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Sufficient and Necessary Conditions for Stochastic Comparability of Jump Processes

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Abstract This note is devoted to the study of the stochastic comparability of jump processes. On the basis of [2] and [3], it is proved that two jump processes are stochastically comparable if and only if their q-pairs are comparable. Meanwhile, the result concerning the uniqueness given in [6] is also improved upon.

Keywords Stochastically comparable, Totally stable, Conservative, Uniqueness
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1 Introduction

In the past twenty years or so, the coupling methods developed rapidly and had a very wide range of applications. It is well known that the order-preserving couplings often play an important role in various applications. One problem arises naturally: Under what conditions for marginal processes does there exist an order-preserving Markovian coupling? If so, then the two marginal processes must be obviously stochastically comparable. Recently, Zhang [1] proved the converse proposition in the context of jump processes. Hence, it is meaningful to study the stochastic comparability of jump processes. In fact, the research on this topic has a long history and some good results have been obtained; refer to [2–5]. The main purpose of this note is to complete the work of [2] and [3], from which the main ideas of the proofs and the notations originate. However, for the readers' convenience, we still recall a few of the notations here.

Suppose that $(E, \mathcal{E})$ is a Polish space endowed with a measurable semi-order $\leq$ and that $F = \{(x_1, x_2) \in E \times E : x_1 \leq x_2\}$ is a measurable closed set on $E \times E$.

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Definition 1.1 *A measurable real function f is called monotone if for all $x \le y$, $f(x) \le f(y)$. Denote by $\mathcal{M}$ the set of all bounded monotone functions. A measurable set A is called monotone if so is its indicator I_A, denoted by $A \in \mathcal{M}$. For two probability measures μ_1 and μ_2, we define $\mu_1 \le \mu_2$, provided for every $f \in \mathcal{M}$, $\mu_1 f \le \mu_2 f$ where $\mu_k f = \int f(x)\, \mu_k(dx)$.*

Definition 1.2 *Two jump processes $P_1(t)$ and $P_2(t)$ are said to be stochastically comparable if $P_1(t)f(x_1) \le P_2(t)f(x_2), f \in \mathcal{M}$, $x_1 \le x_2$, $t \ge 0$, where $P_k(t)f(x) = \int P_k(t,x,dy)f(y)$. We write it as $P_1(t) \le P_2(t)$ briefly. Two q-pairs $(q_1(x), q_1(x,dy))$ and $(q_2(x), q_2(x,dy))$ are called comparable if for all $A \in \mathcal{M}$ and $x_1 \le x_2$, either $x_1, x_2 \in A$ or $x_1, x_2 \notin A$, the following inequality holds:*

$$\Omega_1 I_A(x_1) \le \Omega_2 I_A(x_2), \tag{1.1}$$

where the operators Ω_k are defined by $\Omega_k f(x) = \int q_k(x,dy)f(y) - q_k(x)f(x)$, $x \in E$, $f \in {}_b\mathcal{E}$. Here, denote by ${}_b\mathcal{E}$ the set of all bounded $\mathcal{E}$-measurable functions.

The first main result of this paper can be stated as follows:

Theorem 1.3 *Let $(q_k(x), q_k(x,dy))\,(k=1,2)$ be two bounded totally stable q-pairs, possibly non-conservative. Denote by $P_k(t)\,(k=1,2)$ the corresponding q-process. Then $P_1(t) \le P_2(t)$ if and only if $(q_1(x), q_1(x,dy))$ and $(q_2(x), q_2(x,dy))$ are comparable.*

To study the comparability of q-processes with unbounded q-pairs, we need some hypotheses as follows (cf. [3; Hypothesis 5.30]).

