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A CLASS OF MULTIDIMENSIONAL Q-PROCESSES

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Abstract

In this paper we present some necessary conditions for the uniqueness, recurrence, and ergodicity of a class of multidimensional Q-processes, using the dual Yan–Chen comparison method. Then the coupling method is used to study the multidimensional processes in a specific space. As applications, three models of particle systems are illustrated.

Keywords: Multidimensional Q-process; single-birth process; upwardly skip-free process

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1. Introduction

Some stochastic models for linear master equations of several variables were introduced in [11], [14], and [22]. In probability language, these models correspond to certain multidimensional Q-processes which satisfy the forward Kolmogorov equation. Naturally, it would be interesting to study the following three classical problems relating to this class of multidimensional Q-processes: determination of their uniqueness, their recurrence, and their ergodicity. However, as we know, for multidimensional Q-processes it is more difficult to study these problems directly than it is for one-dimensional Q-processes (see [16] and [22]). In [21], Yan and Chen proposed a method which reduces the multidimensional problems to one-dimensional ones. Yan and Chen's main idea was to compare the multidimensional Q-processes with a single-birth process. Typical examples of applications to keep in mind are Schlögl's model and the Brusselator model. For convenience, we present their results on the uniqueness of multidimensional Q-processes here (see [8, Theorem 3.19]).

Theorem 1.1. *Let E be a countable set and let $Q = (q(x, y) : x, y \in E)$ be a totally stable and conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\bigcup_{k=0}^{\infty} E_k = E$ and the following conditions hold:*

(i) *if $q(x, y) > 0$ and $x \in E_k$ then $y \in \bigcup_{j=0}^{k+1} E_j$ for all $k \geq 0$;*

(ii) *$\sum_{y \in E_{k+1}} q(x, y) > 0$ for all $x \in E_k$ and all $k \geq 0$;*

(iii) *$C_k := \sup\{q(x) : x \in E_k\} < \infty$ for all $k \geq 0$, where $q(x) \equiv -q(x, x) = \sum_{y \neq x} q(x, y)$.*

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Define a totally stable and conservative Q-matrix $(q_{ij}: i, j \in \mathbb{Z}_+)$ as follows:

$$q_{ij} = \begin{cases} \sup\{\sum_{y\in E_j} q(x, y): x \in E_i\} & \text{if } j = i + 1, \\ \inf\{\sum_{y\in E_j} q(x, y): x \in E_i\} & \text{if } j < i, \\ 0 & \text{if } j > i + 1, \end{cases}$$

and $q_i \equiv -q_{ii} = \sum_{j\neq i} q_{ij}$. If the (q_{ij})-process is unique then so is the $(q(x, y))$-process.

Using the comparison method, Yan and Chen presented some sufficient and much more practicable conditions for the uniqueness, recurrence, and ergodicity of this class of multidimensional Q-processes in [21]. Recently, in [13] and [26], Mao and Zhang obtained some sufficient conditions for exponential ergodicity and strong ergodicity in the same way. However, in the study of the Brusselator model we know that this method is valid only in the determination of uniqueness, not recurrence or ergodicity. Fortunately, the dual comparison method can be used to study the necessary condition for the (strong) ergodicity of the Brusselator model.

Keeping this 'new' idea in mind, we aim to obtain some necessary conditions for the uniqueness, recurrence, and ergodicity of the class of Q-processes considered in this paper. Our main results are the following three theorems.

Theorem 1.2. *Let E be a countable set and let $Q = (q(x, y): x, y \in E)$ be a totally stable and conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\bigcup_{k=0}^{\infty} E_k = E$ and the following conditions hold:*

(i) *if $q(x, y) > 0$ and $x \in E_k$ then $y \in \bigcup_{j=0}^{k+1} E_j$ for all $k \geq 0$;*

(ii) $\inf\{\sum_{y\in E_{k+1}} q(x, y): x \in E_k\} > 0$ *for all $k \geq 0$;*

(iii) $C_k := \sup\{q(x): x \in E_k\} < \infty$ *for all $k \geq 0$.*

Define a totally stable and conservative Q-matrix $(q_{ij}: i, j \in \mathbb{Z}_+)$ as follows:

$$q_{ij} = \begin{cases} \inf\{\sum_{y\in E_j} q(x, y): x \in E_i\} & \text{if } j = i + 1, \\ \sup\{\sum_{y\in E_j} q(x, y): x \in E_i\} & \text{if } j < i, \\ 0 & \text{if } j > i + 1, \end{cases} \tag{1.1}$$

and $q_i \equiv -q_{ii} = \sum_{j\neq i} q_{ij}$. If the $(q(x, y))$-process is unique then so is the (q_{ij})-process.

