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F-Sobolev Inequality for General Symmetric Forms*

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Abstract: Some sufficient conditions for the F-Sobolev inequality for symmetric forms are presented in terms of new Cheeger's constants. Meanwhile, an estimate of the F-Sobolev constants is obtained.

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1 Introduction

The F-Sobolev inequality is a generalization of the logarithmic Sobolev inequality. The latter is initiated by [1] in 1976 and it has attracted a great deal of research in the past two decades. Refer to the survey articles [2] and [3]. Very recently, the inequality has been studied in [4] and [5] for general symmetric forms. The method used in the two papers is the Cheeger's technique for unbounded operators developed in [6]. But the proofs are different and a detailed comparison about them is presented in [4]. The purpose of this paper is to study the F-Sobolev inequality for general symmetric forms by combining the proofs of [4] with some techniques in [5].

Let $(E,\ \mathcal{E},\ \pi)$ be a measurable probability space satisfying $\{(x,x): x\in E\}\in\mathcal{E}\times\mathcal{E}$ and denote by $L^2(\pi)$ the usual real L^2-space with norm $\|\cdot\|$. The symmetric form $(D,\mathcal{D}(D))$ considered here on $L^2(\pi)$ is as follows:

$$\begin{gathered} D(f,g)=\frac{1}{2}\int J(dx,dy)[f(x)-f(y)][g(x)-g(y)], \\ f,\,g\in\mathcal{D}(D):=\{f\in L^2(\pi): D(f,f)<\infty\}, \end{gathered} \tag{1.1}$$

where J is a symmetric measure on $E\times E$: $J(dx,dy)=J(dy,dx)$. Without loss of generality, assume that $J(\{(x,x): x\in E\})=0$.

Let $F\in C(0,\infty)$ be such that $\inf_{t\geq 0} tF(t)>-\infty$ and $F(t)>0$ for large t. We say that the F-Sobolev inequality with dimension $p\in(2,\infty]$ (denoted by $FS(p)$) holds for the

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symmetric form $(D,\mathcal{D}(D))$ if there exist two constants $C_1>0$, $C_2\geq 0$ such that

$$\pi\big(|f|^{2p/(p-2)}F(f^2)\big)^{(p-2)/p}\leq C_1D(f,f)+C_2,\qquad f\in\mathcal{D}(D),\ \|f\|=1 \tag{1.2}$$

holds, where $\pi(f)=\displaystyle\int fd\pi$. In particular, we call the inequality tight (denoted by $TFS(p)$) if (1.2) holds for $C_2=0$ and the reciprocal of the smallest possible C_1 is denoted by σ which is called the F-Sobolev constant.

The inequality is a general version of Sobolev inequality and logarithmic Sobolev inequality. When $F=1$, (1.2) is just the classical Sobolev inequality with dimension p

$$\pi\big(|f|^{2p/(p-2)}\big)^{(p-2)/p}\leq C_1D(f,f)+C_2,\qquad f\in\mathcal{D}(D),\ \|f\|=1. \tag{1.3}$$

When $F=\log$ and $p=\infty$, (1.2) becomes the defective logarithmic Sobolev inequality

$$\pi(f^2\log f^2)\leq C_1D(f,f)+C_2,\qquad f\in\mathcal{D}(D),\ \|f\|=1,$$

which is equivalent to the usual logarithmic Sobolev inequality

$$\pi(f^2\log f^2)\leq CD(f,f),\qquad f\in\mathcal{D}(D),\ \|f\|=1, \tag{1.4}$$

whenever Poincaré inequality

$$\pi(f^2)-\pi(f)^2\leq\lambda_1^{-1}D(f,f),\qquad f\in\mathcal{D}(D)$$

holds where λ_1 is the spectral gap.

