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EXPONENTIAL ERGODICITY FOR SINGLE-BIRTH PROCESSES

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Abstract

An explicit, computable, and sufficient condition for exponential ergodicity of single-birth processes is presented. The corresponding criterion for birth–death processes is proved using a new method. As an application, some sufficient conditions are obtained for exponential ergodicity of an extended class of continuous-time branching processes and of multidimensional Q-processes, by comparison methods.

Keywords: Exponentially ergodic; single-birth process; birth–death process

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1. Introduction

Consider a continuous-time, irreducible Markov chain with transition probability matrix $P(t) = (p_{ij}(t))$ on a countable state space $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with stationary distribution $(\pi_i > 0 : i \in \mathbb{Z}_+)$. In the study of the theory of Markov chains, there are traditionally three types of ergodicity: ordinary ergodicity (or positive recurrence), exponential ergodicity, and strong ergodicity (or uniform ergodicity). The main purpose of this paper is to deal with the second of these for single-birth processes, which are also called upwardly skip-free processes (see [1], [12]): $\lim_{t\to\infty} \mathrm{e}^{\beta t}|p_{ij}(t) - \pi_j| = 0$ for some $\beta > 0$.

The Q-matrix of a single-birth process $(q_{ij} : i, j \in \mathbb{Z}_+)$ is as follows: $q_{i,i+1} > 0$ and $q_{i,i+j} = 0$ for all $i \in \mathbb{Z}_+$ and $j \geq 2$. Throughout the paper, we consider only totally stable and conservative Q-matrices: $q_i = -q_{ii} = \sum_{j\neq i} q_{ij} < \infty$ for all $i \in \mathbb{Z}_+$. Define

$$q_n^{(k)} = \sum_{j=0}^{k} q_{nj} \quad \text{for } k = 0, \dots, n-1 \ (n \in \mathbb{Z}_+)$$

and

$$m_0 = \frac{1}{q_{01}}, \qquad m_n = \frac{1}{q_{n,n+1}}\biggl(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k\biggr), \quad n \geq 1,$$

$$F_n^{(n)} = 1, \qquad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n,$$

$$d_0 = 0, \qquad d_n = \frac{1}{q_{n,n+1}}\biggl(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\biggr), \quad n \geq 1.$$

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Then $m_n = q_{01}^{-1} F_n^{(0)} + d_n$ for all $n \in \mathbb{Z}_+$. For birth–death processes (a_i, b_i), these quantities take a simple form:

$$m_n = \frac{1}{\mu_n b_n} \mu[0, n], \qquad F_n^{(0)} = \frac{b_0}{\mu_n b_n}, \qquad d_n = \frac{1}{\mu_n b_n} \mu[1, n], \qquad n \geq 1,$$

where $\mu_0 = 1$, $\mu_i = b_0 b_1 \cdots b_{i-1}/a_1 a_2 \cdots a_i$ $(i \geq 1)$, and $\mu[i, k] = \sum_{j=i}^{k} \mu_j$. The main advantage of single-birth processes is that the exit boundary consists of at most a single extremal point and so explicit criteria are expected. We give here the criteria for several classical problems (see [5], [6], [8], [19], [20]).

First, the process is unique (regular) if and only if $R := \sum_{n=0}^{\infty} m_n = \infty$. Next, assume that the Q-matrix is irreducible; then the process is recurrent if and only if $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$. In the regular case, it is ergodic if and only if $d := \sup_{k \in \mathbb{Z}_+} (\sum_{n=0}^{k} d_n)/(\sum_{n=0}^{k} F_n^{(0)}) < \infty$, and it is strongly ergodic if and only if $\sup_{k \in \mathbb{Z}_+} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty$.

The four criteria are all explicit (depending on the Q-matrix (q_{ij}) only, without using test functions) and computable. This advantage makes single-birth processes a useful tool when studying more complicated processes (see [5, Chapters 3 and 4] and [16]). Now, it is natural to look for an explicit criterion of exponential ergodicity for this class of processes. Meanwhile, there are a number of principal investigations into the exponential ergodicity of birth–death processes (see [8], [11], [14], [17], [18]) and such a criterion has been obtained recently by Mu-Fa Chen (see [7, Theorem 3.5]). But the difficulty is that single-birth processes are in general irreversible. In this paper, we give a partial answer. In fact, Theorem 1.1 is a generalization of the criterion for birth–death processes. For research on the ergodicity of nonhomogeneous Markov chains, see [2].

