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STRONG ERGODICITY FOR SINGLE-BIRTH PROCESSES

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Abstract

An explicit and computable criterion for strong ergodicity of single-birth processes is presented. As an application, some sufficient conditions are obtained for strong ergodicity of an extended class of continuous-time branching processes and multi-dimensional Q-processes by comparison methods respectively. Consequently strong ergodicity of the Q-process corresponding to the finite-dimensional Schlögl model is proven.

Keywords: Strongly ergodic; single-birth Q-process; Schlögl model

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1. Introduction

Consider a continuous-time, irreducible Markov chain with transition probability matrix $P(t) = (p_{ij}(t))$ on a countable state space $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ with stationary distribution $(\pi_i > 0 : i \in \mathbb{Z}_+)$. In the study of the theory of Markov chains, there are traditionally three types of ergodicity: ordinary ergodicity (or positive recurrence), exponential ergodicity and strong ergodicity (or uniform ergodicity). The main purpose of the paper is to deal with the last one: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$ for single-birth processes. For the readers' convenience, we recall some notation and results here.

The Q-matrix of a single-birth process $Q = (q_{ij} : i, j \in \mathbb{Z}_+)$ is as follows: $q_{i,i+1} > 0, q_{i,i+j} = 0$ for all $i \in \mathbb{Z}_+$ and $j \geq 2$. Throughout the paper, we consider only totally stable and conservative Q-matrices: $q_i = -q_{ii} = \sum_{j\neq i} q_{ij} < \infty$ for all $i \in \mathbb{Z}_+$. Define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ for $0 \leq k < n$ $(k, n \in \mathbb{Z}_+)$ and

$$m_0 = \frac{1}{q_{01}}, \qquad m_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k\right), \quad n \geq 1,$$

$$F_n^{(n)} = 1, \qquad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n,$$

$$d_0 = 0, \qquad d_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\right), \quad n \geq 1.$$

It is not difficult to check that $m_n = q_{01}^{-1} F_n^{(0)} + d_n$ for all $n \in \mathbb{Z}_+$. This notation is used to describe the uniqueness, recurrence and ergodicity of single-birth processes. First, the process

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is unique if and only if $R := \sum_{n=0}^{\infty} m_n = \infty$. Next, suppose that the Q-matrix is regular and irreducible, then the process is recurrent if and only if $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$, and is ergodic if and only if

$$d := \sup_{k \in \mathbb{Z}_+} \left[\frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}} \right] < \infty$$

(cf. [3, Theorems 3.16 and 4.54] and [4]). The three criteria are all explicit (depending on the Q-matrix (q_{ij}) only, without using test functions) and computable. This advantage makes single-birth processes a useful tool studying more complicated processes (cf. [3, Chapters 3 and 4]). Now, it is natural to look for some explicit criterion for strong ergodicity of this class of processes. In this paper, we have a complete answer as follows.

Theorem 1.1. *Let $Q = (q_{ij})$ be a regular, irreducible single-birth Q-matrix. Then the Q-process is strongly ergodic if and only if*

$$\sup_{k \in \mathbb{Z}_+} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty. \tag{1.1}$$

We now discuss strong ergodicity for an extended class of time-continuous branching processes. The original branching process can be described as follows. Let $\alpha > 0$ and $(p_j : j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate $\alpha i p_0 : i \to i-1$ $(i \geq 1)$ and growth rate $\alpha i p_{k+1} : i \to i+k$ $(k \geq 1, i \in \mathbb{Z}_+)$. Note that the process absorbs at state 0. In [5], an extended class of branching processes with the following Q-matrix is introduced:

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i-1,\ i \geq 1; \\ r_i p_{k+1}, & j = i+k,\ i, k \geq 1; \\ -r_i(1-p_1), & j = i \geq 1; \\ 0, & \text{otherwise},\ i, j \in \mathbb{Z}_+; \end{cases} \tag{1.2}$$

where $r_i > 0$ for all $i \geq 1$ and $0 < q_0 := \sum_{j \geq 1} q_{0j} < \infty$. The typical case we are interested in is that $q_{0j} = p_j$ or p_{j+1} $(j \geq 1)$ and r_i is a polynomial with degree $\theta \geq 1$. Define $M_1 = \sum_{k=1}^{\infty} k p_k$. Based on the same comparing idea as in [5], some sufficient conditions for strong ergodicity are obtained as follows.

