), f \neq \text{constant} \right\} \\ &= \lambda_1^{(0,p)}. \end{aligned}$$

Now, by Lemma 3, we obtain

$$\lambda_0 \geq \lambda_1^{(0,p)} > \lambda_0^{(0,p)} > \lambda_0.$$

This is a contradiction. \square

Lemma 5 *The a.e. eigenfunction g of λ_0 is either positive or negative everywhere.*

Proof If $\lambda_0 = 0$, then g must be a constant and so the assertion is obvious. Now, let $\lambda_0 > 0$. By Lemma 4, without loss of generality, assume that $g'|_{(0,D)} < 0$ and $g(0) > 0$. We need only to prove that $g \neq 0$ on $(0, D)$. If, otherwise, $g(p) = 0$ for some $p \in (0, D)$, then, since $\lambda_0^{(0,p)}$ is the minimal ND-eigenvalue on $(0, p)$, the eigenequation restricted to $(0, p)$ shows that

$$\lambda_0 \geq \lambda_0^{(0,p)} > \lambda_0,$$

which is a contradiction. \square

Because of (3.4), we have $I(g)^{-1} \equiv \lambda_0$. This explains where the operator I comes from. Next, from (3.3), we have

$$g(x) - g(D) = \lambda_0 \int_x^D \nu(ds) \int_0^s g d\mu. \quad (3.5)$$

When $D < \infty$, since $g(D) = 0$ by our boundary condition, we obtain $II(g)^{-1} \equiv \lambda_0$. This explains the meaning of the operator II . To show that the last assertion holds even for $D = \infty$, it is necessary to prove that $g(\infty) = 0$. This is impossible if $\lambda_0 = 0$ since then g can be an arbitrary non-zero constant.

Lemma 6 *Let $D = \infty$. If $\lambda_0 > 0$, then its a.e. eigenfunction g satisfies $g(\infty) = 0$.*

Proof Without loss of generality, by Lemmas 4 and 5, assume that $g'|_{(0,D)} < 0$ and $g|_{[0,D)} > 0$.

(a) By what we have just seen and the decreasing property of g , we have

$$\frac{g(x) - g(\infty)}{\lambda_0} = \int_x^\infty \nu(ds) \int_0^s g d\mu \geq g(\infty) \int_x^\infty \nu(ds) \int_0^s d\mu.$$

Thus, $g(\infty) = 0$ once

$$\int_0^\infty \nu(ds) \int_0^s d\mu = \infty$$

(which is the uniqueness criterion for the semigroup or the nonexplosive criterion for the minimal process) since the left-hand side is finite.

(b) Otherwise, we have

$$M(x) := \int_x^\infty \nu(ds) \int_0^s d\mu < \infty, \quad x \in (0, D).$$

Let $f = g - g(\infty)$ and suppose that $g(\infty) > 0$. Then $f \in \mathcal{F}_{II}$, and moreover,

$$fII(f)(x) = \lambda_0^{-1}(g(x) - g(\infty)) - g(\infty)M(x) = \lambda_0^{-1}f(x) - g(\infty)M(x).$$

We arrive at

$$\sup_{x \in (0, \infty)} II(f)(x) = \frac{1}{\lambda_0} - g(\infty) \inf_{x \in (0, \infty)} \frac{M(x)}{f(x)}.$$

Since $f(\infty) = 0$ and $M(\infty) = 0$, by Cauchy's mean value theorem, we have

$$\begin{aligned}
& \inf_{x \in (0, \infty)} \frac{M(x)}{f(x)} \\
& \geq \inf_{x \in (0, \infty)} \frac{M'(x)}{f'(x)} \\
& = \inf_{x \in (0, \infty)} \left(-\frac{e^{-C(x)}}{g'(x)} \int_0^x \frac{e^{C(u)}}{a(u)} du \right) \\
& \geq \inf_{x \in (0, \infty)} \left(-\frac{e^{-C(x)}}{g(0)g'(x)} \int_0^x g d\mu \right) \quad (\text{since } g' < 0 \text{ and } g > 0 \text{ on } (0, D)) \\
& = \inf_{x \in (0, \infty)} \frac{1}{g(0)} I(g)(x) \\
& = \frac{1}{\lambda_0 g(0)} \\
& > 0.
\end{aligned}$$

Inserting this into the previous equation, it follows that

$$\lambda_0 < \inf_{x \in (0, \infty)} II(f)(x)^{-1}.$$

But

$$\inf_{x \in (0, \infty)} II(f)(x)^{-1} \leq \lambda_0$$

is a part of Theorem 1 (2) and will be proved soon below, without using the properties of the a.e. eigenfunction g . We have thus obtained a contradiction. \square

From now on in this section, we assume that the a.e. eigenfunction (say g) satisfies $g > 0$ and $g' < 0$ on $(0, D)$, $g'(0) = 0$, and $g(D) = 0$ (recall that $g(D) = \lim_{x \rightarrow D} g(x)$ if $D = \infty$).

Proof of Theorem 1 and Proposition 1 Similar to the proof of [5; Theorem 2.4 and Proposition 2.5], we can prove the assertions by two circle arguments.