We now look for a new Cheeger's constant which is hoped to best describe $FS(p)$. Following [4] and [5], take and fix a non-negative and symmetric function $r\in\mathcal{E}\times\mathcal{E}$ such that

$$J^{(1)}(dx,E)/\pi(dx)\leq 1,\qquad \pi\text{-a.s.}, \tag{1.5}$$

where $J^{(\alpha)}(dx,dy)=I_{\{r(x,y)^\alpha>0\}}J(dx,dy)/r(x,y)^\alpha$, $\alpha\geq 0$. We adopt the convention that $r^0=1$ for all $r\geq 0$. Replacing J with $J^{(\alpha)}$ in (1.1), we have the symmetric forms $(D^{(\alpha)},\mathcal{D}(D^{(\alpha)}))$ generated by $J^{(\alpha)}$. Suppose that F satisfies the following conditions:

$$F\in C[0,\infty),\ F\text{ is increasing},\ F(0)=1,\ F'\text{ is piecewise continuous and}$$
$$c_1:=\sup_{t\geq 0}\frac{tF'_{\pm}(t)}{F(t)}<\infty, \tag{1.6}$$

where $F'_{\pm}$ denote the right- and left-derivatives of F. Define

$$\xi^\delta=\inf_{\pi(A)>0}\frac{J^{(1/2)}(A\times A^c)+\delta\pi(A)}{\pi(A)^{(p-1)/p}F(\pi(A)^{-1})^{(p-2)/(2p)}},\qquad \delta>0,$$

$$\kappa=\inf_{\pi(A)>0}\frac{J(A\times A^c)}{\pi(A)^{(p-2)/p}F(\pi(A)^{-1})^{(p-2)/p}},$$

$$C(p,c_1)=\Big(\frac{p-1}{p}\Big)^{2(p-1)/p}\Big(\frac{p-1}{p-2}+\frac{p-1}{p}c_1\Big)^{2/p}\Big(\frac{p}{p-2}+c_1\Big)^{2(p-1)/p}.$$

Put $\xi^0:=\lim_{\delta\to 0}\xi^\delta=\inf_{\delta>0}\xi^\delta$. Some sufficient conditions are presented as follows.

Theorem 1.1 *Under (1.6), the following conclusions hold:*

(1) If there exists some $\delta>0$ such that $\xi^\delta>0$, then $FS(p)$ holds for $C_1=4C(p,c_1)(2+\delta)(\xi^\delta)^{-2}$ and $C_2=C_1\delta$;

(2) If $\xi^0>0$, then $TFS(p)$ holds and $\kappa\geq\sigma\geq(\xi^0)^2/(8C(p,c_1))$.

Remark 1.1 In particular, if $F(r) = 1 + \log^+ r = 1 + \max\{0, \log r\}, p = \infty$ and $\xi^\delta > 0$ for some $\delta > 0$ in Theorem 1.1, then the logarithmic Sobolev inequality (1.4) holds and $\kappa \geq \sigma \geq \lambda_1/\left[1 + 16\inf_{\delta>0}(2+\delta)(\lambda_1+\delta)(\xi^\delta)^{-2}\right]$. See [4, Theorem 1.1].

To prove Theorem 1.1, consider the following general symmetric form as in [4]:

$$\begin{gathered}\overline{D}(f,g) = \frac{1}{2}\int J(dx,dy)[f(x)-f(y)][g(x)-g(y)] + \int K(dx)f(x)g(x),\\ f,\, g \in \mathcal{D}(\overline{D}) := \{f \in L^2(\pi) : \overline{D}(f,f) < \infty\},\end{gathered} \tag{1.7}$$

where J is the same as before and K is a measure. Choose a non-negative, symmetric function $\bar{r} \in \mathcal{E} \times \mathcal{E}$ and a non-negative function $s \in \mathcal{E}$ such that

$$[\bar{J}^{(1)}(dx,E) + K^{(1)}(dx)]/\pi(dx) \leq 1, \qquad \pi\text{-}a.s, \tag{1.8}$$