One of the most important problems for possible applications is the bounding of the rate of convergence (see [7], [11], [12], [14], [17], [18]). So the lower bound of the rate of exponential convergence for single-birth processes is studied in Theorem 1.1. Denote the rate of exponential convergence by

$$\widehat{\alpha} = \sup\{\alpha : |p_{ij}(t) - \pi_j| = O(\mathrm{e}^{-\alpha t}) \text{ as } t \to \infty \text{ for all } i, j \in \mathbb{Z}_+\}.$$

Theorem 1.1. *Let the single-birth Q-matrix be regular and irreducible. If*

$$q := \inf_{i \geq 0} q_i > 0 \quad \text{and} \quad M := \sup_{i > 0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty, \tag{1.1}$$

then the process is exponentially ergodic and $\widehat{\alpha} \geq (4M)^{-1} \wedge q_0$. In addition, if $q_0 \geq \inf_{i>0} q_i$ then $\widehat{\alpha} \geq (4M)^{-1}$. The condition (1.1) is necessary for the exponential ergodicity of birth–death processes (a_i, b_i). Equivalently,

$$\delta := \sup_{i > 0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{j=i}^{\infty} \mu_j < \infty. \tag{1.2}$$

Now we discuss exponential ergodicity for a class of multidimensional Q-processes. In [5, Theorem 4.58] and [16], a method that reduces the multidimensional problems to one-dimensional ones is proposed. By keeping the idea in mind, some sufficient conditions for strong ergodicity of multidimensional Q-processes are obtained.

Theorem 1.2. *Let E be a countable set and let $(q(x, y) : x, y \in E)$ be a conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\sum_{k=0}^{\infty} E_k = E$ with $E_0 = \{\theta\}$, where $\theta \in E$ is a reference point. Next, suppose that*

(i) *if $q(x, y) > 0$ and $x \in E_k$, then $y \in \sum_{j=0}^{k+1} E_j$ for all $k \geq 0$;*

(ii) *for all $x \in E_k$ and all $k \geq 0$,*

$$\sum_{y \in E_{k+1}} q(x, y) > 0;$$

(iii) *for all $k \geq 0$,*

$$C_k := \sup\{q(x) : x \in E_k\} < \infty.$$

Define a conservative Q-matrix $(q_{ij} : i, j \in \mathbb{Z}_+)$ as follows:

$$q_{ij} = \begin{cases} \sup\left\{ \sum_{y \in E_j} q(x, y) : x \in E_i \right\} & \text{if } j = i + 1, \\ \inf\left\{ \sum_{y \in E_j} q(x, y) : x \in E_i \right\} & \text{if } j < i, \\ 0, & \text{otherwise if } j \neq i. \end{cases}$$

Moreover, suppose that both $(q(x, y))$ and (q_{ij}) are irreducible and that (q_{ij}) is regular. If $M < \infty$, where M is as defined in Theorem 1.1, then the (q_{ij})-process and the $(q(x, y))$-process are both exponentially ergodic.

The remainder of the paper is organized as follows. In the next section, the proofs of Theorems 1.1 and 1.2 are given and some examples are illustrated along with some remarks. As applications, in Section 3, some sufficient conditions for exponential ergodicity for an extended class of time-continuous branching processes are presented.

2. Proofs of Theorems 1.1 and 1.2

In this section, we present the proofs of Theorems 1.1 and 1.2 in detail.

2.1. Proof of Theorem 1.1

In view of Theorem 4.45(2) of [5], the condition $q > 0$ is indeed necessary. We divide the rest of the proof into three parts.

(a) From [5, Theorem 4.45(2)], the single-birth process is exponentially ergodic if and only if, for some λ with $0 < \lambda < q_i$ for all $i \in \mathbb{Z}_+$, the system of inequalities

$$\sum_j q_{ij} y_j \leq -\lambda y_i - 1, \qquad i \geq 1, \tag{2.1}$$

has a nonnegative finite solution (y_i). We need to construct a solution (g_i) to the equation (2.1) for a fixed λ with $0 < \lambda < q$. First, define an operator

$$II_i(f) = \frac{1}{f_i} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{k=j+1}^{\infty} \frac{f_k}{q_{k,k+1} F_k^{(0)}}, \qquad i \geq 1.$$