Theorem 1.2. *Let $Q = (q_{ij})$ an irreducible Q-matrix as given by (1.2). Assume that $M_1 \leq 1$. If $\sum_{i=1}^{\infty} i/r_i < \infty$, then the Q-process is strongly ergodic. The condition can be weakened to $\sum_{i=1}^{\infty} 1/r_i < \infty$ provided $M_1 < 1$ and the r_i are increasing.*

It should be pointed out that for birth–death processes, which are special cases of single-birth processes, Theorem 1.1 and the second part of Theorem 1.2 have been obtained in [8] and [6] respectively, using a different approach. See Remark 2.5(i) and Remark 3.1(i) for further comments.

Now we discuss strong ergodicity of multi-dimensional Q-processes. In [7] and [3, Theorem 4.58], methods which reduce the multi-dimensional problems to one-dimensional problems are proposed. By keeping this idea in mind, some sufficient conditions for strong ergodicity of multi-dimensional Q-processes are obtained.

Theorem 1.3. *Let E be a countable set and $Q = (q(x, y) : x, y \in E)$ a conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\sum_{k=0}^{\infty} E_k = E$ with $E_0 = \{\theta\}$ and E_k $(k \geq 1)$ being finite, where $\theta \in E$ is a reference point. Next, suppose that:*

(i) *if $q(x, y) > 0$ and $x \in E_k$, then $y \in \sum_{j=0}^{k+1} E_j$ for all $k \geq 0$;*

(ii) *$\sum_{y \in E_{k+1}} q(x, y) > 0$ for all $x \in E_k$ and all $k \geq 0$;*

(iii) *$C_k := \sup\{q(x) : x \in E_k\} < \infty$ for all $k \geq 0$.*

Define a conservative Q-matrix $Q = (q_{ij} : i, j \in \mathbb{Z}_+)$ as follows:

$$q_{ij} = \begin{cases} \sup\{\sum_{y \in E_j} q(x, y) : x \in E_i\}, & \text{if } j = i + 1; \\ \inf\{\sum_{y \in E_j} q(x, y) : x \in E_i\}, & \text{if } j < i; \\ 0, & \text{other cases of } j \neq i. \end{cases} \tag{1.3}$$

Moreover, suppose that both $(q(x, y))$ and (q_{ij}) are irreducible and that (q_{ij}) is regular. If

$$\sup_{k \in \mathbb{Z}_+} \sum_{n=0}^{k} (F_n^{(0)} \hat{d} - d_n) < \infty \tag{1.4}$$

where $\hat{d} := \sup_{k \in \mathbb{Z}_+} d_k / F_k^{(0)}$, then the $(q(x, y))$-process is strongly ergodic.

Theorem 1.3 is rather practical. For example, we can apply it directly to the finite-dimensional Schlögl model. Let S be a finite set and $E = \mathbb{Z}_+{}^S$. The model is defined by the following Q-matrix $Q = (q(x, y) : x, y \in E)$:

$$q(x, y) = \begin{cases} \lambda_1 \binom{x(u)}{2} + \lambda_4, & \text{if } y = x + e_u, \\ \lambda_2 \binom{x(u)}{3} + \lambda_3 x(u), & \text{if } y = x - e_u, \\ x(u) p(u, v), & \text{if } y = x - e_u + e_v, \\ 0, & \text{other } y \neq x, \end{cases} \tag{1.5}$$

$$q(x) = -q(x, x) = \sum_{y \neq x} q(x, y),$$

where $x = (x(u) : u \in S)$, $\lambda_1, \ldots, \lambda_4$ are positive constants, $(p(u, v) : u, v \in S)$ is a transition probability matrix on S and e_u is the element in E having value 1 at u and 0 elsewhere (cf. [2] and [3]). In fact, we have the following conclusion.

Theorem 1.4. *The Q-process corresponding to the finite-dimensional Schlögl model is strongly ergodic.*

The proof of Theorem 1.1 is given in the next section. In addition, Corollary 2.4, which is devoted to birth–death processes, is presented there. Applying this corollary and making a comparison of the 'extended' branching process with some birth–death processes, we prove Theorem 1.2 in Section 3. Besides, some examples show that the conditions of Theorem 1.2 are necessary. Then the proofs of Theorems 1.3 and 1.4 are completed.

2. Proof of Theorem 1.1

The following lemma is the key to proving Theorem 1.1.