where $\bar{J}^{(\alpha)}(dx,\ dy)$ is defined as $J^{(\alpha)}(dx,\ dy)$ but replacing r with $\bar{r}$, and $K^{(\alpha)}(dx) = I_{\{s(x)^\alpha>0\}}K(dx)/s(x)^\alpha$, $\alpha \geq 0$. As in (1.7), we have the symmetric forms $(\overline{D}^{(\alpha)}, \mathcal{D}(\overline{D}^{(\alpha)}))$ generated by $(\bar{J}^{(\alpha)}, K^{(\alpha)})$. Define $\lambda_0^{*(\alpha)} = \inf\{\overline{D}^{(\alpha)}(f,f) : \|f\| = 1\}$. Next, set

$$\xi^* = \inf_{\pi(A)>0} \frac{\bar{J}^{(1/2)}(A \times A^c) + K^{(1/2)}(A)}{\pi(A)^{(p-1)/p}F(\pi(A)^{-1})^{(p-2)/(2p)}}.$$

Now we can state another main result of the paper.

Theorem 1.2 *Set* $\sigma(F) = \inf\{\overline{D}(f,f)/\pi\left(|f|^{2p/(p-2)}F(f^2)\right)^{(p-2)/p} : \|f\| = 1\}$. *Then*

$$\inf_{\pi(A)>0} \frac{J(A\times A^c) + K(A)}{\pi(A)^{(p-2)/p}F(\pi(A)^{-1})^{(p-2)/p}} \geq \sigma(F) \geq \frac{\xi^{*2}}{4C(p,c_1)(2-\lambda_0^{*(1)})}. \tag{1.9}$$

Remark 1.2 Theorem 1.2 is an extension of [4, Theorem 2.1] which is in the case of $p = \infty$. As pointed out in [4], Theorem 1.2 is true in the general case, i.e., π is a reference measure (See [5]).

2 Proofs

In this section, the proofs of the two theorems are presented in details.

Proof of Theorem 1.2. Put $E_F(f) = \pi\left(|f|^{2p/(p-2)}F(f^2)\right)$. The first inequality is obtained directly by computing $\overline{D}(f,f)/E_F(f)^{(p-2)/p}$ for $f = I_A/\sqrt{\pi(A)}$ with $\pi(A) > 0$.

a) To prove the second inequality, we need more notations (this is the part a) of the proof of [4, Theorem 2.1]).

When $K(dx) \neq 0$, it is convenient to enlarge the space E by letting $E^* = E \cup \{\infty\}$. For any $f \in \mathcal{E}$, define $f^* = fI_E \in \mathcal{E}^*$. Next, define a symmetric measure $J^{*(\alpha)}$ on $E^* \times E^*$ by

$$J^{*(\alpha)}(C) = \begin{cases} J^{(\alpha)}(C), & C \in \mathcal{E}\times\mathcal{E},\\ K^{(\alpha)}(A), & C = A\times\{\infty\} \text{ or } \{\infty\}\times A, A \in \mathcal{E},\\ 0, & C = \{\infty\}\times\{\infty\}.\end{cases}$$

Then, we have $J^{*(\alpha)}(dx,dy) = J^{*(\alpha)}(dy,dx)$ and

$$\int_E \bar{J}^{(\alpha)}(dx,E)f(x)^2 + K^{(\alpha)}(f^2) = \int_{E^*} J^{*(\alpha)}(dx,E^*)f^*(x)^2,$$

$$\overline{D}^{(\alpha)}(f,f) = \frac{1}{2}\int_{E^*\times E^*} J^{*(\alpha)}(dx,dy)[f^*(y)-f^*(x)]^2,$$

$$\frac{1}{2}\int_{E\times E} \bar{J}^{(\alpha)}(dx,dy)|f(y)-f(x)| + \int_E K^{(\alpha)}(dx)|f(x)|$$
$$= \frac{1}{2}\int_{E^*\times E^*} J^{*(\alpha)}(dx,dy)|f^*(y)-f^*(x)|.$$