To prove the lower estimates, we adopt the following circle arguments:

$$\begin{aligned}
\lambda_0 & \geq \tilde{\lambda}_0 \\
& \geq \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1} \\
& = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} II(f)(x)^{-1} \\
& = \sup_{f \in \mathcal{F}_{II}} \inf_{x \in (0, D)} II(f)(x)^{-1} \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
& \geq \sup_{h \in \mathcal{H}} \inf_{x \in (0, D)} R(h)(x) \\
& \geq \lambda_0, \tag{3.7}
\end{aligned}$$

$$\begin{aligned}
\lambda_0 &\leq \inf_{f \in \tilde{\mathcal{F}}_I \cup \tilde{\mathcal{F}}'_I} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} \\
&= \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1} \\
&= \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1}
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&= \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0,D)} I(f)(x)^{-1} \\
&= \inf_{f \in \tilde{\mathcal{F}}'_I} \sup_{x \in (0,D)} I(f)(x)^{-1}
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\leq \inf_{h \in \tilde{\mathcal{H}}} \sup_{x \in (0,D)} R(h)(x) \\
&\leq \lambda_0.
\end{aligned} \tag{3.10}$$

In fact, most of the proof here are parallel to those in the discrete case (see [5; Section 2]). Actually, one can follow the cited proofs with some changes illustrated here. For instance, to prove

$$\tilde{\lambda}_0 \geq \sup_{f \in \tilde{\mathcal{F}}_I} \inf_{x \in (0,D)} II(f)(x)^{-1},$$

following [5; Part I (a) of proof of Theorem 2.4 and Proposition 2.5], let g (irrelated to the eigenfunction) be a test function of $\tilde{\lambda}_0$: $g \in \mathcal{C}^1(0,D) \cap \mathcal{C}[0,D]$, $g(D) = 0$, $g'(0) = 0$, and $\mu(g^2) = 1$. Then for every h with $h|_{(0,D)} > 0$, we have

$$\begin{aligned}
1 &= \mu(g^2) \\
&= \int_0^D \frac{e^{C(x)}}{a(x)} \left(\int_x^D g'(t) dt \right)^2 dx \\
&\leq \int_0^D \frac{e^{C(x)}}{a(x)} dx \int_x^D \frac{e^{C(t)}}{h(t)} g'(t)^2 dt \int_x^D \frac{h(s)}{e^{C(s)}} ds \\
&\hspace{15em} \text{(by Cauchy-Schwarz's inequality)} \\
&= \int_0^D \frac{e^{C(t)}}{h(t)} g'(t)^2 dt \int_0^t \frac{e^{C(x)}}{a(x)} dx \int_x^D \frac{h(s)}{e^{C(s)}} ds \quad \text{(by Fubini's Theorem)} \\
&\leq D(g) \sup_{t \in (0,D)} \frac{1}{h(t)} \int_0^t \frac{e^{C(x)}}{a(x)} dx \int_x^D \frac{h(s)}{e^{C(s)}} ds \\
&=: D(g) \sup_{t \in (0,D)} H(t).
\end{aligned}$$

For $f \in \tilde{\mathcal{F}}_I$ satisfying

$$\sup_{x \in (0,D)} II(f)(x) < \infty,$$

we specify

$$h(t) = \int_0^t a(s)^{-1} e^{C(s)} f(s) ds.$$

Then by Cauchy’s mean value theorem, it follows that

$$\sup_{t \in (0, D)} H(t) \leq \sup_{x \in (0, D)} \frac{1}{f(x)} \int_x^D e^{-C(s)} ds \int_0^s \frac{e^{C(u)}}{a(u)} f(u) du = \sup_{x \in (0, D)} II(f)(x).$$

Hence,

$$\inf_{x \in (0, D)} II(f)(x)^{-1} \leq \inf_{t \in (0, D)} H(t)^{-1} \leq D(g).$$

Making infimum with respect to g , we obtain the required assertion. We have also completed the proof of Lemma 6.

As mentioned before Lemma 6, the operators I and II are all from the eigenequation. Here, we show that so is the operator R . Rewrite the eigenequation as

$$-\frac{Lg}{g} = \lambda_0$$

which is meaningful since $g > 0$. To simplify the left-hand side, in the discrete case, one uses the ratio $g(x + 1)/g(x)$. However, this is useless in the present continuous situation. What instead is using the function $h = g'/g$. Then

$$-\frac{Lg}{g} = -(ah^2 + bh + ah') = R(h).$$

The conditions $g > 0$ and $g' < 0$ on $(0, D)$ lead to the restraint $h < 0$ in defining \mathcal{H} . Note that the inverse transform $h \rightarrow g$ is unique up to a positive constant:

$$g(x) = \exp \left[\int_0^x h(u) du \right].$$

The restraint allowing $h = 0$ in the definition of \mathcal{H} is to include the degenerated case that $g' \equiv 0$ when $\lambda_0 = 0$ (then $D = \infty$ by hypothesis (2.1)). Clearly, the use of R is essentially the use of L . For this reason, we make the continuous condition on a and b once concerning with R . Because of this point, we need two additional terms in the circle arguments above: the right-hand side of (3.6) is not less than λ_0 and the right-hand side of (3.9) is no more than λ_0 . This is rather easy since for the a.e eigenfunction g , we have $I(g)^{-1} \equiv \lambda_0$ and $II(g)^{-1} \equiv \lambda_0$ by (3.4), (3.5), and Lemma 6. Actually, the required assertion was also contained in the corresponding proof of the discrete situation.