Lemma 2.1. *Let $Q = (q_{ij})$ be a regular, irreducible and recurrent single-birth Q-matrix. Then the equation*

$$y_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i}, \quad i \geq 1; \qquad y_0 = 0, \tag{2.1}$$

has a bounded non-negative solution if and only if (1.1) holds. If so,

$$d = \lim_{k \to \infty} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}}$$

and the unique solution of (2.1) is as follows:

$$y_0 = 0, \qquad y_1 = d, \qquad y_{n+1} = y_n + F_n^{(0)} y_1 - d_n, \quad n \geq 1. \tag{2.2}$$

Proof. First, assume that (1.1) holds and define (y_i) by (2.2). Then, it should be easy to verify that (y_i) is a bounded non-negative solution of (2.1).

Next, let (y_i) be a bounded non-negative solution of (2.1) and define $v_n = y_{n+1} - y_n$ for $n \geq 0$. From (2.1), it is not difficult to derive

$$v_n = \frac{1}{q_{n,n+1}} \left(\sum_{k=0}^{n-1} q_n^{(k)} v_k - 1 \right), \qquad n \geq 1.$$

By induction, we can easily prove that $v_n = F_n^{(0)} v_0 - d_n$ for all $n \geq 0$. Note that $v_0 = y_1$. From these facts, it follows that

$$y_{k+1} = \sum_{n=0}^{k} v_n = \sum_{n=0}^{k} (F_n^{(0)} v_0 - d_n), \qquad k \in \mathbb{Z}_+. \tag{2.3}$$

Now, on the one hand, by (2.3) and $y_{k+1} \geq 0$, it follows that $v_0 \geq \sum_{n=0}^{k} d_n \big/ \sum_{n=0}^{k} F_n^{(0)}$ for all $k \in \mathbb{Z}_+$. Hence $v_0 \geq d$. On the other hand, by (2.3) again,

$$\frac{y_{k+1}}{\sum_{n=0}^{k} F_n^{(0)}} = v_0 - \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}}, \qquad k \in \mathbb{Z}_+. \tag{2.4}$$

Note that (y_i) is bounded and $\sum_{n=0}^{k} F_n^{(0)} \to +\infty$ as $k \to \infty$ (by recurrence). Letting $k \to \infty$ in (2.4), we see that the second part on the right-hand side of (2.4) tends to the limit v_0, and furthermore $v_0 \leq d$. Hence, we have

$$y_1 = v_0 = d = \lim_{k \to \infty} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}},$$

Combining this with (2.3), it follows that the solution (y_i) must have the representation (2.2) and hence is unique. Finally, by the boundedness of (y_i) and (2.3), condition (1.1) follows.

The proof of Theorem 1.1 will be completed by using [3, Theorem 4.45] and Lemma 2.1.

Proof of Theorem 1.1. From [1, §6.3, Proposition 3.3] or [3, Theorem 4.45], we know that the Q-process is strongly ergodic if and only if the following equation has a bounded non-negative solution:

$$\sum_j q_{ij} y_j \le -1, \quad i \notin H; \qquad \sum_{i \in H} \sum_{j \ne i} q_{ij} y_j < \infty,$$

where H is a non-empty finite subset of $\mathbb{Z}_+$. Let $H = \{0\}$. For single-birth processes, the last equation is reduced to

$$\sum_j q_{ij} y_j \le -1, \qquad i \ne 0, \tag{2.5}$$

since $\sum_{j \ne 0} q_{0j} y_j = q_{01} y_1 < \infty$.

Assume that the single-birth process is strongly ergodic. Then there exists a bounded non-negative solution (u_i) of (2.5), i.e.,

$$u_i \ge \sum_{j \ne i} \frac{q_{ij}}{q_i} u_j + \frac{1}{q_i}, \quad i \ge 1; \qquad u_0 \ge 0.$$

Denote by (u_i^*) the minimal non-negative solution of (2.1). By the Comparison Theorem [3, Theorem 2.6], we have $u_i \ge u_i^*$ for all $i \ge 0$. Thus, (u_i^*) is bounded and (2.1) has a bounded non-negative solution. By Lemma 2.1, (1.1) holds.

Conversely, let (1.1) hold. Define (y_i) by (2.2). By Lemma 2.1, (y_i) is a bounded non-negative solution of (2.1). Clearly (y_i) is also a bounded non-negative solution of (2.5). This implies strong ergodicity by the criterion quoted above.