This is an analogue of the operator $I(f)$ used many times in [7]. It indicates a key point in this proof, which comes from the study of the first eigenvalue. Next, define

$$\varphi_i = q_{01}^{-1} \sum_{j=0}^{i-1} F_j^{(0)}, \qquad i \geq 1.$$

Then φ_i is increasing in i and $\varphi_1 = q_{01}^{-1}$. Let $f_i = cq_{10}\sqrt{q_{01}\varphi_i}$ for some $c > 1$. Then f_i is increasing and $f_1 = cq_{10}$. Finally, define $g_i = f_i II_i(f)$. Then g_i is increasing and

$$g_1 = \sum_{k=1}^{\infty} \frac{f_k}{q_{k,k+1}F_k^{(0)}} \geq \frac{f_1}{q_{12}F_1^{(0)}} = c > 1.$$

By Lemma 3.6 of [7], it follows that

$$\begin{aligned} g_i &= cq_{10}\sqrt{q_{01}} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{k=j+1}^{\infty} \frac{\sqrt{\varphi_k}}{q_{k,k+1}F_k^{(0)}} \\ &\leq \frac{2Mcq_{10}}{\sqrt{q_{01}}} \sum_{j=0}^{i-1} F_j^{(0)} \varphi_{j+1}^{-1/2} \\ &\leq 2Mcq_{10} \sum_{j=0}^{i-1} F_j^{(0)} < \infty, \qquad i \geq 1. \end{aligned}$$

Let $g_0 = 1$. Then $1 \leq g_i < \infty$ for all $i \geq 0$. We now determine λ using (2.1). When $i = 1$, we get $\lambda \leq (c-1)c^{-1}II_1(f)^{-1}$. When $i \geq 2$, we should have that

$$\lambda g_i \leq \sum_{k=0}^{i-1} q_i^{(k)} F_k^{(0)} \sum_{j=k+1}^{\infty} \frac{f_j}{q_{j,j+1}F_j^{(0)}} - q_{i,i+1}F_i^{(0)} \sum_{k=i+1}^{\infty} \frac{f_k}{q_{k,k+1}F_k^{(0)}}. \tag{2.2}$$

For this, it suffices that

$$\begin{aligned} \lambda g_i &\leq \sum_{k=0}^{i-1} q_i^{(k)} F_k^{(0)} \sum_{j=i}^{\infty} \frac{f_j}{q_{j,j+1}F_j^{(0)}} - q_{i,i+1}F_i^{(0)} \sum_{k=i+1}^{\infty} \frac{f_k}{q_{k,k+1}F_k^{(0)}} \\ &= q_{i,i+1}F_i^{(0)} \sum_{k=i}^{\infty} \frac{f_k}{q_{k,k+1}F_k^{(0)}} - q_{i,i+1}F_i^{(0)} \sum_{k=i+1}^{\infty} \frac{f_k}{q_{k,k+1}F_k^{(0)}} \\ &= f_i. \end{aligned}$$

In other words, for (2.1), we need only $\lambda \leq f_i/g_i = II_i(f)^{-1}$ for all $i \geq 2$ and $\lambda \leq (c-1)c^{-1}II_1(f)^{-1}$. Then we can take any λ such that

$$0 < \lambda < \lambda(c) := \left(\frac{c-1}{c} II_1(f)^{-1}\right) \wedge \left(\inf_{i\geq 2} II_i(f)^{-1}\right) \wedge q, \tag{2.3}$$

provided the right-hand side of (2.3) is positive or, equivalently, $\sup_{i\geq 2} II_i(f) < \infty$. To prove the last property, define another operator

$$I_i(f) = \frac{F_i^{(0)}}{f_{i+1} - f_i} \sum_{k=i+1}^{\infty} \frac{f_k}{q_{k,k+1}F_k^{(0)}}, \qquad i \geq 1,$$

which is exactly the one used many times in [7]. By the proportion property, we get

$$\sup_{i\geq 1} II_i(f) \leq \sup_{i\geq 1} I_i(f).$$

By Lemma 3.6 of [7] and the condition $M < \infty$, it follows that

$$I_i(f) = \frac{F_i^{(0)}}{\sqrt{\varphi_{i+1}} - \sqrt{\varphi_i}} \sum_{k=i+1}^{\infty} \frac{\sqrt{\varphi_k}}{q_{k,k+1} F_k^{(0)}} \leq \frac{2MF_i^{(0)}}{q_{01}(\sqrt{\varphi_{i+1}} - \sqrt{\varphi_i})\sqrt{\varphi_{i+1}}} \leq 4M$$

for all $i \geq 1$. Therefore, $\sup_{i\geq 1} II_i(f) \leq 4M < \infty$ as required. We have thus constructed a solution (g_i) to (2.1) with $1 \leq g_i < \infty$ for all i. This implies the exponential ergodicity of the single-birth process.