Theorem 1.3. *Let the assumptions of Theorem 1.2 hold and suppose that $E_0 = \{\theta\}$, where $\theta \in E$ is a reference point. Define a Q-matrix $(q_{ij}: i, j \in \mathbb{Z}_+)$ as in (1.1). Moreover, suppose that both $(q(x, y))$ and (q_{ij}) are irreducible and that $(q(x, y))$ is regular. If the $(q(x, y))$-process is recurrent then so is the (q_{ij})-process.*

Theorem 1.4. *Let the assumptions of Theorem 1.2 hold and suppose both that $E_0 = \{\theta\}$, where $\theta \in E$ is a reference point, and that E_k is finite for all $k \geq 1$. Define a Q-matrix $(q_{ij}: i, j \in \mathbb{Z}_+)$ as in (1.1). Moreover, suppose that both $(q(x, y))$ and (q_{ij}) are irreducible and that $(q(x, y))$ is regular. Define $\tau = \inf\{t \geq 0: X(t) = \theta\}$, where $X(t)$ denotes the $(q(x, y))$-process.*

1. *Choose an $x_i \in E_i$ such that $f_i := \mathrm{E}_{x_i} \tau = \min_{x\in E_i} \mathrm{E}_x \tau$. If f_i is an increasing function of i and the $(q(x, y))$-process is ergodic or strongly ergodic, then the (q_{ij})-process is also ergodic or, respectively, strongly ergodic.*

2. *Choose some* $\lambda \in (0, \inf_{x\in E} q(x))$ *and an* $x_i \in E_i$ *such that* $g_i := \mathrm{E}_{x_i} \mathrm{e}^{\lambda\tau} = \min_{x\in E_i} \mathrm{E}_x \mathrm{e}^{\lambda\tau}$. *If* g_i *is an increasing function of* i *and the* $(q(x, y))$*-process is exponentially ergodic, then the* (q_{ij})*-process is also exponentially ergodic.*

The (q_{ij})-process defined in (1.1) is in fact a single-birth process (also called an upwardly skip-free process or a birth–death process with catastrophe), i.e. (1.1) defines a single-birth Q-matrix. For single-birth processes there are explicit criteria for uniqueness, recurrence, and ergodicity. (For details, we refer the reader to [1]–[5], [8], [13], [15], [26]–[28].) In other words, necessary conditions on the (q_{ij})-process in the above theorems are explicitly known.

Two other points should be mentioned. First, in view of [8, Theorem 4.45(2)], in Theorem 1.4.2 the implicit condition $\inf_{x\in E} q(x) > 0$ is indeed necessary for exponential ergodicity. Second, Theorem 1.3 has been proved previously (see [18, Theorem 1.6]). Here, for convenience, we present and prove the result again.

The remainder of the paper is organized as follows. We will prove Theorems 1.2–1.4 and present some corollaries in the next section. In Section 3 the multidimensional Q-processes in a specific space are studied using coupling methods. Finally, some applications are illustrated, in Section 4.

2. Proofs

First we present the proofs of Theorems 1.2–1.4 in detail. Then two corollaries are obtained.

Proof of Theorem 1.2. To prove the uniqueness of the (q_{ij})-process, by the uniqueness criterion it suffices to show that the equation

$$(\lambda + q_i)w_i = \sum_{j\neq i} q_{ij} w_j, \qquad 0 \le w_i \le 1,\ i \ge 0, \tag{2.1}$$

has only the trivial solution. Suppose that (2.1) has a nontrivial solution $w = \{w_i : i \ge 0\}$. By [8, Theorem 3.16, Proof a)], w_i is increasing in i. Let $w(x) = w_i$ if $x \in E_i$. Then $w(x)$, $x \in E$, is nonzero. For $x \in E_i$, $i \ge 0$, we have

$$\begin{aligned}\sum_{y\neq x} q(x, y)(w(y) - w(x)) &= \sum_{j=0}^{i-1} \sum_{y\in E_j} q(x, y)(w_j - w_i) + \sum_{y\in E_{i+1}} q(x, y)(w_{i+1} - w_i)\\ &\ge \sum_{j=0}^{i-1} q_{ij}(w_j - w_i) + q_{i,i+1}(w_{i+1} - w_i)\\ &= \sum_{j\neq i} q_{ij}(w_j - w_i)\\ &= \lambda w_i\\ &= \lambda w(x),\end{aligned}$$

that is,

$$(\lambda + q(x))w(x) \le \sum_{y\neq x} q(x, y)w(y), \qquad 0 \le w(x) \le 1,\ x \in E.$$

According to the comparison lemma [8, Lemma 3.14] and the uniqueness criterion, this shows that the $(q(x, y))$-process is not unique. The statement of the theorem follows by contraposition.