Hypothesis 1.4 *There exists a sequence of G_δ sets $\{E_n\}_1^\infty$ such that $E_n \uparrow E\,(n \to \infty)$ and for all $n \ge 1$, (1) $\sup_{x \in E_n} q_k(x) < \infty$, $k = 1,2$, (2) The set $H_n = \{y \in E \setminus E_n :$ there exists $x \in E_n$ such that $x \le y\} \in \mathcal{M}$. Moreover, if $H_n \ne \emptyset$, then there is a point $b_n \in H_n$ so that $x \le b_n$ for all $x \in E_n$.*

In the cases where $E = \mathbf{R}^d$, $\mathbf{Z}^d$, $\mathbf{R}_+^d$ or $\mathbf{Z}_+^d$ with the ordinary semi-order and where the two q-pairs are locally bounded, Hypothesis 1.4 is trivial; refer to [3]. The following is the second main result of this paper.

Theorem 1.5 *Let $(q_k(x), q_k(x,dy))\,(k=1,2)$ be two regular q-pairs. Denote by $P_k(t)\,(k=1,2)$ the correponding q-process. Under Hypothesis 1.4, $P_1(t) \le P_2(t)$ if and only if $(q_1(x), q_1(x,dy))$ and $(q_2(x), q_2(x,dy))$ are comparable.*

In addition, we can improve the result concerning the uniqueness given in [6] for jump processes. The new conclusion is as follows.

Theorem 1.6 *Let $(q_1(x), q_1(x,dy))$ be a totally stable and conservative q-pair and $(q_2(x), q_2(x,dy))$ be a regular q-pair. Suppose that Hypothesis 1.4 holds and $q_k(x, E \setminus E_n) = q_k(x, H_n)$, $x \in E$, $k=1,2$. If $(q_1(x), q_1(x,dy))$ and $(q_2(x), q_2(x,dy))$ are comparable, then $(q_1(x), q_1(x,dy))$ is regular, and furthermore $P_1(t) \le P_2(t)$.*

Because Theorem 1.6 can be derived from the proof of Theorem 1.5 and [6] directly, we omit the details of its proof in this note.

2 Proofs

To prove Theorem 1.3, we need two lemmas as follows.

Lemma 2.1 *Let $P_k(x_k, dy_k)\,(k=1,2)$ be two transition probabilities on $(E,\mathcal{E})$. If $P_1(x_1,dy_1) \le P_2(x_2,dy_2)$ for all $x_1 \le x_2$, then $P_1^n(x_1,dy_1) \le P_2^n(x_2,dy_2)$ for all $x_1 \le x_2$ and $n \ge 1$.*

Proof Assume that the conclusion holds for every $n \le m$. By [1; Theorem 2.3], there exists a coupling transition probability $P(x_1,x_2;dy_1,dy_2)$ on $(E\times E, \mathcal{E}\times\mathcal{E})$ of $P_1(x_1,dy_1)$ and $P_2(x_2,dy_2)$ such that $P(x_1,x_2;F)=1$ for all $x_1 \le x_2$. Therefore, for every $f \in \mathcal{M}$ and $x_1 \le x_2$, we have

$$\begin{aligned} P_1^{m+1}f(x_1) - P_2^{m+1}f(x_2) &= \int_{E\times E} P(x_1,x_2;dy_1,dy_2)\big(P_1^m f(y_1) - P_2^m f(y_2)\big) \\ &= \int_F P(x_1,x_2;dy_1,dy_2)\big(P_1^m f(y_1) - P_2^m f(y_2)\big) \le 0. \end{aligned}$$

So the assertion follows, by Definition 1.1 and induction.

For an arbitrarily given point $\triangle \notin E$, set $E_\triangle = E \cup \{\triangle\}$ and $\mathcal{E}_\triangle = \sigma(\mathcal{E}\cup\{\triangle\})$. Let $(q(x), q(x,dy))$ be a q-pair on $(E,\mathcal{E})$, define a new conservative q-pair $(\tilde q(x), \tilde q(x,dy))$ on $(E_\triangle, \mathcal{E}_\triangle)$ as follows:

$$\tilde q(x,A) = I_E(x)\big(q(x,A\backslash\{\triangle\}) + (q(x)-q(x,E))I_A(\triangle)\big), \quad \tilde q(x) = I_E(x)q(x), \quad x\in E_\triangle,\ A\in\mathcal{E}_\triangle.$$

Denote by $P^{\min}(\lambda)$ and $\widetilde{P}^{\min}(\lambda)$ the Laplace transforms of the minimal q-processes determined by the two q-pairs respectively. By Proof a) of [3; Theorem 3.2], we have the following lemma.