By the definition of $F_i^{(0)}$, we have $q_{i,i+1}F_i^{(0)} \leq q_i^{(i-1)} \sum_{j=0}^{i-1} F_j^{(0)}$ for all $i \geq 1$. Hence, $\inf_{i\geq 1} II_i(f)^{-1} \leq M^{-1} \leq \inf_{i>0} q_i$, so we can rewrite (2.3) as

$$0 < \lambda < \lambda(c) := \left(\frac{c-1}{c} II_1(f)^{-1}\right) \wedge \left(\inf_{i\geq 2} II_i(f)^{-1}\right) \wedge q_0. \tag{2.4}$$

So, by (2.4) and the above discussion, we obtain that the exponential convergence rate

$$\widehat{\alpha} \geq \lim_{c\to\infty} \lambda(c) = \inf_{i\geq 1} II_i(f)^{-1} \wedge q_0 \geq (4M)^{-1} \wedge q_0.$$

If $q_0 \geq \inf_{i>0} q_i$ then $q_0 \geq M^{-1}$. Hence, we have $\widehat{\alpha} \geq (4M)^{-1}$.

For the remainder of this proof, we consider birth–death processes only.

(b) Let $\sigma_0 = \inf\{t$ at or following the first jumping time such that $X_t = 0\}$. Suppose that the process is exponentially ergodic. By [5, Theorem 4.44(2)], there exists a λ with $0 < \lambda < q_i$ for all i such that $E_0 \mathrm{e}^{\lambda\sigma_0} < \infty$. Define

$$e_{i0}(\lambda) = \int_0^{\infty} \mathrm{e}^{\lambda t}\, \mathrm{P}_i[\sigma_0 > t]\, \mathrm{d}t, \qquad i \in \mathbb{Z}_+.$$

Then $E_i \mathrm{e}^{\lambda\sigma_0} = \lambda e_{i0}(\lambda) + 1$. By [5, p. 148], $e_{i0}(\lambda) < \infty$ for all $i \geq 1$. Furthermore, $E_i \mathrm{e}^{\lambda\sigma_0} < \infty$ for all $i \geq 1$. Note that if the starting point is not 0, then σ_0 is equal to the first hitting time:

$$\tau_0 = \inf\{t > 0 : X(t) = 0\}.$$

Then $E_i \mathrm{e}^{\lambda\tau_0} < \infty$ for all $i \geq 1$. Define $m_i^{(n)} = E_i \tau_0^n$. The Taylor expansion

$$\infty > E_i \mathrm{e}^{\lambda\tau_0} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} m_i^{(n)} \tag{2.5}$$

leads us to estimate the moments $m_i^{(n)}$. By a result due to Wang (see [15, p. 525]), we have

$$m_i^{(1)} = \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k, \qquad m_i^{(n)} = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k m_k^{(n-1)}, \qquad n \geq 2. \tag{2.6}$$

Obviously, $m_k^{(n)} \geq m_i^{(n)}$ if $k \geq i$. By (2.6), it follows that

$$m_i^{(n)} \geq n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k m_k^{(n-1)} \geq n \left(\sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right) m_i^{(n-1)}, \qquad n \geq 2,$$

and

$$m_i^{(1)} \geq \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k.$$

Hence, by induction,

$$m_i^{(n)} \geq n! \left(\sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right)^n, \qquad n \geq 1.$$

Combining this with (2.5), we have

$$\sum_{n=1}^{\infty} \left(\lambda \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right)^n < \infty,$$

which implies that

$$\lambda \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k < 1.$$

Taking the supremum over i, we obtain $\delta \leq \lambda^{-1} < \infty$. Hence, the necessity is proved.