Corollary 2.2. *Let $Q = (q_{ij})$ be a regular irreducible single-birth Q-matrix. If (1.4) holds, then the Q-process is strongly ergodic and (2.1) has (uniquely) a bounded non-negative and increasing solution as follows:*

$$y_0 = 0, \quad y_1 = \hat{d}, \quad y_{n+1} = y_n + F_n^{(0)} y_1 - d_n, \quad n \ge 1 \tag{2.6}$$

Moreover, $d = \hat{d} = \lim_{k\to\infty} (\sum_{n=0}^k d_n)/(\sum_{n=0}^k F_n^{(0)})$.

Proof. Note that $d \le \hat{d}$. So strong ergodicity can be obtained directly by Theorem 1.1. Next, as in the proof of Lemma 2.1, it can be checked that (y_i) in (2.6) is a bounded, non-negative and increasing solution of (2.1) since (1.4) holds (cf. [3, Lemma 4.56]). This implies the last assertion by Lemma 2.1. An alternative proof goes as follows. By the definition of d, we have $\sum_{n=0}^k (F_n^{(0)} \hat{d} - d_n) \ge (\hat{d} - d) \sum_{n=0}^k F_n^{(0)}$. Then the assertion follows from the assumption and since $\sum_{n=0}^\infty F_n^{(0)} = \infty$.

Remarks 2.3. (i) There is a non-negative increasing solution to (2.5) if and only if $\hat{d} < \infty$ (cf. [3, Lemma 4.56]).

(ii) Suppose that the process is recurrent and that $\lim_{k\to\infty} d_k/F_k^{(0)} = \hat{d}$. Then

$$\lim_{k\to\infty} \frac{\sum_{n=0}^k d_n}{\sum_{n=0}^k F_n^{(0)}} = \lim_{k\to\infty} \frac{d_k}{F_k^{(0)}}$$

(by Stolz's theorem) and hence $d \ge \hat{d}$. So we also have $d = \hat{d}$.

We now turn to birth–death processes. The Q-matrix of a birth–death process is as follows: $q_{i,i+1} = b_i,\ i \geq 0$; $q_{i,i-1} = a_i,\ i \geq 1$ and $q_{ij} = 0$ for all $|i-j| \geq 2$. Correspondingly,

$$F_n^{(0)} = \frac{a_1 \ldots a_n}{b_1 \ldots b_n}, \qquad d_n = \frac{1}{b_n} + \sum_{i=1}^{n-1} \frac{a_{i+1} \ldots a_n}{b_i \ldots b_n}, \qquad n \geq 1.$$

For a regular recurrent birth–death Q-matrix, we see that

$$\lim_{k\to\infty} \frac{d_k}{F_k^{(0)}} = \lim_{k\to\infty} \frac{1}{b_0} \sum_{n=0}^{k-1} \frac{b_0 \ldots b_n}{a_1 \ldots a_{n+1}} = \frac{1}{b_0} \sum_{n=0}^{\infty} \frac{b_0 \ldots b_n}{a_1 \ldots a_{n+1}} = \hat{d}.$$

By Remark 2.3(ii), we have $d = \hat{d}$. Note that

$$\sup_{k\in\mathbb{Z}_+} \sum_{i=0}^{k} (F_i^{(0)} d - d_i) = \sum_{n=1}^{\infty} \left(\frac{1}{a_n} + \sum_{i=1}^{n} \frac{b_i \ldots b_n}{a_i \ldots a_{n+1}} \right) =: S'.$$

We can obtain the following criterion for strong ergodicity of birth–death processes by Theorem 1.1. The condition given is well known, but used as a sufficient condition for exponential ergodicity (cf. [1, §6.6, Proposition 6.6] and [3, Corollary 4.51]).

Corollary 2.4. *A regular birth–death process is strongly ergodic if and only if*

$$S := \sum_{n=1}^{\infty} \left(\frac{1}{a_{n+1}} + \sum_{k=1}^{n} \frac{b_k \ldots b_n}{a_k \ldots a_{n+1}} \right) < \infty. \tag{2.7}$$

Note that $d \leq S' = S + 1/a_1$ and (1.1), (1.4) and (2.7) are equivalent for birth–death processes.