As another illustration of the proof when moving from the discrete case to the continuous one, we consider a proof for the upper estimates. For instance, we prove that

$$\lambda_0 \leq \inf_{f \in \tilde{\mathcal{F}}_II \cup \tilde{\mathcal{F}}'_II} \sup_{x \in \text{supp}(f)} II(f)(x)^{-1}.$$

Before moving to the details, let us mention that, for the upper estimates of λ_0 , we are actually using a comparison between λ_0 and $\lambda_0^{(0, x_0)}$. Thus, for the upper estimates of λ_0 , we indeed use the restriction on $[0, x_0]$ for the test functions, ignoring their behaviors out of $[0, x_0]$.

Given $f \in \widetilde{\mathcal{F}}_H$ with $f = f\mathbb{1}_{[0,x_0]}$ for some $x_0 \in (0, D)$, let

$$g = fH(f)\mathbb{1}_{\text{supp}(f)}.$$

Then $g \in L^2(\mu)$. Since

$$[e^C g'](x) = - \int_0^x f d\mu \quad \text{on } [0, x_0),$$

by the integration by parts formula, we have

$$D(g) = \int_0^D e^{C(x)} g'(x)^2 dx = \int_0^{x_0} f g d\mu.$$

Hence,

$$D(g) \leq \int_0^{x_0} g^2 d\mu \sup_{(0,x_0)} \frac{f}{g} = \mu(g^2) \sup_{x \in (0,x_0)} H(f)(x)^{-1}.$$

Since $g \in L^2(\mu)$, it follows that

$$\lambda_0 \leq \frac{D(g)}{\mu(g^2)} \leq \sup_{\text{supp}(f)} H(f)^{-1} \quad (3.11)$$

for every $f \in \widetilde{\mathcal{F}}_H$. It remains to show that the same assertion holds for every $f \in \widetilde{\mathcal{F}}'_H$. Recall that in the proof above, the conclusion $g \in L^2(\mu)$ comes from the finiteness of x_0 . Otherwise, if $x_0 = D = \infty$, then $f \in \widetilde{\mathcal{F}}'_H$ means that the function $g = fH(f)$ is assumed to be in $L^2(\mu)$, and the proof above still works. Therefore, we obtain the required assertion.

Hopefully, we have explained enough the difference between the discrete and the continuous cases. Now, one may follow [5; Proof of Theorem 2.4 and Proposition 2.5] (quite long and technical) to complete the whole proof. \square

Before moving further, let us mention a fact about the localizing procedures used in Theorem 3 (2). Instead of the approximating to the infinite state space ($D = \infty$) by finite ones, it seems more natural to use the truncating procedure for the test function $f: f^{(n)} = f\mathbb{1}_{[0,x_n]}$ with $x_n \uparrow \infty$. The next result shows that such a procedure is not practical in general.

Remark 2 Assume that hypothesis (2.1) holds. Let $D = \infty$, let g be the eigenfunction of $\lambda_0 > 0$, and define $g^{(n)} = g\mathbb{1}_{[0,x_n]}$ for some $x_n \in (0, \infty)$. Then

$$\inf_{x \in \text{supp}(g^{(n)})} H(g^{(n)})(x) = 0.$$

In particular, $\inf_{x \in \text{supp}(g^{(n)})} H(g^{(n)})(x)$ does not converge to λ_0 as $x_n \rightarrow \infty$.

Proof By definition of $g^{(n)}$, we have

$$\begin{aligned}
 \inf_{x \in \text{supp}(g^{(n)})} II(g^{(n)})(x) &= \inf_{x \in [0, x_n]} \frac{1}{g^{(n)}(x)} \int_x^{x_n} e^{-C(s)} ds \int_0^s g^{(n)} d\mu \\
 &= \inf_{x \in [0, x_n]} \frac{1}{g(x)} \int_x^{x_n} e^{-C(s)} ds \int_0^s g d\mu \\
 &= \inf_{x \in [0, x_n]} \frac{1}{g(x)} \int_x^{x_n} (-\lambda_0^{-1} g'(s)) ds \quad (\text{by (3.4)}) \\
 &= \inf_{x \in [0, x_n]} \frac{1}{\lambda_0 g(x)} (g(x) - g(x_n)) \\
 &= \inf_{x \in [0, x_n]} \frac{1}{\lambda_0} \left(1 - \frac{g(x_n)}{g(x)} \right) \\
 &= 0 \quad (\text{since } g \in \mathcal{C}[0, D] \text{ and } g \downarrow \downarrow). \quad \square
 \end{aligned}$$

Proof of Proposition 2 (1) Let $g \in \mathcal{C}^1(0, D)$ with $g > 0$ and $g' < 0$ on $(0, D)$, and let

$$\bar{h}(x) = -e^{-C(x)} \int_0^x g d\mu.$$

Then $\bar{h} \in \mathcal{H}_*$ and

$$\bar{R}(\bar{h})(x) = -\frac{(a\bar{h}' + b\bar{h})'(x)}{\bar{h}(x)} = \frac{g'(x)}{\bar{h}(x)} > 0.$$

This clearly implies that

$$\sup_{h \in \mathcal{H}_*} \inf_{x \in (0, D)} \bar{R}(h)(x) \geq 0.$$

(2) Without loss of generality, assume that $\lambda_0 > 0$. Since $a, b \in \mathcal{C}^1(0, D)$, there exists an eigenfunction g such that $\bar{h} := g' \in \mathcal{H}_*$ and

$$\bar{R}(\bar{h})(x) = -\frac{(Lg)'(x)}{g'(x)} \equiv \lambda_0.$$

Thus,

$$\sup_{h \in \mathcal{H}_*} \inf_{x \in (0, D)} \bar{R}(h)(x) \geq \sup_{h \in \mathcal{H}_*} \inf_{x \in (0, D)} \bar{R}(h)(x) \geq \lambda_0.$$

Now, one can complete the proof following that of the discrete case ([5; Proof of Proposition 2.7]). \square

To prove Theorem 2, we need the following result.