(c) To complete the proof of the theorem, it suffices to show that

$$\inf_i q_i = 0 \implies \delta = \infty. \tag{2.7}$$

From Proposition 5.13 of [9], we know that, if $\inf_i q_i = 0$, then the first eigenvalue (spectral gap) λ_1 is 0. For birth–death processes, by Theorem 3.5 of [7], we have $(4\delta)^{-1} \leq \lambda_1 \leq (\sum_{j=0}^{\infty} \mu_j)\delta^{-1}$, so the proof of (2.7) is trivial.

Remark 2.1. For birth–death processes, Theorem 9.21 of [5] tells us that the first eigenvalue λ_1 coincides with the exponential convergence rate $\widehat{\alpha}$. So Theorem 3.5 of [7] gives us the estimates of the exponential convergence rate, and at the same time it does indeed give us an explicit criterion for exponential ergodicity of birth–death processes for the first time. In the proof of Theorem 1.1, we have presented another proof of the criterion.

Remark 2.2. In the above proof, not only do we prove the exponential ergodicity, but also we obtain an increasing solution (which is very important) to the equation (2.1) for single-birth processes. In particular, for a birth–death process, once the process is exponentially ergodic, an increasing solution to (2.1) is obtained as follows:

$$g_0 = 1, \qquad g_i = f_i II_i(f) = \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k f_k, \quad i \geq 1,$$

where $f_i = c a_1 \sqrt{b_0 \varphi_i}$ and $\varphi_i = \sum_{j=0}^{i-1} (\mu_j b_j)^{-1}$ for all $i \geq 1$ and some $c > 1$.

Remark 2.3. Since birth–death processes are special cases of single-birth processes, by (1.2) and the criterion for strong ergodicity of single-birth processes, it seems that a more reasonable sufficient condition should be

$$\bar{M} := \sup_{i\geq 0}\sum_{j=0}^{i} F_j^{(0)}\left(d - \frac{d_i}{F_i^{(0)}}\right) < \infty.$$

For birth–death processes, since $d - d_i/F_i^{(0)} = \sum_{j=i+1}^{\infty}(q_{j,j+1}F_j^{(0)})^{-1} = \sum_{j=i+1}^{\infty}\mu_j$ for all $i \geq 0$, it follows that $\bar{M} = M = \delta$.

Example 2.1. Let $q_{n,n+1} = 1$ for all $n \geq 0$, $q_{10} = 1$, $q_{n,n-2} = 1$ for all $n \geq 2$, and $q_{ij} = 0$ for other $i \neq j$. Then the single-birth process is exponentially ergodic and not strongly ergodic. Moreover, we have $\widehat{\alpha} \geq (4C)^{-1}$, where C is given in (2.8), below.

We prove this assertion in detail as follows. Obviously, the process is regular and recurrent. By computation, we know that $\{F_n^{(0)}\}$ are Fibonacci numbers:

$$F_n^{(0)} = \frac{1}{\sqrt{5}}[A^{n+1} - (-B)^{n+1}] =: F_n, \qquad n \geq 0,$$

where $A = (\sqrt{5}+1)/2$ and $B = (\sqrt{5}-1)/2$. Note that Fibonacci numbers have the property that $\sum_{k=0}^{n} F_n = F_{n+2} - F_1$ for $n \geq 0$. So $d_n = F_{n+1} - F_1$ for $n \geq 0$. By the facts that $AB = 1$, $A - B = 1$, and $A + B = \sqrt{5}$, it is not difficult to show that

$$A > \frac{\sum_{k=0}^{n} d_k}{\sum_{k=0}^{n} F_k^{(0)}} = \frac{F_{n+3} - (n+3)F_1}{F_{n+2} - F_1} \to A, \qquad n \to \infty.$$

So $d = A$. In addition, we can prove that $d_n/F_n^{(0)} \uparrow A$, implying that $\hat{d} := \sup_{n\geq 0}(d_n/F_n^{(0)}) = A = d$.

Therefore,

$$\begin{aligned}\sup_{k\geq 0}\sum_{n=0}^{k}(F_n^{(0)}d - d_n) &= \sup_{k\geq 0}\sum_{n=0}^{k}(F_nA - F_{n+1} + F_1)\\ &= \sup_{k\geq 0}\left((k+1) + \frac{(1+B^2)(1-(-B)^{k+1})}{\sqrt{5}(1+B)}\right) = \infty,\end{aligned}$$

which implies that the process is not strongly ergodic. As pointed out by a referee, the absence of strong ergodicity for the process seems to be well known. This follows from the skip-free property and from boundedness of the Q-matrix.