Remarks 2.5. (i) Under the restriction of stochastic monotonicity, [8] discusses the uniformly polynomial convergence for time-continuous Markov chains in terms of Feller transition functions. The convergence means that there exist two constants $v > 0$ and $C > 0$ so that $\sup_{i,j\in E} t^v |p_{ij}(t) - \pi_j| \leq C$ for all $t \geq 0$. For a birth–death process, [8] proves that it is uniformly polynomial convergent if and only if $S < \infty$. Corollary 2.4 clarifies the equivalence of uniformly polynomial convergence and strong ergodicity. In fact, by [3, Theorem 4.43], the chain is strongly ergodic if and only if $\sup_i \sum_j |p_{ij}(t) - \pi_j| = O(\mathrm{e}^{-\rho t})$ as $t \to \infty$ for some $\rho > 0$ (that is, the chain is uniformly exponential ergodic). Combining this result with the definition of strong ergodicity, we can easily see that the uniformly polynomial convergence is equivalent to the strong ergodicity. The techniques in [8] are ineffective for single-birth processes because the processes are not stochastically monotone in general. This is one of the motivations for writing this paper.

(ii) For a strongly ergodic birth–death process, by Corollary 2.2, the equation

$$a_i(y_{i-1} - y_i) + b_i(y_{i+1} - y_i) = -1, \qquad i \neq 0, \qquad y_0 = 0 \tag{2.8}$$

has uniquely a bounded non-negative (increasing) solution as follows:

$$y_0 = 0, \qquad y_i = \sum_{k=0}^{i-1} \left(\frac{1}{a_{k+1}} + \sum_{j=k+1}^{\infty} \frac{b_{k+1} \ldots b_j}{a_{k+1} \ldots a_{j+1}} \right), \qquad i \geq 1.$$

By Corollary 2.4, we can easily check that the following examples hold.

Example 2.6. (i) Take $a_n = b_n,\ n \ge 1$. Then the process is strongly ergodic if and only if $\sum_{n=1}^{\infty} n/a_n < \infty$.

(ii) Take $a_n = \lambda b_n,\ n \ge 1,\ \lambda > 1$. Then the process is strongly ergodic if and only if $\sum_{n=1}^{\infty} 1/a_n < \infty$.

3. Proofs of Theorem 1.2–1.4

Proof of Theorem 1.2. First, under the assumption, by [5, Theorems 1.2 and 1.3], we know that the process is unique and recurrent. Let $\Gamma = \sum_{k=1}^{\infty} kp_{k+1}$. It is easy to check that $\Gamma = M_1 + p_0 - 1$. Hence $M_1 < 1 \iff \Gamma < p_0$ and $M_1 = 1 \iff \Gamma = p_0$.

(a) Next, consider the birth–death process $(\bar{p}_{ij}(t))$ with $a_i = b_i = r_i p_0$ for $i \ge 1$, where b_0 is some positive constant. By Example 2.6(i) and the assumption, the process $(\bar{p}_{ij}(t))$ is strongly ergodic. By Remark 2.5(ii), the equation (2.8) has a unique bounded non-negative solution as follows: $y_0 = 0$ and

$$y_i = \frac{1}{p_0}\Bigg(i\sum_{j=i+1}^{\infty}\frac{1}{r_j} + \sum_{j=1}^{i}\frac{j}{r_j}\Bigg) \le \frac{1}{p_0}\sum_{j=1}^{\infty}\frac{j}{r_j} =: c_1 < \infty, \qquad i \ge 1. \tag{3.1}$$

Since $y_{i+1} - y_i = \sum_{j=i+1}^{\infty}(r_j p_0)^{-1}$ for $i \ge 1$, $y_{i+1} - y_i$ is decreasing as $i \to \infty$. Thus, on the one hand, we have

$$\begin{aligned}\sum_j q_{ij}y_j &= \sum_j q_{ij}(y_j - y_i) = q_{i,i-1}(y_{i-1} - y_i) + \sum_{k=1}^{\infty} q_{i,i+k}(y_{i+k} - y_i)\\ &\le r_i p_0(y_{i-1} - y_i) + \sum_{k=1}^{\infty} r_i p_{k+1} k(y_{i+1} - y_i) = r_i p_0(y_{i-1} - y_i) + r_i\Gamma(y_{i+1} - y_i)\\ &\le a_i(y_{i-1} - y_i) + b_i(y_{i+1} - y_i) = -1, \qquad i \ne 0.\end{aligned}$$

On the other hand, by (3.1), we have $\sum_{i=1}^{\infty} q_{0i}y_i \le c_1 q_0 < \infty$. Combining these facts with [1, §6.3, Proposition 3.3] or [3, Theorem 4.45], it follows that $(p_{ij}(t))$ is strongly ergodic.