Lemma 7 *Given two nonnegative, measurable, and locally integrable functions m and n on $[0, D]$, suppose that*

$$\int_0^D n(y) dy < \infty, \quad c := \sup_{x \in (0, D)} \int_0^x m(y) dy \int_x^D n(y) dy < \infty.$$

Set

$$\psi(x) = \int_x^D n(y)dy.$$

Then for every $r \in (0, 1)$, we have

$$\int_0^x m(y)\psi^r(y)dy \leq \frac{c}{1-r} \psi^{r-1}(x), \quad x \in (0, D).$$

Proof Let

$$M(x) = \int_0^x m(y)dy.$$

Noticing that $M'(x) = m(x)$ and $M\psi \leq c$, we obtain the assertion by using the integration by parts formula. \square

Proof of Theorem 2 To prove the lower estimate, without loss of generality, assume that $\delta < \infty$. Applying Lemma 7 to

$$m(x) = \frac{e^{C(x)}}{a(x)}, \quad n(x) = e^{-C(x)},$$

we get

$$\int_0^x \varphi^r(y)\mu(dy) = \int_0^x \varphi^r(y)m(y)dy \leq \frac{\delta}{1-r} \varphi^{r-1}(x), \quad x \in (0, D).$$

Put $f = \varphi^r$. Then $f \in \mathcal{F}_I$ and $I(f)(x) \leq \delta/(r - r^2)$. Optimizing the inequality with respect to r , it follows that

$$I(f)(x) \leq \inf_{0 < r < 1} \frac{\delta}{r - r^2} = 4\delta. \quad (3.12)$$

We have thus proved the lower estimate.

For the upper estimate, we choose the test function as $f = \nu(x_0 \vee \cdot, x_1)\mathbb{1}_{[0, x_1]}$ for some $x_0, x_1 \in [0, D)$ with $x_0 < x_1$. Then, the assertion follows by using either the variational formula for upper estimate given by Theorem 1 (1):

$$\lambda_0 \leq \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0, D)} I(f)(x)^{-1}$$

or the classical variational formula:

$$\lambda_0^{-1} = \lambda_*^{-1} \geq \sup_{x_0, x_1: x_0 < x_1} \frac{\|f_1^{x_0, x_1}\|}{D(f_1^{x_0, x_1})}$$

and then letting $x_1 \rightarrow D$.

At last, if $\nu(0, D) = \infty$, then we have $\nu(x, D) = \infty$ because of hypothesis (2.1). Furthermore, $\mu(0, x)\nu(x, D) = \infty$ for every $x \in (0, D)$. Therefore, $\delta = \infty$ and $\lambda_0 = 0$. If

$$\int_0^\infty \mu(0, x)\nu(dx) < \infty,$$

then for each $x \in (0, D)$, we have

$$\mu(0, x)\nu(x, D) = \int_x^D \mu(0, x)\nu(dt) < \int_x^\infty \mu(0, t)\nu(dt) < \int_0^\infty \mu(0, x)\nu(dx) < \infty.$$

Hence, $\delta < \infty$ and $\lambda_0 > 0$. \square

Proof of Theorem 3 and Corollary 1 Simply follow [5; Proof of Theorem 3.2 and Corollary 3.3]. We mention that the proof of ' $\delta'_1 \leq 2\delta$ ' and the computation of δ'_1 are not easy. \square

4 DN-case

We now turn to study the DN-case. As Section 2, we use the same notation $\mathcal{C}[0, D]$, $\mathcal{C}^k(0, D)$, and the operator L . The main different point for the eigenequation $Lg = -\lambda_0 g$ is the boundary condition: $g(0) = 0$ and $g'(D) = 0$ if $D < \infty$. Now, define

$$\lambda_0 = \inf \left\{ \frac{D(f)}{\mu(f^2)} : f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], D(f) < \infty, f(0) = 0, f \neq 0 \right\}, \quad (4.1)$$

where

$$D(f) = \int_0^D a f'^2 d\mu, \quad \mu(dx) = \frac{e^{C(x)}}{a(x)} dx, \quad C(x) = \int_0^x \frac{b(u)}{a(u)} du.$$

Again, define

$$\nu(dx) = e^{-C(x)} dx.$$

Here, we have used the hypothesis (2.1). The restraint ' $D(f) < \infty$ ' in (4.1) is to avoid ∞/∞ since we allow $\mu(f^2) = \infty$. Then the restraint ' $f \neq 0$ ' is needed to avoid $0/0$. Note that the restriction on the set \mathcal{C}_K for test functions disappears in (4.1). This means that the maximal Dirichlet form or the maximal process is used here, instead of the minimal one used in Section 2. In other words, we do not assume the uniqueness of the semigroup, which is different from what we studied earlier in [1–4,9]. The constant λ_0 defined above describes the optimal constant $C = \lambda_0^{-1}$ in the following *weighted Hardy inequality*:

$$\mu(f^2) \leq CD(f), \quad f(0) = 0.$$

(See [4; Section 5.2]). In other words, we are studying the weighted Hardy inequality in this section. To save the notation, we use the same notation λ_0 , I , II , R and so on as before, each of them plays a similar role but may have different meaning in different context.