Proof of Theorem 1.3. To prove the recurrence of the (q_{ij})-process, it suffices to show that the equation

$$h_i = \sum_{j \neq 0} \Pi_{ij} h_j, \qquad 0 \le h_i \le 1,\ i \ge 0, \tag{2.2}$$

where

$$\Pi_{ij} = \mathbf{1}_{\{q_i \neq 0\}} (1 - \delta_{ij}) \frac{q_{ij}}{q_i} + \mathbf{1}_{\{q_i = 0\}} \delta_{ij},$$

has only the trivial solution. Suppose that (2.2) has a nontrivial solution $\{h_i : i \ge 0\}$, which is equivalent to supposing that

$$h_0 = 1, \qquad h_i = \sum_{j \neq 0} \Pi_{ij} h_j, \quad i \ge 0, \tag{2.3}$$

has a nonnegative, bounded solution. By [8, Theorem 4.54, Proof a)], h_i is increasing in i. Let $h(x) = h_i$ if $x \in E_i$. Then $h(x),\ x \in E$, is nonzero. For $x \in E_i,\ i \ge 1$, we have

$$\begin{aligned}
\sum_{y \neq x, \theta} q(x, y)(h(y) - h(x)) &= \sum_{j=1}^{i-1} \sum_{y \in E_j} q(x, y)(h_j - h_i) + \sum_{y \in E_{i+1}} q(x, y)(h_{i+1} - h_i) \\
&\ge \sum_{j=1}^{i-1} q_{ij}(h_j - h_i) + q_{i,i+1}(h_{i+1} - h_i) \\
&= \sum_{j \neq i, 0} q_{ij}(h_j - h_i) \\
&= q_{i0} h_i \\
&\ge q(x, \theta) h(x),
\end{aligned}$$

$$\sum_{y \neq \theta} q(\theta, y)(h(y) - h(\theta)) = \sum_{y \in E_1} q(\theta, y)(h(y) - h(\theta)) = \sum_{y \in E_1} q(\theta, y)(h_1 - h_0) = 0.$$

The last equality is obtained from the fact that $h_1 = h_0 = 1$ in (2.3). From the above results, we have

$$h(\theta) = 1, \qquad h(x) \le \sum_{y \neq \theta} \Pi(x, y) h(y), \quad x \in E,$$

where

$$\Pi(x, y) = \mathbf{1}_{\{q(x) \neq 0\}} (1 - \delta_{xy}) \frac{q(x, y)}{q(x)} + \mathbf{1}_{\{q(x) = 0\}} \delta_{xy}.$$

Note that $h(x)$ is bounded. According to the comparison lemma [8, Lemma 3.14], the equation

$$f(x) = \sum_{y \neq \theta} \Pi(x, y) f(y), \qquad 0 \le f(x) \le 1,\ x \in E,$$

has a nontrivial solution, meaning that the $(q(x, y))$-process is not recurrent. The statement of the theorem follows by contraposition.

Proof of Theorem 1.4. First we prove assertion 1. Let $u(x) = \mathrm{E}_x \tau$. It is already known that $\sum_{y \neq x} q(x, y)(u(y) - u(x)) = -1$ for all $x \neq \theta$ and that $u(\theta) = 0$. Thus, for $x_i \in E_i$ such that f_i is increasing in i (where $f_i = u(x_i)$ for all $i \geq 0$), we obtain

$$\begin{aligned}
-1 &= \sum_{y \neq x_i} q(x_i, y)(u(y) - u(x_i)) \\
&= \sum_{j=0}^{i-1} \sum_{y \in E_j} q(x_i, y)(u(y) - u(x_i)) + \sum_{y \in E_{i+1}} q(x_i, y)(u(y) - u(x_i)) \\
&\quad + \sum_{y \in E_i,\, y \neq x_i} q(x_i, y)(u(y) - u(x_i)) \\
&\geq \sum_{j=0}^{i-1} \sum_{y \in E_j} q(x_i, y)(u(x_j) - u(x_i)) + \sum_{y \in E_{i+1}} q(x_i, y)(u(x_{i+1}) - u(x_i)) \\
&\geq \sum_{j=0}^{i-1} q_{ij}(f_j - f_i) + q_{i,i+1}(f_{i+1} - f_i), \qquad i \geq 1.
\end{aligned}$$