Lemma 2.2 *$P^{\min}(\lambda)$ and $\widetilde{P}^{\min}(\lambda)$ satisfy $P^{\min}(\lambda)I_A(x) = \widetilde{P}^{\min}(\lambda)I_A(x), A\subset E, x\in E$.*

Proof of Theorem 1.3 The necessity has been proved by [3; Part (i) of Lemma 5.29]. We only need to show the sufficiency. For this, we enlarge the state space E to $E_\triangle$ on which the semi-order $\le$ is extended so that $\triangle \le x$ for all $x\in E$.

Define two q-pairs $(\tilde q_k(x), \tilde q_k(x,dy))\,(k=1,2)$ on $(E_\triangle,\mathcal{E}_\triangle)$ as above. Obviously, no matter whether the original q-pairs are conservative or not, the new q-pairs are bounded, totally stable and conservative. Denote by $\mathcal{M}_\triangle$ the set of all bounded monotone functions on $(E_\triangle,\mathcal{E}_\triangle)$. Set $\widetilde{P}_k^{(\lambda)} = I + \frac{1}{\lambda}\widetilde{\Omega}_k\,(k=1,2)$ where $\lambda \ge \sup_{x\in E} q_1(x) + \sup_{x\in E} q_2(x)$. We need to show that

$$\widetilde{P}_1^{(\lambda)} I_A(x_1) \le \widetilde{P}_2^{(\lambda)} I_A(x_2), \qquad x_1,x_2\in E_\triangle,\ x_1\le x_2,\ A\in\mathcal{M}_\triangle. \tag{2.1}$$

In the case of $\triangle\notin A$, if $x_1=\triangle$, then $\widetilde{P}_1^{(\lambda)}I_A(x_1) = 0 \le \widetilde{P}_2^{(\lambda)}I_A(x_2)$; if $x_1\ne\triangle$, then $x_1, x_2 \in E$; besides, because of $\triangle\notin A$, we have $A\in\mathcal{M}$. For these x_1 and x_2, whenever $x_1, x_2\in A$ or $x_1, x_2\notin A$, by (1.1), we have $\widetilde{P}_1^{(\lambda)}I_A(x_1) = I_A(x_1) + \frac{1}{\lambda}\Omega_1 I_A(x_1) \le I_A(x_2) + \frac{1}{\lambda}\Omega_2 I_A(x_2) = \widetilde{P}_2^{(\lambda)}I_A(x_2)$. Next, if $x_1\notin A$ and $x_2\in A$, due to the selection of λ, we have $\widetilde{P}_1^{(\lambda)}I_A(x_1) = \frac{1}{\lambda}q_1(x_1,A) \le 1 + \frac{1}{\lambda}(q_2(x_2,A) - q_2(x_2)) = \widetilde{P}_2^{(\lambda)}I_A(x_2)$. Finally, in the case of $\triangle\in A$, we have $A = E_\triangle$, and hence $\widetilde{P}_1^{(\lambda)}I_A(x_1) = 1 = \widetilde{P}_2^{(\lambda)}I_A(x_2)$. Therefore, (2.1) always holds. Note that $\widetilde{P}_k^{(\lambda)}\,(k=1,2)$ are two transition probabilities on the Polish space $(E_\triangle,\mathcal{E}_\triangle)$. By Lemma 2.1 and [3; Lemma 5.28], it follows that $(\widetilde{P}_1^{(\lambda)})^m I_A(x_1) \le (\widetilde{P}_2^{(\lambda)})^m I_A(x_2), m\ge 1$. Moreover, because $(\tilde q_k(x), \tilde q_k(x,dy))\,(k=1,2)$ are bounded q-pairs, we have $\widetilde{P}_k(t) = \exp(t\widetilde{\Omega}_k) = e^{-\lambda t}\sum_{m=0}^{\infty}\frac{(\lambda t)^m}{m!}(\widetilde{P}_k^{(\lambda)})^m$. Hence, it follows that $\widetilde{P}_1(t)I_A(x_1) \le \widetilde{P}_2(t)I_A(x_2)$ for all $x_1, x_2 \in E_\triangle$ with $x_1 \le x_2$ and $A \in \mathcal{M}_\triangle$. Thus, by Lemma 2.2, we have $P_1(t)I_A(x_1) \le \widetilde{P}_2(t)I_A(x_2)$ for all $x_1, x_2 \in E$ with $x_1 \le x_2$ and $A \in \mathcal{M}$. Hence, by [3; Lemma 5.28] and Definition 1.2, we deduce that $P_1(t) \le P_2(t)$.