Before going to our main text, we note that in definition of λ_0 , one may replace $\mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]$ by $\mathcal{A}[0, D]$ as shown by Lemma 2.

Now, we review some notation defined originally in [3,9] and introduce some new ones as follows:

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)} \int_x^D f d\mu \quad (\text{single integral form}),$$

$$II(f)(x) = \frac{1}{f(x)} \int_0^x \nu(ds) \int_s^D f d\mu \quad (\text{double integral form}),$$

$$R(h)(x) = -(ah^2 + bh + ah')(x) \quad (\text{differential form}).$$

The domains of I , II , and R , respectively, are as follows:

$$\mathcal{F}_I = \{f: f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], f(0) = 0, f'|_{(0, D)} > 0\},$$

$$\mathcal{F}_{II} = \{f: f \in \mathcal{C}[0, D], f(0) = 0, f|_{(0, D)} > 0\},$$

$$\mathcal{H} = \left\{ h: h \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D], h|_{(0, D)} > 0, \int_{0+} h(u) du = \infty \right\},$$

where \int_{0+} means \int_0^ε for sufficiently small $\varepsilon > 0$. These sets are used for the estimates on lower bounds of λ_0 . For the upper bounds, we have the following domains:

$$\widetilde{\mathcal{F}}_I = \{f: \exists x_0 \in (0, D), f \in \mathcal{C}^1(0, x_0) \cap \mathcal{C}[0, D], f(0) = 0, f = f(\cdot \wedge x_0), f'|_{(0, x_0)} > 0\},$$

$$\widetilde{\mathcal{F}}_{II} = \{f: \exists x_0 \in (0, D), f \in \mathcal{C}[0, x_0], f(0) = 0, f = f(\cdot \wedge x_0), f|_{(0, x_0)} > 0\},$$

$$\widetilde{\mathcal{H}} = \left\{ h: \exists x_0 \in (0, D), h \in \mathcal{C}^1(0, x_0) \cap \mathcal{C}[0, D], h|_{(0, x_0)} > 0, \int_{0+} h(u) du = \infty, h|_{[x_0, D]} = 0, \sup_{(0, x_0)} (ah^2 + bh + ah') < 0 \right\}.$$

Besides, we also need

$$\widetilde{\mathcal{F}}'_{II} = \{f: f > 0, f \in \mathcal{C}[0, D], fII(f) \in L^2(\mu)\}.$$

Under hypothesis (2.1), if $\mu(0, D) = \infty$, then λ_0 defined by (4.1) is trivial. Indeed, let

$$f = \mathbb{1}_{(\delta, D]} + h\mathbb{1}_{[0, \delta]},$$

where h is chosen such that $h(0) = 0$ and $f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]$ (for example, $h(x) = -x^2 \cdot \delta^{-2} + 2x \cdot \delta^{-1}$). Then $D(f) \in (0, \infty)$ and $\mu(f^2) = \infty$. It follows that $\lambda_0 = 0$.

Otherwise, $\mu(0, D) < \infty$. Then for every f with $\mu(f^2) = \infty$, by setting $f^{(x_0)} = f(\cdot \wedge x_0) \in L^2(\mu)$, we have

$$\infty > D(f^{(x_0)}) \uparrow D(f), \quad \infty > \mu(f^{(x_0)^2}) \rightarrow \mu(f^2) \quad \text{as } x_0 \rightarrow D.$$

In other words, for each non-square-integrable function f , both $\mu(f^2)$ and $D(f)$ can be approximated by a sequence of square-integrable ones. Hence, we can rewrite λ_0 as follows:

$$\lambda_0 = \inf\{D(f) : \mu(f^2) = 1, f(0) = 0, f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]\}. \tag{4.2}$$

In this case, as will be seen soon but not obvious, we also have

$$\begin{aligned} \lambda_0 &= \inf\{D(f) : \mu(f^2) = 1, f(0) = 0, f = f(\cdot \wedge x_0), \\ &\quad f \in \mathcal{C}^1(0, x_0) \cap \mathcal{C}[0, x_0] \text{ for some } x_0 \in (0, D)\} \\ &=: \tilde{\lambda}_0. \end{aligned}$$

Now, we introduce our main results. Their relations are very much the same as that indicated in Section 2, except that the test function used in Theorem 5 is $\nu(0, x)^\gamma$ but not $\nu(x, D)^\gamma$ ($\gamma = 1/2$ or 1).

Theorem 4 *Let hypothesis (2.1) hold. Assume that $\mu(0, D) < \infty$. Then λ_0 defined by (4.1) or (4.2) coincides with $\tilde{\lambda}_0$ and the following variational formulas hold.*

(1) *Single integral forms:*

$$\inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0, D)} I(f)(x)^{-1} = \lambda_0 = \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} I(f)(x)^{-1}.$$

(2) *Double integral forms:*

$$\begin{aligned} \lambda_0 &= \inf_{f \in \tilde{\mathcal{F}}_I} \sup_{x \in (0, D)} II(f)(x)^{-1} \\ &= \inf_{f \in \tilde{\mathcal{F}}_II} \sup_{x \in (0, D)} II(f)(x)^{-1} \\ &= \inf_{f \in \tilde{\mathcal{F}}_II \cup \tilde{\mathcal{F}}'_II} \sup_{x \in (0, D)} II(f)(x)^{-1}, \\ \lambda_0 &= \sup_{f \in \mathcal{F}_I} \inf_{x \in (0, D)} II(f)(x)^{-1} = \sup_{f \in \mathcal{F}_II} \inf_{x \in (0, D)} II(f)(x)^{-1}. \end{aligned}$$