Note that if we set $r^*(x,y) = \bar{r}(x,y)$, $r^*(x,\infty) = r^*(\infty,x) = s(x)$ for all $x,y \in E$ and $r^*(\infty,\infty) = 0$, then $J^{*(\alpha)}(dx,dy)$ can be also expressed by $I_{\{r^*(x,y)^\alpha>0\}}J^*(dx,dy)/r^*(x,y)^\alpha$.

b) Following [4] and [5], take $\varphi(t) = t^{(p-1)/(p-2)}\sqrt{F(t)}$ and $\eta(t) = \varphi(t^2)$. Note that φ is a strictly increasing function and so is η. Moreover,

$$\varphi'(t) = t^{1/(p-2)}\sqrt{F(t)}\left[\frac{p-1}{p-2} + \frac{tF'(t)}{2F(t)}\right], \quad \eta'(t) = \frac{2\eta(t)}{t}\left[\frac{p-1}{p-2} + \frac{t^2F'(t^2)}{2F(t^2)}\right],$$

except a finite number of points on each finite interval. By (1.6), we have $\eta'(t) \le c_2\eta(t)/t = c_2t^{p/(p-2)}\sqrt{F(t^2)}$ where $c_2 := 2(p-1)/(p-2) + c_1$. Given $s < t$, label the discontinuous points in $[s,t]$ by $s = t_1 < \cdots < t_m = t$. Then, by the Mean Value Theorem and the monotonicity of $\eta(t)/t$, there exist $\theta_i \in (t_i, t_{i+1})$ such that

$$0 \le \eta(t) - \eta(s) = \sum_{i=1}^{m-1}[\eta(t_{i+1}) - \eta(t_i)] = \sum_{i=1}^{m-1}\eta'(\theta_i)(t_{i+1}-t_i)$$
$$\le c_2\sum_{i=1}^{m-1}\frac{\eta(\theta_i)}{\theta_i}(t_{i+1}-t_i) \le c_2\frac{\eta(t)}{t}\sum_{i=1}^{m-1}(t_{i+1}-t_i) = c_2\frac{\eta(t)}{t}(t-s).$$

c) Let $f \ge 0$, $\|f\| = 1$ and set $g^* = \varphi(f^{*2})$. Then by b), we have $|g^*(y) - g^*(x)| \le c_2|f^*(y)-f^*(x)|\dfrac{\eta(f^*(x)\vee f^*(y))}{f^*(x)\vee f^*(y)}$. Thus, by Cauchy-Schwarz inequality and (1.8), we have

$$\begin{aligned}
I^* &:= \frac{1}{2}\int J^{*(1/2)}(dx,dy)|g^*(y)-g^*(x)| \\
&\le \frac{c_2}{2}\Big[\int J^*(dx,dy)|f^*(y)-f^*(x)|^2\Big]^{1/2}\Big[\int J^{*(1)}(dx,dy)\Big[f^*(y)^{p/(p-2)}\sqrt{F(f^*(y)^2)} \\
&\quad + f^*(x)^{p/(p-2)}\sqrt{F(f^*(x)^2)}\Big]^2\Big]^{1/2} \\
&= \frac{c_2}{\sqrt{2}}\sqrt{\overline{D}(f,f)}\Big[\int J^{*(1)}(dx,dy)\Big[2f^*(y)^{2p/(p-2)}F(f^*(y)^2) + 2f^*(x)^{2p/(p-2)}F(f^*(x)^2) \\
&\quad - \Big(f^*(y)^{p/(p-2)}\sqrt{F(f^*(y)^2)} - f^*(x)^{p/(p-2)}\sqrt{F(f^*(x)^2)}\Big)^2\Big]\Big]^{1/2} \\
&= \frac{c_2}{\sqrt{2}}\sqrt{\overline{D}(f,f)}\Big[4E_F(f) - 2\overline{D}^{(1)}(f^{p/(p-2)}\sqrt{F(f^2)}, f^{p/(p-2)}\sqrt{F(f^2)})\Big]^{1/2} \\
&\le c_2\sqrt{(2-\lambda_0^{*(1)})\overline{D}(f,f)E_F(f)}. \qquad (2.1)
\end{aligned}$$