Note that $F_n \geq (A^{n+1}-1)/\sqrt{5} \geq A^n/\sqrt{5}$ for $n \geq 1$. Thus,

$$M \leq \sup_{i>0}\frac{A(A^2 - (-B/A)^i B^2 - \sqrt{5}/A^i)}{A-1} \leq A^2(A^2 + B^4 + \sqrt{5}B) =: C < \infty. \tag{2.8}$$

So, by Theorem 1.1, it follows that the process is exponentially ergodic and $\widehat{\alpha} \geq (4M)^{-1} \geq (4C)^{-1}$. In addition,

$$\bar{M} = \sup_{i\geq 0}\frac{(A^2 - (-B/A)^{i+1}B^2 - \sqrt{5}/A^{i+1})(1-(-B)^{i+1})}{1-(-B/A)^{i+1}} \leq C < \infty.$$

Example 2.2. (i) Take $a_n = b_n$ for $n \geq 1$. Then the process is exponentially ergodic if and only if $\sup_{i>0} \sum_{j=i}^{\infty} i/a_j < \infty$.

(ii) Take $a_n = \nu b_n$ for $n \geq 1$, where $\nu > 1$. Then the process is exponentially ergodic if and only if $\sup_{i>0} \sum_{j=i}^{\infty} 1/(\nu^{j-i} a_j) = \sup_{i>0} \sum_{j=0}^{\infty} 1/(\nu^j a_{i+j}) < \infty$. If, in addition, $\inf_{i>0} a_i > 0$, then the process must be exponentially ergodic.

2.2. Proof of Theorem 1.2

By Theorems 3.19 and 4.58 of [5], the $(q(x, y))$-process is regular. At present, (q_{ij}) is a regular, irreducible single-birth Q-matrix. By Theorem 1.1, the (q_{ij})-process is exponentially ergodic. To prove exponential ergodicity of the $(q(x, y))$-process, we need only to show that the equations

$$\sum_{y \neq x} q(x, y)(g(y) - g(x)) \leq -\lambda g(x), \qquad x \neq \theta,$$
$$\sum_{y \neq \theta} q(\theta, y) g(y) < \infty$$

have a finite solution $(g(y))$ with $g \geq 1$ for some $\lambda > 0$. For this, by the assumptions and Remark 2.2, let (g_k) be an increasing solution to (2.1) with $g_k \geq 1$ for all $k \geq 0$, and take $g(x) = g_k$ for $x \in E_k$, $k \geq 0$. Now, for $x \neq \theta$, there exists exactly one $k \geq 1$ such that $x \in E_k$. Hence, on the one hand, by the definition of (q_{ij}),

$$\begin{aligned}\sum_{y \neq x} q(x, y)(g(y) - g(x)) &= -\sum_{j=0}^{k-1} \sum_{y \in E_j} q(x, y)(g_k - g_j) + \sum_{y \in E_{k+1}} q(x, y)(g_{k+1} - g_k) \\ &\leq -\sum_{j=0}^{k-1} q_{kj}(g_k - g_j) + q_{k,k+1}(g_{k+1} - g_k) \\ &= \sum_{j \neq k} q_{kj}(g_j - g_k) \\ &\leq -\lambda g_k = -\lambda g(x)\end{aligned}$$

and, on the other hand,

$$\sum_{y \neq \theta} q(\theta, y) g(y) = \sum_{y \in E_1} q(\theta, y) g_1 = q_{01} g_1 < \infty.$$

We have thus constructed a solution, as desired.

3. Applications

In this section, we discuss exponential ergodicity for an extended class of time-continuous branching processes. The original branching process can be described as follows. Let $\alpha > 0$ and let $(p_j : j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate $\alpha i p_0 : i \mapsto i - 1$ (for $i \geq 1$) and growth rate $\alpha i p_{k+1} : i \mapsto i + k$ (for $k \geq 1, i \in \mathbb{Z}_+$). Note that the process absorbs at state 0. In [10], an extended class of branching processes with the following Q-matrix is introduced (see also [3], [4]):