Moreover, if $a, b \in \mathcal{C}[0, D]$, then we also have

(3) *differential forms:*

$$\inf_{h \in \tilde{\mathcal{H}}} \sup_{x \in (0, D)} R(h)(x) = \lambda_0 = \sup_{h \in \mathcal{H}} \inf_{x \in (0, D)} R(h)(x).$$

Theorem 5 (Criterion and basic estimates) *Let hypothesis (2.1) hold. Then λ_0 defined by (4.1) (or equivalently, $\tilde{\lambda}_0$ provided $\mu(0, D) < \infty$) is positive if and only if*

$$\delta := \sup_{x \in (0, D)} \nu(0, x)\mu(x, D) < \infty.$$

More precisely, we have

$$(4\delta)^{-1} \leq \lambda_0 \leq \delta^{-1}.$$

In particular, we have $\lambda_0 = 0$ if $\mu(0, D) = \infty$, and $\lambda_0 > 0$ if

$$D < \infty \quad \text{or} \quad \int_0^D (a(u)^{-1}e^{C(u)} + e^{-C(u)})du < \infty.$$

Proof The result was proved in [2; Theorem 1.1] except the case that $\mu(0, D) = \infty$, which implies $\lambda_0 = 0$ ($\delta = \infty$) and so the assertion is trivial. \square

Theorem 6 (Approximating procedure) *Let hypothesis (2.1) hold. Assume that $\mu(0, D) < \infty$ and $\delta < \infty$. Set $\varphi(x) = \nu(0, x)$ for $x \in (0, D)$.*

(1) *Define*

$$f_1 = \sqrt{\varphi}, \quad f_n = f_{n-1}H(f_{n-1}), \quad n \geq 2,$$

and let

$$\delta_n = \sup_{x \in (0, D)} H(f_n)(x), \quad n \geq 1.$$

Then δ_n is decreasing in n and

$$\lambda_0 \geq \delta_n^{-1} \geq (4\delta)^{-1}, \quad n \geq 1.$$

(2) *For fixed $x_0 \in (0, D)$, define*

$$f_1^{(x_0)} = \varphi(\cdot \wedge x_0), \quad f_n^{(x_0)} = (f_{n-1}^{(x_0)}H(f_{n-1}^{(x_0)}))(\cdot \wedge x_0), \quad n \geq 2,$$

and let

$$\delta'_n = \sup_{x_0 \in (0, D)} \inf_{x \in (0, D)} H(f_n^{(x_0)})(x).$$

Then δ'_n is increasing in n and

$$\delta^{-1} \geq \delta_n'^{-1} \geq \lambda_0, \quad n \geq 1.$$

Next, define

$$\bar{\delta}_n = \sup_{x_0 \in (0, D)} \frac{\|f_n^{(x_0)}\|}{D(f_n^{(x_0)})}, \quad n \geq 1.$$

Then $\bar{\delta}_n^{-1} \geq \lambda_0$, $\bar{\delta}_{n+1} \geq \delta'_n$ for every $n \geq 1$ and $\bar{\delta}_1 = \delta_1$.

Corollary 2 (Improved estimates) *We have the following estimates:*

$$\delta^{-1} \geq \delta_1'^{-1} \geq \lambda_0 \geq \delta_1^{-1} \geq (4\delta)^{-1},$$

where

$$\begin{aligned} \delta_1 &= \sup_{x \in (0, D)} \frac{1}{\sqrt{\varphi(x)}} \int_0^D \varphi(x \wedge \cdot) \sqrt{\varphi} \, d\mu \\ &= \sup_{x \in (0, D)} \left(\frac{1}{\sqrt{\varphi(x)}} \int_0^x \varphi^{3/2} \, d\mu + \sqrt{\varphi(x)} \int_x^D \sqrt{\varphi} \, d\mu \right), \end{aligned}$$

$$\delta'_1 = \sup_{x \in (0, D)} \frac{1}{\varphi(x)} \int_0^D \varphi(\cdot \wedge x)^2 d\mu \in [\delta, 2\delta].$$

Since the proofs of the results above are either known from [2,3] or parallel to [5], here we make some remarks only.

Remark 3 (1) As mentioned in [5], the original proofs given in [2,3] are still suitable to support the idea using the maximal Dirichlet form instead of the uniqueness assumption.

(2) As discussed in the last section, it is natural to extend a and b from continuous to measurable when using operators I and II only.

(3) About the duality. Recall that

$$L = \frac{d}{d\mu} \frac{d}{d\nu}.$$

The dual operator of L is simply defined as

$$L^* = \frac{d}{d\mu^*} \frac{d}{d\nu^*}, \quad \mu^* := \nu, \quad \nu^* := \mu.$$

For the boundaries, simply exchange the names of Dirichlet and Neumann. The basic results for these operators are $\lambda_0(L) = \lambda_0(L^*)$ and $\delta = \delta^*$, where $\lambda_0(L)$ and δ are defined in Section 2, and $\lambda_0(L^*)$ and δ^* are defined in this section by replacing L with L^* . The proof goes as follows.