(b) Assume that $M_1 < 1$. Consider the birth–death process $(\hat{p}_{ij}(t))$ with $a_i = r_i p_0$ and $b_i = r_i\Gamma$ for $i \ge 1$ and b_0 being some positive constant. Since $\Gamma < p_0$, by Example 2.6(ii) and the assumption, the process $(\hat{p}_{ij}(t))$ is strongly ergodic. Using 'r_i is increasing', similar to (a), we can easily prove the second part of Theorem 1.2.

Remarks 3.1. (i) Take $a_i = b_i = r_i p_0$ $(i \ge 1)$ in Example 2.6(i) and $a_i = \lambda b_i = \lambda r_i p_2$ $(i \ge 1, \lambda > 1)$ in Example 2.6(ii) respectively. Then the birth–death processes can be regarded as a special case of the extended branching processes. In the former case, we have $p_0 = p_2$, $p_j = 0$ for $j \ge 2$, and so $M_1 = p_1 + 2p_2 = p_0 + p_1 + p_2 = 1$. In the latter one, $p_0 = \lambda p_2 > p_2$ and $p_j = 0$ for $j \ge 2$, hence $M_1 < 1$. Therefore, by Example 2.6, the sufficient conditions of Theorem 1.2 are all necessary. We point out that the second part of Theorem 1.2 for the minimal process has also been proven in [6] by the same technique.

(ii) Let $M_1 < 1$ and r_i $(i \ge 1)$ satisfy

$$\sum_{i=1}^{\infty}\frac{1}{r_i} < \infty \quad \text{and} \quad \sum_{j=0}^{\infty}\left(\frac{\Gamma}{p_0}\right)^j\frac{1}{r_{j+i+2}} \le \frac{p_0}{(p_0 - \Gamma)r_{i+1}}, \quad i \ge 1.$$

Then the process is strongly ergodic. We need only to replace 'r_i is increasing' by the last inequality.

Proof of Theorem 1.3. By [3, Theorems 3.19 and 4.58], the $(q(x, y))$-process is regular. Hence (q_{ij}) is a regular, irreducible single-birth Q-matrix. By Corollary 2.2, the (q_{ij})-process is strongly ergodic. To prove strong ergodicity of the $(q(x, y))$-process, by [3, Theorem 4.45], we need only show that the equation

$$\sum_{y\in E} q(x, y)u(y) + 1 \le 0, \quad x \neq \theta, \qquad \sum_{y\neq\theta} q(\theta, y)u(y) < \infty$$

has a bounded non-negative solution. For this, by the assumptions and Corollary 2.2, let (u_k) be defined by (2.6), which is a bounded non-negative and increasing solution of (2.1), and take $u(x) = u_k$ for $x \in E_k, k \ge 0$. Now, for $x \neq \theta$, there exists some (exactly one) $k \ge 1$ so that $x \in E_k$. From this, it is easy to verify that we have constructed a desired solution.

Proof of Theorem 1.4. For $x = (x(u) : u \in S) \in E$, put $|x| := \sum_{u\in S} x(u)$. Take $E_k = \{x \in E : |x| = k\}$ for all $k \in \mathbb{Z}_+$. Then conditions (i)–(iii) of Theorem 1.3 hold. Note that the Q-matrix (q_{ij}) corresponding to (1.5) is a birth–death Q-matrix. Let $|S|$ denote the cardinality of S. Note that $\sup\{\sum_{u\in S} x(u)^2 : |x| = k\} = k^2$ and $(\sum_{u\in S} x(u)/|S|)^3 \le \sum_{u\in S} x(u)^3/|S|$. We have

$$b_k = \sup\left\{\sum_{u\in S}\left(\lambda_1\binom{x(u)}{2} + \lambda_4\right) : |x| = k\right\} = \frac{\lambda_1}{2}(k^2 - k) + \lambda_4|S|, \tag{3.2}$$

$$\begin{aligned} a_k &= \inf\left\{\sum_{u\in S}\left(\lambda_2\binom{x(u)}{3} + \lambda_3 x(u)\right) : |x| = k\right\} \\ &\ge \frac{\lambda_2}{6}\left(\inf_{|x|=k}\sum_{u\in S} x(u)^3 - 3\sup_{|x|=k}\sum_{u\in S} x(u)^2\right) + \left(\lambda_3 + \frac{\lambda_2}{3}\right)k \\ &\ge \frac{\lambda_2}{6}\left(\frac{k^3}{|S|^2} - 3k^2\right) + \left(\lambda_3 + \frac{\lambda_2}{3}\right)k. \end{aligned} \tag{3.3}$$

By (3.2) and (3.3), it is not difficult to check that (2.7) holds. The conclusion follows from Theorem 1.3.

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