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0, \\ -q_0, & j = i = 0, \\ r_i p_0, & j = i-1,\ i \geq 1, \\ r_i p_{k+1}, & j = i+k,\ i,k \geq 1, \\ -r_i(1-p_1), & j = i \geq 1, \\ 0 & \text{otherwise for } i, j \in \mathbb{Z}_+, \end{cases} \tag{3.1}$$

where $r_i > 0$ for all $i \geq 1$ and $0 < q_0 := \sum_{j\geq 1} q_{0j} < \infty$. The typical case we are interested in is where $q_{0j} = p_j$ or p_{j+1} $(j \geq 1)$ and r_i is a polynomial with degree θ (≥ 1). Define $M_1 = \sum_{k=1}^{\infty} kp_k$ and $\Gamma = \sum_{k=1}^{\infty} kp_{k+1}$. It is easy to check that $\Gamma = M_1 + p_0 - 1$. Hence, $M_1 < 1$ if and only if $\Gamma < p_0$, and $M_1 = 1$ if and only if $\Gamma = p_0$. Let $\nu = p_0/\Gamma$.

Based on the same comparison idea as in [10], some sufficient conditions for exponential ergodicity are obtained as follows.

Theorem 3.1. *Let (q_{ij}) be the irreducible Q-matrix given by (3.1). Assume that $M_1 \leq 1$. If $\sup_{i>0} \sum_{j=i}^{\infty} i/r_j < \infty$ and*

$$\sum_{i=1}^{\infty} q_{0i} \sum_{k=1}^{\infty} \frac{\sqrt{k}}{r_k}(i \wedge k) < \infty, \tag{3.2}$$

then the (q_{ij})-process is exponentially ergodic.

In addition, under the same assumption, if

$$\sup_{i>0} \sum_{j=i}^{\infty} \frac{1}{\nu^{j-i} r_j} < \infty,$$

$$\sup_{i>1} \frac{\nu^i r_i}{\sqrt{i}} \sum_{k=i+1}^{\infty} \frac{\sqrt{k}}{\nu^k r_k} \leq \frac{1}{\nu - 1}, \tag{3.3}$$

and

$$\sum_{i=1}^{\infty} q_{0i} \sum_{k=1}^{\infty} \frac{\sqrt{k}\,\nu^{i\wedge k}}{\nu^k r_k} < \infty, \tag{3.4}$$

then the (q_{ij})-process is exponentially ergodic.

Proof. First, under the assumption, by Theorems 1.2 and 1.3 of [10], we know that the process is unique and recurrent. Next, consider the birth–death process $(\bar{p}_{ij}(t))$ with $a_i = b_i = r_i p_0$ for $i \geq 1$ and b_0 some positive constant. By Example 2.2(i) and the assumption, the process $(\bar{p}_{ij}(t))$ is exponentially ergodic; furthermore, by Remark 2.2, we have the following increasing solution (g_i) to (2.1) for some λ with $0 < \lambda < q$:

$$g_0 = 1, \qquad g_i = cr_1 \sum_{j=0}^{i-1} \sum_{k=j+1}^{\infty} \frac{\sqrt{k}}{r_k}, \quad i \geq 1.$$

Since

$$g_{i+1} - g_i = cr_1 \sum_{k=i+1}^{\infty} \frac{\sqrt{k}}{r_k}, \qquad i \geq 1,$$

as $i \to \infty$, $g_{i+1} - g_i$ is decreasing. Thus, on the one hand, for $i \geq 1$,

$$\begin{aligned}\sum_{j \neq i} q_{ij}(g_j - g_i) &= q_{i,i-1}(g_{i-1} - g_i) + \sum_{k=1}^{\infty} q_{i,i+k}(g_{i+k} - g_i) \\ &\leq r_i p_0(g_{i-1} - g_i) + \sum_{k=1}^{\infty} r_i p_{k+1} k(g_{i+1} - g_i) \\ &= r_i p_0(g_{i-1} - g_i) + r_i \Gamma(g_{i+1} - g_i) \\ &\leq a_i(g_{i-1} - g_i) + b_i(g_{i+1} - g_i) \\ &\leq -\lambda g_i,\end{aligned}$$

and, on the other hand, by (3.2), we have $\sum_{i=1}^{\infty} q_{0i} g_i < \infty$. By these facts, it follows that $(p_{ij}(t))$ is exponentially ergodic.