(a) Reduce to finite D . By an approximating procedure we have used many times before, it suffices to prove the assertion for finite D . The point is that for $\lambda_0(L)$, one needs to consider only the test functions having compact support; for $\lambda_0(L^*)$, it suffices to consider the test function $f = f(\cdot \wedge x_0)$, where x_0 varies over $(0, D)$.

(b) By a standard smoothing procedure, one may assume that a and b are smooth.

(c) The identity of $\lambda_0(L)$ and $\lambda_0(L^*)$ is a combination of Proposition 2 (2) and Theorem 4 (3). The discrete case was given in [5; Section 5]. An alternative proof of this assertion was presented in [7] based on isospectral. Note that in the last proof, the finiteness of D is crucial, otherwise, the domains of L and L^* are essential different unless the Dirichlet form corresponding to L^* is assumed to be regular.

(4) When $D < \infty$, one may simply reverse the variable to obtain one from the other of between ND- and DN-cases. In this sense, the identity $\lambda_0(L) = \lambda_0(L^*)$ stated in (3) is quite natural even though the duality is not a ‘reverse transform’. When $D = \infty$, these two cases are certainly different since the Dirichlet boundary at 0 is touchable but not the one at ∞ . We mention that the variational formulas and then the approximating procedure in this section are different from those deduced by the dual approach. It is interesting that in the discrete situation, the approximating procedure given by Theorem 6 is

often less powerful than those given by Theorem 3 in terms of duality. Similar phenomenon happens in the continuous situation as shown in [6] with $D < \infty$.

5 Supplement to NN-case

Everything is the same as those in the last section except the mixed eigenvalue λ_0 is replaced by

$$\lambda_1 = \inf\{D(f) : \mu(f) = 0, \mu(f^2) = 1, f \in \mathcal{C}^1(0, D) \cap \mathcal{C}[0, D]\}. \quad (5.1)$$

Let us repeat that throughout this section, we assume that hypothesis (2.1) holds and $\mu(0, D) < \infty$.

The supplement consists of three parts. The first one is using the maximal Dirichlet form instead of the uniqueness assumption of the semigroup. The second one is using the ‘a.e. eigenfunction’ instead of ‘eigenfunction’. These two parts have already been studied in the previous sections. See also [9] for some supplement to the original paper. The third part is about the monotonicity of an approximating procedure which we are going to study below.

Define

$$\bar{f} = f - \pi(f), \quad f_1 = \sqrt{\varphi}, \quad f_n = \bar{f}_{n-1} H(\bar{f}_{n-1}), \quad \eta_n = \sup_{x \in (0, D)} I(\bar{f}_n)(x),$$

where $\pi = \mu/\mu(0, D)$. Here, our main question is about the monotonicity of $\{\eta_n\}$. Unlike the sequences $\{\delta_n\}$ and $\{\delta'_n\}$ defined in Theorems 3 and 6, their monotonicity results from simply twice applications of Cauchy’s mean value theorem, the method does not work for the sequence $\{\eta_n\}$ since each \bar{f}_n can be zero in $(0, D)$. We were unable to solve this problem for years until the appearance of the recent paper [5; Section 6], in which the problem was solved in the discrete context. Note that $\lambda_1 > 0$ if and only if

$$\delta := \sup_{x \in (0, D)} \nu(0, x) \mu(x, D) < \infty$$

by [2; Theorem 3.7], [5; Theorem 6.2], and Theorem 5.

Proposition 3 *Let hypothesis (2.1) hold and assume that $\delta < \infty$. Then the sequence $\{\eta_n\}$ defined above (i.e., $\{\eta'_n\}$ in [3; Theorem 1.4]) is non-decreasing.*

Proof (a) First, we show that $f_1 \in L^1(\mu)$. Recall that $\varphi(x) = \nu(0, x)$. Clearly, for arbitrarily fixed $x_0 \in (0, D)$, we have

$$\mu(\sqrt{\varphi}) = \int_0^{x_0} \sqrt{\varphi} \, d\mu + \int_{x_0}^D \sqrt{\varphi} \, d\mu \leq \int_0^{x_0} \sqrt{\varphi} \, d\mu + \frac{2\delta}{\sqrt{\varphi(x_0)}} < \infty.$$

Hence, $\sqrt{\varphi} \in L^1(\mu)$.

(b) Define two sequences $\{h_n\}$ and $\{\tilde{f}_n\}$ by the same recurrence $h_n = h_{n-1}II(h_{n-1})$ but different initial condition:

$$h_0 = 1, \quad \tilde{f}_1 = f_1 = \sqrt{\varphi}.$$

We now study $\{\tilde{f}_n\}$ first. From [3; Theorem 1.2 (1)], we have known that $\tilde{f}_2 \leq 4\delta\tilde{f}_1$. Assume that $\tilde{f}_{n-1} \leq (4\delta)^{n-2}\tilde{f}_1$ for some $n \geq 3$. Then

$$\begin{aligned} \tilde{f}_n &= \int_0^\cdot \nu(dy) \int_y^D \tilde{f}_{n-1} d\mu \\ &\leq (4\delta)^{n-2} \int_0^\cdot \nu(dy) \int_y^D \tilde{f}_1 d\mu \\ &= (4\delta)^{n-2} \tilde{f}_2 \\ &\leq (4\delta)^{n-1} \tilde{f}_1. \end{aligned}$$

By induction, this estimate holds for $n \geq 2$. Hence, $\tilde{f}_n \in L^1(\mu)$ for $n \geq 1$ by (a).