That is,

$$f_0 = 0, \qquad \sum_{j \neq i} q_{ij}(f_j - f_i) \leq -1, \quad i \geq 1. \tag{2.4}$$

If the $(q(x, y))$-process is ergodic then it follows from [8, Theorem 4.44] that $u(x) < \infty$; furthermore, $f_i < \infty$. Hence, the conclusion follows from [8, Theorem 4.45] and (2.4). If the $(q(x, y))$-process is strongly ergodic then it follows from [8, Theorem 4.44] that $\sup_{x \in E} u(x) < \infty$; furthermore, $\sup_{i \geq 0} f_i < \infty$. Therefore, by [8, Theorem 4.45] and (2.4), we see that the (q_{ij})-process is strongly ergodic.

Now we prove the second assertion. Let $v(x) = \mathrm{E}_x\, \mathrm{e}^{\lambda\tau}$. If the $(q(x, y))$-process is exponentially ergodic then $v(x) < \infty$, $\sum_{y \neq x} q(x, y)(v(y) - v(x)) = -\lambda v(x)$ for some $\lambda \in (0, \inf_{x \in E} q(x))$ and all $x \neq \theta$, and $v(\theta) = 1$. Note that we choose $x_i \in E_i$ such that g_i is increasing in i (where $g_i = v(x_i)$ for all $i \geq 0$). We find that $g_0 = 1$, that $1 \leq g_i < \infty$ for all $i \geq 1$, and that

$$\begin{aligned}
-\lambda g_i = -\lambda v(x_i) &= \sum_{y \neq x_i} q(x_i, y)(v(y) - v(x_i)) \\
&= \sum_{j=0}^{i-1} \sum_{y \in E_j} q(x_i, y)(v(y) - v(x_i)) + \sum_{y \in E_{i+1}} q(x_i, y)(v(y) - v(x_i)) \\
&\quad + \sum_{y \in E_i,\, y \neq x_i} q(x_i, y)(v(y) - v(x_i)) \\
&\geq \sum_{j=0}^{i-1} \sum_{y \in E_j} q(x_i, y)(v(x_j) - v(x_i)) + \sum_{y \in E_{i+1}} q(x_i, y)(v(x_{i+1}) - v(x_i)) \\
&\geq \sum_{j=0}^{i-1} q_{ij}(g_j - g_i) + q_{i,i+1}(g_{i+1} - g_i) \\
&= \sum_{j \neq i} q_{ij}(g_j - g_i), \qquad i \geq 1.
\end{aligned}$$

Thus, from [8, Theorem 4.45] it follows that the (q_{ij})-process is exponentially ergodic.

From Theorem 1.4, we can obtain the following two corollaries.

Corollary 2.1. *Let E be a countable set and let $Q = (q(x, y)\colon x, y \in E)$ be a totally stable and conservative Q-matrix. Denote the $(q(x, y))$-process by $X(t)$. Define $\tau = \inf\{t \ge 0\colon X(t) = \theta\}$, where $\theta \in E$ is a reference point. Let $F(x) = \mathrm{E}_x\,\tau$ and $E_k = \{x \in E\colon F(x) \in (k-1, k]\}$, $k \ge 0$. Suppose that conditions (i)–(iii) of Theorem 1.1 hold and that E_k is finite for all $k \ge 0$. Define the same Q-matrix (q_{ij}) as in (1.1). Moreover, suppose that both $(q(x, y))$ and (q_{ij}) are irreducible and that $(q(x, y))$ is regular. If the $(q(x, y))$-process is ergodic (or strongly ergodic), then so is the (q_{ij})-process.*

Corollary 2.2. *Let the conditions of Corollary 2.1 be satisfied with $F(x) = \mathrm{E}_x\,\tau$ replaced by $F(x) = \mathrm{E}_x\,\mathrm{e}^{\lambda\tau}$, $\lambda \in (0, \inf_{x\in E} q(x))$. If the $(q(x, y))$-process is exponentially ergodic then so is the (q_{ij})-process.*

Let $f_i := F(x_i) = \min_{x\in E_i} F(x)$. Then, by the definition of E_i, we see that f_i is increasing in i. The above corollaries thus follow immediately from Theorem 1.4.

3. Coupling methods

In this section we consider multidimensional Q-processes in the specific configuration space $E = (\mathbb{Z}_+^k)^S$, where $k \in \mathbb{N}$ and S is a given set. We denote a generic configuration by $x = ((x_1(u), x_2(u), \dots, x_k(u))\colon u \in S) \in E$. Define the reference point $\theta = ((0, 0, \dots, 0)\colon u \in S)$ and let $E_n = \{x \in E\colon |x| = n\}$, $n \ge 0$, where $|x| = \sum_{u\in S}\sum_{i=1}^k x_i(u)$. Let $|S|$ denote the cardinality of S. For $|S| = \infty$, when determining the existence of Q-processes we only consider the multidimensional Q-processes in $E_* = \{x \in E\colon |x| < \infty\}$. Note that $E = E_*$ for $|S| < \infty$.