Assume now that $M_1 < 1$. Consider the birth–death process $(\hat{p}_{ij}(t))$ with $a_i = r_i p_0$, $b_i = r_i \Gamma$ for $i \geq 1$, and b_0 some positive constant. Since $\Gamma < p_0$, by Example 2.2(ii) and the assumption, the process $(\hat{p}_{ij}(t))$ is exponentially ergodic. As above, we have the following increasing solution (g_i) to (2.1):

$$g_0 = 1, \qquad g_i = c r_1 \sum_{j=0}^{i-1} \nu^j \sum_{k=j+1}^{\infty} \frac{\sqrt{k}}{\nu^{k-1} r_k}, \quad i \geq 1.$$

Note that

$$g_{i+1} - g_i = c r_1 \nu^{i+1} \sum_{k=i+1}^{\infty} \frac{\sqrt{k}}{\nu^k r_k}, \qquad i \geq 1.$$

By (3.3), we know that $g_{i+1} - g_i$ is decreasing as $i \to \infty$. Thus, it follows similarly that

$$\sum_{j \neq i} q_{ij}(g_j - g_i) \leq a_i(g_{i-1} - g_i) + b_i(g_{i+1} - g_i) \leq -\lambda g_i, \qquad i \geq 1.$$

By (3.3) and (3.4), we have $\sum_{i=1}^{\infty} q_{0i} g_i < \infty$. Putting these facts together, it follows that $(p_{ij}(t))$ is exponentially ergodic.

From Theorem 3.1, we obtain the following corollary.

Corollary 3.1. *Let (q_{ij}) be the irreducible Q-matrix given by (3.1). Assume that $M_1 \leq 1$. If $\delta := \sup_{i>0} \sum_{j=i}^{\infty} i/r_j < \infty$ and $\sum_{i=1}^{\infty} i q_{0i} < \infty$, then the (q_{ij})-process is exponentially ergodic.*

In addition, under the same assumption, if $\sup_{i>0} \sum_{j=i}^{\infty} 1/\nu^{j-i} r_j < \infty$, $\sqrt{i}/r_i$ is decreasing in $i \geq 2$, and $\sum_{i=1}^{\infty} i q_{0i} < \infty$, then the (q_{ij})-process is exponentially ergodic.

Proof. As in the first part of the proof of Theorem 3.1, by Lemma 3.6 of [7], we obtain that

$$g_i \leq 2(\delta p_0^{-1}) c a_1 \sqrt{b_0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j \sqrt{\varphi_{j+1}}} = 2\delta c r_1 \sum_{j=0}^{i-1} \frac{1}{\sqrt{j+1}} \leq 2\delta c r_1 i.$$

By the assumptions (rather than by (3.2)), we obtain $\sum_{i=1}^{\infty} q_{0i} g_i < \infty$. The first assertion follows easily. By the assumptions, both (3.3) and (3.4) hold. Thus, the second assertion follows from Theorem 3.1.

Example 3.1. Let (q_{ij}) be the irreducible Q-matrix given by (3.1), where $r_i = i^\theta$ and $M_1 \le 1$.

(i) By [3] and [20, Theorem 1.2], we know that if $M_1 = 1$ and $\theta > 2$, then the process is strongly ergodic, and when $M_1 < 1$, the process is strongly ergodic if and only if $\theta > 1$. Note that, by [3] and [13, Corollary 2.2], if $M_1 = 1$ and $0 < \theta \le 1$, then the process cannot be strongly ergodic.

(ii) Assume that $\sum_{i=1}^{\infty} iq_{0i} < \infty$. By Corollary 3.1, the process is exponentially ergodic provided that either $M_1 = 1$ and $\theta \ge 2$, or $M_1 < 1$ but $\theta \ge \frac{1}{2}$.

Remark 3.1. By [5, Corollary 4.49], Theorem 1.4(iii) of [10] tells us that, if $M_1 < 1$, $\limsup_{i\to\infty} i/r_i < \infty$ and $\sum_{i=1}^{\infty} iq_{0i} < \infty$, then the process is exponentially ergodic. Compared with Corollary 3.1, this result does not need the 'decreasing' property. However, applying it in Example 3.1, we obtain only that the process is exponentially ergodic, provided that $M_1 < 1$ and $\theta \ge 1$.

Remark 3.2. In fact, we can prove that if $M_1 < 1$ and the r_i are increasing, then the process is strongly ergodic if and only if $\sum_{i=1}^{\infty} 1/r_i < \infty$.

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