Next, we study the sequence $\{h_n\}$. Fix $x_0 \in (0, D)$. For $x > x_0$, we have

$$\begin{aligned} h_1(x) &= h_1(x_0) + \int_{x_0}^x \nu(dy) \mu(y, D) \\ &\leq h_1(x_0) + \frac{1}{\sqrt{\varphi(x_0)}} \tilde{f}_2(x) \\ &\leq h_1(x_0) + \frac{4\delta}{\sqrt{\varphi(x_0)}} \tilde{f}_1(x). \end{aligned}$$

By induction, it is not difficult to verify that

$$h_n(x) \leq \sum_{k=1}^n \frac{(4\delta)^k}{\sqrt{\varphi(x_0)}} h_1^{n-k}(x_0) \tilde{f}_1(x) + h_1^n(x_0).$$

Hence, $h_n \in L^1(\mu)$ for $n \geq 1$.

(c) Now, we look for the relationship between f_n and \tilde{f}_n . We begin with

$$f_1 = \tilde{f}_1 = \sqrt{\varphi}, \quad f_2 = \int_0^\cdot \nu(dy) \int_y^D \tilde{f}_1 d\mu = \tilde{f}_2 - \pi(f_1) h_1.$$

By induction, we have, in general,

$$f_n = \tilde{f}_n - \sum_{k=1}^{n-1} h_{n-k} \pi(f_k), \quad n \geq 2.$$

Thus, $f_n \in L^1(\mu)$ for every $n \geq 1$ by (b).

(d) We now come to the central part of the proof: showing the monotonicity of η_n . By definition of f_n , we have

$$\eta_n = \sup_{x \in (0, D)} \frac{e^{-C(x)}}{\bar{f}_n'(x)} \int_x^D \bar{f}_n \, d\mu = \sup_{x \in (0, D)} \left(\int_x^D \bar{f}_n \, d\mu \right) \left(\int_x^D \bar{f}_{n-1} \, d\mu \right)^{-1}. \quad (5.2)$$

Thus, $\eta_n \leq \eta_{n-1}$ if and only if

$$\int_x^D (\bar{f}_n - \eta_{n-1} \bar{f}_{n-1}) \, d\mu \leq 0, \quad x \in [0, D).$$

That is,

$$\int_x^D (f_n - \eta_{n-1} f_{n-1}) \, d\mu \leq (\pi(f_n) - \eta_{n-1} \pi(f_{n-1})) \mu(x, D),$$

or equivalently,

$$S(x) := \frac{1}{\mu(x, D)} \int_x^D (\eta_{n-1} f_{n-1} - f_n) \, d\mu \geq \eta_{n-1} \pi(f_{n-1}) - \pi(f_n) = S(0). \quad (5.3)$$

This is our key observation and leads to the study on the monotonicity of S .

(e) In view of (5.3), we have reduced our proof to showing non-decreasing property of S . For this, it is enough to show that

$$\mu(y, D) \int_x^D (\eta_{n-1} f_{n-1} - f_n) \, d\mu \leq \mu(x, D) \int_y^D (\eta_{n-1} f_{n-1} - f_n) \, d\mu$$

for any $x, y \in [0, D)$ with $x < y$. By separating f_n and f_{n-1} , the last inequality is equivalent to the following one:

$$\eta_{n-1} \int_y^D \mu(du) \int_x^y (f_{n-1}(t) - f_{n-1}(u)) \mu(dt) \leq \int_y^D \mu(du) \int_x^y (f_n(t) - f_n(u)) \mu(dt). \quad (5.4)$$

To see this, it suffices to check that

$$f_n(u) - f_n(t) \leq \eta_{n-1} (f_{n-1}(u) - f_{n-1}(t)), \quad u \geq t.$$

To check the last inequality, consider $n \geq 3$ first. Then

$$\begin{aligned} f_n(u) - f_n(t) &= \int_t^u \nu(dy) \int_y^D \bar{f}_{n-1} \, d\mu \quad (\text{by definition of } f_n) \\ &\leq \eta_{n-1} \int_t^u \nu(dy) \int_y^D \bar{f}_{n-2} \, d\mu \quad (\text{by (5.2)}) \\ &= \eta_{n-1} (f_{n-1}(u) - f_{n-1}(t)) \quad (\text{by definition of } f_{n-1}), \quad u \geq t. \end{aligned}$$

It remains to check the required inequality for $n = 2$. By definition of η_1 , we have

$$\frac{e^{-C(y)}}{\bar{f}'_1(y)} \int_y^D \bar{f}_1 d\mu = I(\bar{f}_1)(y) \leq \eta_1.$$

It follows that

$$f_2(u) - f_2(t) = \int_t^u \nu(dy) \int_y^D \bar{f}_1 d\mu \leq \eta_1 \int_t^u f'_1(y) dy \leq \eta_1 (f_1(u) - f_1(t)), \quad u \geq t.$$

We have thus completed the proof of the monotonicity of $\{\eta_n\}$ in the continuous context. \square

The monotonicity of $\{\eta_n\}$ means that we can theoretically improve our lower estimates of λ_1 step by step. There is a similar result for the upper estimates but omitted here. It is regretted that the convergence of $\{\eta_n^{-1}\}$ to λ_1 (as $n \rightarrow \infty$) remains open. All examples we have ever computed support the convergence.

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