We will study the class of multidimensional Q-processes using coupling methods. Our first result is as follows.

Theorem 3.1. *Let $Q = (q(x, y)\colon x, y \in E)$ be a totally stable, conservative, and irreducible Q-matrix. Suppose that the following conditions hold:*

(i) *if $q(x, y) > 0$ and $x \in E_k$ then $y \in \bigcup_{j=k-1}^{\infty} E_j$ for all $k \ge 0$;*

(ii) *$\sum_{y\in E_{k-1}} q(x, y) > 0$ for some $x \in E_k$ and all $k \ge 1$;*

(iii) *$\inf\{\sum_{y\in E_{k+1}} q(x, y)\colon x \in E_k\} > 0$ for all $k \ge 0$.*

Define a conservative birth–death Q-matrix (a_i, b_i) as follows:

$$a_i = \sup\left\{\sum_{y\in E_{i-1}} q(x, y)\colon x \in E_i\right\}, \qquad i \ge 1,$$

$$b_i = \inf\left\{\sum_{y\in E_{i+1}} q(x, y)\colon x \in E_i\right\}, \qquad i \ge 0.$$

Moreover, assume that both Q-matrices are regular. If the $(q(x, y))$-process is ergodic, exponentially ergodic, or strongly ergodic, then so, respectively, is the birth–death process.

Proof. Denote the $(q(x, y))$-process and the birth–death process by

$$X(t) = ((X_{1t}(u), X_{2t}(u), \dots, X_{kt}(u))\colon u \in S) \quad \text{and} \quad Y(t),$$

respectively. Define

$$\tau = \inf\{t \geq 0\colon X(t) = \theta\} = \inf\{t \geq 0\colon |X(t)| = 0\}, \qquad \tau^* = \inf\{t \geq 0\colon Y(t) = 0\}.$$

In the following we construct a coupling Q-matrix for $(q(x, y))$ and (a_i, b_i). Note that both marginal Q-matrices are regular. Hence, their coupling Q-matrices

$$\bar{Q} = (\bar{q}(x, i; y, j)\colon (x, i), (y, j) \in E \times \mathbb{Z}_+)$$

are all totally stable, conservative, and regular (see [7], [9], and [24]). As in [25], it is easily verified that the marginality is equivalent to the following conditions:

$$\begin{aligned} \sum_{j=0}^{\infty} \bar{q}(x, i; y, j) &= q(x, y), \qquad && y \neq x, \\ \sum_{y \in E} \bar{q}(x, i; y, j) &= q_{ij}, && j \neq i. \end{aligned} \tag{3.1}$$

Let $F := \{(x, i) \in E \times \mathbb{Z}_+\colon |x| \geq i\}$. Our aim is to construct the Q-process $Z(t) = (X(t), Y(t))$ on $E \times \mathbb{Z}_+$ whose transition probability function, $P(t) \equiv P(t; \cdot, \cdot; \cdot)$, satisfies

$$P(t; x, i; F) = 1, \qquad t \geq 0,\ (x, i) \in F. \tag{3.2}$$

Similar to [8, Theorem 5.26], it can be proved that (3.2) is equivalent to the condition that $\bar{Q}$ satisfy

$$\bar{q}(x, i; F^{\mathrm{c}}) := \sum_{\{(y,j)\colon |y| < j\}} \bar{q}(x, i; y, j) = 0, \qquad (x, i) \in F. \tag{3.3}$$