d) Define $h(t) = \pi(f^{*2} > t)$, $A_t = \{g^* > t\} = \{\varphi(f^{*2}) > t\}$. Then $h(t) \le 1\wedge t^{-1}$, $\pi(A_t) = \pi(f^{*2} > \varphi^{-1}(t)) = h\circ\varphi^{-1}(t)$, where φ^{-1} denotes the inverse function of φ. By the definition of ξ^*, we have

$$
\begin{aligned}
I^{*p/(p-1)} &= \Big[\int_0^\infty J^{*(1/2)}(A_t\times A_t^c)dt\Big]^{p/(p-1)}\\
&\ge \xi^{*p/(p-1)}\Big[\int_0^\infty \pi(A_t)^{(p-1)/p}F(\pi(A_t)^{-1})^{(p-2)/(2p)}dt\Big]^{p/(p-1)}\\
&= \xi^{*p/(p-1)}\Big[\int_0^\infty h(s)^{(p-1)/p}F(h(s)^{-1})^{(p-2)/(2p)}\varphi'(s)ds\Big]^{p/(p-1)}\\
&\ge \xi^{*p/(p-1)}\Big[\int_0^\infty h(t)^{(p-1)/p}F(t)^{(p-2)/(2p)}\varphi'(t)dt\Big]^{p/(p-1)}\\
&= \frac{p}{p-1}\xi^{*p/(p-1)}\int_0^\infty h(t)^{(p-1)/p}F(t)^{(p-2)/(2p)}\varphi'(t)\\
&\qquad\cdot\Big[\int_0^t h(s)^{(p-1)/p}F(s)^{(p-2)/(2p)}\varphi'(s)ds\Big]^{1/(p-1)}dt\\
&\ge \frac{p}{p-1}\xi^{*p/(p-1)}\int_0^\infty h(t)^{(p-1)/p}F(t)^{(p-2)/(2p)}\varphi'(t)\\
&\qquad\cdot\Big[h(t)^{(p-1)/p}\int_0^t F(s)^{(p-2)/(2p)}\varphi'(s)ds\Big]^{1/(p-1)}dt\\
&= \frac{p}{p-1}\xi^{*p/(p-1)}\int_0^\infty h(t)F(t)^{(p-2)/(2p)}\varphi'(t)\Big[\int_0^t F(s)^{(p-2)/(2p)}\varphi'(s)ds\Big]^{1/(p-1)}dt.
\end{aligned}
$$

Next, by (1.6) and the absolute continuity of F, we can obtain

$$\int_0^t F(s)^{(p-2)/(2p)}\varphi'(s)ds\ge c_3t^{(p-1)(p-2)}F(t)^{(p-1)/p},\qquad t\ge 0,$$

where $c_3=\Big(\frac{p-1}{p-2}+\frac{1}{2}c_1\Big)\Big/\Big(\frac{p-1}{p-2}+\frac{p-1}{p}c_1\Big)>\frac{p}{2(p-1)}$. Hence,

$$
\begin{aligned}
I^{*p/(p-1)} &\ge \frac{p}{p-1}c_3^{1/(p-1)}\xi^{*p/(p-1)}\int_0^\infty h(t)t^{1/(p-2)}\sqrt{F(t)}\varphi'(t)dt\\
&= \frac{p}{p-1}c_3^{1/(p-1)}\xi^{*p/(p-1)}\int d\pi\int_0^{f^{*2}} t^{1/(p-2)}\sqrt{F(t)}\varphi'(t)dt.
\end{aligned}
$$

By (1.6) and the absolute continuity of F once again, we can derive

$$\int_0^r t^{1/(p-2)}\sqrt{F(t)}\varphi'(t)dt\ge c_4r^{p/(p-2)}F(r),\qquad r\ge 0,$$

where $c_4=\Big(\frac{p-1}{p-2}+\frac{1}{2}c_1\Big)\Big/\Big(\frac{p}{p-2}+c_1\Big)>\frac{1}{2}$. Thus,

$$
\begin{aligned}
I^{*p/(p-1)} &\ge \frac{p}{p-1}c_3^{1/(p-1)}c_4\xi^{*p/(p-1)}\int d\pi f^{*2p/(p-2)}F(f^{*2})\\
&= \frac{p}{p-1}c_3^{1/(p-1)}c_4\xi^{*p/(p-1)}E_F(f).
\end{aligned}
\tag{2.2}
$$

e) Combining (2.1) with (2.2), we get

$$\Big[\frac{p}{p-1}c_3^{1/(p-1)}c_4\xi^{*p/(p-1)}E_F(f)\Big]^{(p-1)/p}\le c_2\sqrt{(2-\lambda_0^{*(1)})\overline{D}(f,f)E_F(f)}.$$

That is,

$$\frac{\overline{D}(f,f)}{E_F(f)^{(p-2)/p}}\ge\Big(\frac{p}{p-1}\Big)^{2(p-1)/p}\cdot\frac{c_3^{2/p}c_4^{2(p-1)/p}\xi^{*2}}{c_2^2(2-\lambda_0^{*(1)})}=\frac{\xi^{*2}}{4C(p,c_1)(2-\lambda_0^{*(1)})}.$$

So the second inequality of Theorem 1.2 holds. The proof is completed.

Proof of Theorem 1.1. Let (1.5) holds for some symmetric function r. Fix $\delta > 0$ and take $K(dx) = \delta\pi(dx)$. Next, take $\bar{r} = (1+\delta)r$ and $s = 1+\delta$ so that (1.8) holds. Then

$$\xi^* = \frac{\xi^\delta}{\sqrt{1+\delta}}, \qquad \lambda_0^{*(\alpha)} = \frac{\delta}{(1+\delta)^\alpha}.$$

Therefore, for the symmetric form $(D, \mathcal{D}(D))$ given in (1.1), by Theorem 1.2, we have

$$\sigma(F) = \inf_{\|f\|=1} \frac{D(f,f)+\delta}{E_F(f)^{(p-2)/p}} \geq \frac{(\xi^\delta)^2/(1+\delta)}{4C(p,c_1)(2-\delta/(1+\delta))} = \frac{(\xi^\delta)^2}{c_\delta},$$

where $c_\delta = 4C(p,c_1)(2+\delta)$. Then we have

$$\pi(|f|^{2p/(p-2)}F(f^2))^{(p-2)/p} \leq c_\delta(\xi^\delta)^{-2}[D(f,f)+\delta], \qquad f \in \mathcal{D}(D),\ \|f\| = 1. \tag{2.3}$$

By (2.3), it is easily seen that Theorem 1.1(i) holds and $C_1 = c_\delta(\xi^\delta)^{-2}$, $C_2 = C_1\delta$. Let $\delta \to 0$ in (2.3). Then it follows that $TFS(p)$ holds and $\sigma \geq (\xi^0)^2/(8C(p,c_1))$. Take $f = I_A/\sqrt{\pi(A)}$ $(\pi(A) > 0)$ and compute $D(f,f)/E_F(f)^{(p-2)/p}$. We have $\sigma \leq \kappa$. The proof is completed. By the way, the estimate for the F-Sobolev constant can be obtained by taking K be zero-measure in (1.9) directly.

Remark 2.1 Note that it is easily proven that

$$\xi^0 = \inf_{\pi(A)>0} \frac{J^{(1/2)}(A \times A^c)}{\pi(A)^{(p-1)/p}F(\pi(A)^{-1})^{(p-2)/(2p)}}.$$

So the qualitative conclusion of Theorem 1.1(ii) is obtained by [5, Theorem 1.1].

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