For $x \in E_n$ and $i \geq 0$, let

$$\bar{q}(x, i; y, j) = \begin{cases} \Big(\sum_{z \in E_{n+1}} q(x, z) \wedge b_i\Big) \dfrac{q(x, y)}{\sum_{z \in E_{n+1}} q(x, z)} & \text{if } y \in E_{n+1} \text{ and } j = i + 1, \\ \Big(\sum_{z \in E_{n+1}} q(x, z) - b_i\Big)^+ \dfrac{q(x, y)}{\sum_{z \in E_{n+1}} q(x, z)} & \text{if } y \in E_{n+1} \text{ and } j = i, \\ \Big(b_i - \sum_{z \in E_{n+1}} q(x, z)\Big)^+ & \text{if } y = x \text{ and } j = i + 1, \\ \Big(\sum_{z \in E_{n-1}} q(x, z) \wedge a_i\Big) \dfrac{q(x, y)}{\sum_{z \in E_{n-1}} q(x, z)} & \text{if } y \in E_{n-1} \text{ and } j = i - 1, \\ \Big(\sum_{z \in E_{n-1}} q(x, z) - a_i\Big)^+ \dfrac{q(x, y)}{\sum_{z \in E_{n-1}} q(x, z)} & \text{if } y \in E_{n-1} \text{ and } j = i, \\ \Big(a_i - \sum_{z \in E_{n-1}} q(x, z)\Big)^+ & \text{if } y = x \text{ and } j = i - 1, \\ q(x, y) & \text{if } y \in E_m \text{ with } m \geq n + 2 \\ & \qquad \text{and } j = i, \\ q(x, y) & \text{if } y \in E_n,\ y \neq x, \text{ and } j = i, \\ 0 & \text{otherwise,} \\ & \qquad \text{unless } (y, j) = (x, i), \end{cases} \tag{3.4}$$

and for $(x, i) \in E \times \mathbb{Z}_+$ let $\bar{q}(x, i) \equiv -\bar{q}(x, i; x, i) = \sum_{(y,j) \neq (x,i)} \bar{q}(x, i; y, j)$. It is easy to verify that the Q-matrix defined by (3.4) satisfies (3.1) and (3.3). It is thus a coupling Q-matrix and (3.2) holds for the transition probability function of the corresponding coupling Q-process, i.e.

$$\mathrm{P}^{(x,i)}\{|X(t)| \geq Y(t)\} = 1, \qquad t \geq 0,\ (x, i) \in F.$$

Furthermore, from the right continuity of paths it follows that $\mathrm{P}^{(x,i)}\{|X(t)| \geq Y(t)\colon t \geq 0\} = 1$ for $(x, i) \in F$. It then follows that $\mathrm{P}^{(x,|x|)}\{\tau \geq \tau^*\} = 1$. Hence, we see that $\mathrm{E}_x\, \tau \geq \mathrm{E}_{|x|}\, \tau^*$ and $\mathrm{E}_x\, \mathrm{e}^{\lambda\tau} \geq \mathrm{E}_{|x|}\, \mathrm{e}^{\lambda\tau^*}$ for all $x \in E$, and by [8, Theorem 4.44] the assertion holds.

Our second, dual, result is as follows.

Theorem 3.2. *Let $Q = (q(x, y)\colon x, y \in E)$ be a totally stable, conservative, and irreducible Q-matrix. Suppose that the following conditions hold:*

(i) *if $q(x, y) > 0$ and $x \in E_k$ then $y \in \bigcup_{j=0}^{k+1} E_j$ for all $k \geq 0$;*

(ii) *$\sum_{y \in E_{k+1}} q(x, y) > 0$ for some $x \in E_k$ and all $k \geq 0$;*

(iii) *$\inf\{\sum_{y \in E_{k-1}} q(x, y)\colon x \in E_k\} > 0$ for all $k \geq 1$.*

Define a conservative birth–death Q-matrix (a_i, b_i) as follows:

$$a_i = \inf\Big\{ \sum_{y \in E_{i-1}} q(x, y)\colon x \in E_i \Big\}, \qquad i \geq 1,$$

$$b_i = \sup\Big\{ \sum_{y \in E_{i+1}} q(x, y)\colon x \in E_i \Big\}, \qquad i \geq 0.$$

Moreover, assume that both Q-matrices are regular. If the birth–death process is ergodic, exponentially ergodic, or strongly ergodic, then so, respectively, is the $(q(x, y))$-process.

Proof. Let $F := \{(x, i) \in E \times \mathbb{Z}_+\colon |x| \leq i\}$. By replacing $m \geq n+2$ by $0 \leq m \leq n-1$ in the seventh case in (3.4), with $x \in E_n$ and $i \geq 0$, we construct the coupling Q-matrix $\bar{Q}$ such that

$$\bar{q}(x, i; F^{\mathrm{c}}) := \sum_{\{(y,j)\colon |y| > j\}} \bar{q}(x, i; y, j) = 0, \qquad (x, i) \in F.$$

As in the previous proof, we can prove that this condition is equivalent to

$$P(t; x, i; F) = 1, \qquad t \geq 0,\ (x, i) \in F,$$

where $P(t) \equiv P(t; \cdot, \cdot; \cdot)$ is the transition probability function of the coupling Q-process $Z(t) = (X(t), Y(t))$. Hence, we have

$$\mathrm{P}^{(x,i)}\{|X(t)| \leq Y(t)\} = 1, \qquad t \geq 0,\ (x, i) \in F.$$

From the right continuity of paths it follows that $\mathrm{P}^{(x,i)}\{|X(t)| \leq Y(t)\colon t \geq 0\} = 1$ for $(x, i) \in F$. Thus, $\mathrm{P}^{(x,|x|)}\{\tau \leq \tau^*\} = 1$ and we have $\mathrm{E}_x\, \tau \leq \mathrm{E}_{|x|}\, \tau^*$ and $\mathrm{E}_x\, \mathrm{e}^{\lambda\tau} \leq \mathrm{E}_{|x|}\, \mathrm{e}^{\lambda\tau^*}$ for all $x \in E$. The result then follows from [8, Theorem 4.44].

In applying the coupling method, we have had to use birth–death processes to preserve some specific ‘order’ of state space. In applying the comparison method in Section 2, we constructed a single-birth process which seems to be more versatile than the birth–death processes. However, we have to verify that f_i and/or g_i are increasing functions of i. The two methods thus have their different advantages.

Note that an alternative proof of Theorem 3.2 can be found in [26], since it is a special case of [26, Theorem 1.3]. See the proof there for details.

4. Applications

The first application is to the finite-dimensional Schlögl model (see [8]). Let S be a finite set and let $E = \mathbb{Z}_+^S$, the configuration space, have elements $x = (x(u)\colon u \in S)$. The model is defined by the Q-matrix $Q = (q(x, y)\colon x, y \in E)$ with

$$q(x,y) = \begin{cases} \lambda_1 \binom{x(u)}{2} + \lambda_4 & \text{if } y = x + e_u, \\ \lambda_2 \binom{x(u)}{3} + \lambda_3 x(u) & \text{if } y = x - e_u, \\ x(u)p(u,v) & \text{if } y = x - e_u + e_v, \\ 0 & \text{otherwise, unless } y = x, \end{cases}$$

and $q(x) \equiv -q(x,x) = \sum_{y \neq x} q(x,y)$. Here $(p(u,v)\colon u, v \in S)$ is a transition probability matrix on S, $\lambda_1, \dots, \lambda_4$ are positive constants, and e_u is the element of E having the value 1 at u and the value 0 elsewhere. This model was introduced by Schlögl [17] as a typical model of a nonequilibrium system; see [11] and [8] for related references. It was shown in [8], as an example of Corollary 4.49 there, that the model is exponentially ergodic. Actually, it is strongly ergodic, which was shown for the first time in [26].

Theorem 4.1. *The Q-process corresponding to the finite-dimensional Schlögl model is strongly ergodic.*

Proof. It can be shown that the process is regular (see [21]). In fact, it is easy to check that the birth–death Q-matrix defined in Theorem 3.2 satisfies

$$a_k \geq \frac{\lambda_2}{6}\left(\frac{k^3}{|S|^2} - 3k^2\right) + \left(\lambda_3 + \frac{\lambda_2}{3}\right)k, \quad b_k = \frac{\lambda_1}{2}(k^2 - k) + \lambda_4|S|, \qquad k \geq 1.$$

From the above inequality, we easily see that the birth–death process is regular and that

$$\sum_{i=0}^{\infty} \frac{1}{\mu_i b_i} \sum_{j=i+1}^{\infty} \mu_j < \infty, \tag{4.1}$$

where $\mu_0 = 1$ and $\mu_i = b_0 b_1 \cdots b_{i-1}/a_1 a_2 \cdots a_i$, $i \geq 1$. This means that the birth–death process is strongly ergodic (see [10], [23], and [26]), and the assertion follows from Theorem 3.2.

The second application is to the Brusselator model (see [8]), which is a typical model of a reaction-diffusion process with several species. Let S be a finite or numerable set and let

$E = (\mathbb{Z}_+^2)^S$. The model is described by the Q-matrix $Q = (q(x, y) \colon x, y \in E)$ with

$$q(x, y) = \begin{cases} \lambda_1 a(u) & \text{if } y = x + e_{u1}, \\ \lambda_2 b(u) x_1(u) & \text{if } y = x - e_{u1} + e_{u2}, \\ \lambda_3 \binom{x_1(u)}{2} x_2(u) & \text{if } y = x + e_{u1} - e_{u2}, \\ \lambda_4 x_1(u) & \text{if } y = x - e_{u1}, \\ x_k(u) p_k(u, v) & \text{if } y = x - e_{uk} + e_{vk},\ k = 1, 2, \text{ with } v \neq u, \\ 0 & \text{otherwise, unless } y = x, \end{cases}$$

and $q(x) \equiv -q(x, x) = \sum_{y \neq x} q(x, y)$. Here a and b are positive functions, $\lambda_1, \dots, \lambda_4$ are positive constants, $p_k(u, v)$, $k = 1, 2$, is the transition probability on S, and

$$e_{ui}(v, j) = \begin{cases} 1 & \text{if } v = u \text{ and } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

The model's ergodicity was proved in [12] in the case in which S is a singleton, and the current authors proved that the model is not strongly ergodic and that super-Poincaré inequalities do not hold for it in [19] and [20]. For general finite S, the model's exponential ergodicity was proved in [6]. Here we demonstrate the following result.

Theorem 4.2. *Assume that $a = \sum_{u \in S} a(u) < \infty$. Then the Q-process corresponding to the Brusselator model on E_* is not strongly ergodic.*

Proof. It can be shown that the process is regular (see [21]). Consider the birth–death Q-matrix in Theorem 3.1 with

$$a_i = \lambda_4 i, \quad i \geq 1, \qquad b_i = \lambda_1 a, \quad i \geq 0.$$

It is not difficult to check that this birth–death process is regular but that (4.1) fails. This means that the birth–death process is not strongly ergodic (see [10], [23], and [26]). Thus, by Theorem 3.1, the process is not strongly ergodic.

Note that the birth–death process is exponentially ergodic, since it satisfies

$$\sup_{i>0} \sum_{j=0}^{i-1} (\mu_j b_j)^{-1} \sum_{j=i}^{\infty} \mu_j < \infty$$

(see [10]).

The third application is to epidemic processes (see [1]). Let $E = \mathbb{Z}_+^2$. The process is defined by the Q-matrix $Q = (q((m, n), (m', n')) \colon (m, n), (m', n') \in E)$ with

$$q((m, n), (m', n')) = \begin{cases} \alpha & \text{if } (m', n') = (m + 1, n), \\ \gamma m & \text{if } (m', n') = (m - 1, n), \\ \beta & \text{if } (m', n') = (m, n + 1), \\ \delta n & \text{if } (m', n') = (m, n - 1), \\ \varepsilon m n & \text{if } (m', n') = (m - 1, n + 1), \\ 0 & \text{otherwise, unless } (m, n) = (m', n'), \end{cases}$$

and

$$q((m,n)) \equiv -q((m,n),(m,n)) = \sum_{(m',n')\neq(m,n)} q((m,n),(m',n')),$$

where α, γ, β, δ, and ε are nonnegative constants. Assume that γ and δ are strictly positive. The process is regular and positive recurrent when $\alpha+\beta > 0$ (see [1] and also [16]). Our result on this process is as follows.

Theorem 4.3. *Assume that $\alpha+\beta$, γ, and δ are strictly positive. Then the epidemic process is exponentially ergodic and not strongly ergodic.*

Proof. Let $E_k = \{(m,n) \in E\colon m+n=k\}$, $k \geq 0$. It is obvious that the Q-matrix corresponding to the epidemic processes satisfies the conditions of Theorem 3.1 and Theorem 3.2.

We first prove that the epidemic process is exponentially ergodic. Let

$$a_i = \inf\bigg\{\sum_{(m',n')\in E_{i-1}} q((m,n),(m',n'))\colon (m,n)\in E_i\bigg\},$$
$$b_i = \sup\bigg\{\sum_{(m',n')\in E_{i+1}} q((m,n),(m',n'))\colon (m,n)\in E_i\bigg\}.$$

Then

$$a_i = (\gamma \wedge \delta)i, \quad i \geq 1, \qquad b_i = \alpha+\beta, \quad i \geq 0.$$

This type of birth–death matrix has just been seen in the proof of Theorem 4.2. As mentioned, the birth–death process is exponentially ergodic. Thus, the assertion follows from Theorem 3.2.

We now prove that the epidemic process is not strongly ergodic. Let

$$a_i = \sup\bigg\{\sum_{(m',n')\in E_{i-1}} q((m,n),(m',n'))\colon (m,n)\in E_i\bigg\},$$
$$b_i = \inf\bigg\{\sum_{(m',n')\in E_{i+1}} q((m,n),(m',n'))\colon (m,n)\in E_i\bigg\}.$$

Then

$$a_i = (\gamma \vee \delta)i, \quad i \geq 1, \qquad b_i = \alpha+\beta, \quad i \geq 0.$$

This is again the same type of birth–death matrix as in the proof of Theorem 4.2. From there we know that the birth–death process is not strongly ergodic. Thus, the assertion follows from Theorem 3.1.

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