Proof of Theorem 1.5 The necessity simply follows from the backward Kolmogorov equation; refer to [4]. To prove the sufficiency, define (cf. [3; Theorem 5.31])

$$q_k^{(n)}(x,B) = q_k(x,B), \quad q_k^{(n)}(x,\{b_n\}) = q_k(x,H_n), \qquad x \in E_n,\, B \in E_n \cap \mathcal{E},$$
$$q_k^{(n)}(x) = I_{E_n}(x)q_k(x), \qquad n \ge 1,\, k = 1,2.$$

Then we have $\sup_{x\in E} q_k^{(n)}(x) = \sup_{x\in E_n} q_k(x) < \infty$, $n \ge 1$, $q_k^{(n)}(x) \to q_k(x)$, $q_k^{(n)}(x,A) \to q_k(x,A)$, $n \to \infty$, $x \in E$, $A \in \mathcal{E}$, $k = 1,2$. Thus, $(q_k^{(n)}(x), q_k^{(n)}(x,dy))$ uniquely determines a q-process, denoted by $P_k^{(n)}(t)$ $(k = 1,2)$. We need to show that $(q_1^{(n)}(x), q_1^{(n)}(x,dy))$ and $(q_2^{(n)}(x), q_2^{(n)}(x,dy))$ are comparable on the state space $(E_n \cup \{b_n\}, (E_n \cup \{b_n\}) \cap \mathcal{E})$.

Note that the semi-order on E induces a semi-order on $(E_n \cup \{b_n\})$. Let $\mathcal{M}_n$ denote the set of all bounded monotone functions on $(E_n \cup \{b_n\}, (E_n \cup \{b_n\}) \cap \mathcal{E})$. Clearly, if $B \in \mathcal{M}_n$, then $b_n \in B$ and $B \cup H_n \in \mathcal{M}$. And it is not difficult to prove that

$$\Omega_k^{(n)} I_B(x) = \Omega_k I_{B\cup H_n}(x), \qquad x \in E_n,\, k = 1,2. \tag{2.2}$$

For all $x_1, x_2 \in E_n \cup \{b_n\}$ with $x_1 \le x_2$ and $B \in \mathcal{M}_n$, if $x_2 \ne b_n$, then $x_1, x_2 \in E_n$. Combining (2.2) with (1.1), whenever $x_1, x_2 \in B$ or $x_1, x_2 \notin B$ (i.e. $x_1, x_2 \in E_n \setminus B$), we have $\Omega_1^{(n)} I_B(x_1) \le \Omega_2^{(n)} I_B(x_2)$. If $x_2 = b_n$, then $x_2 \in B$. Thus, we have $\Omega_1^{(n)} I_B(x_1) \le 0 = \Omega_2^{(n)} I_B(x_2)$ for all $x_1 \in B$. Therefore, $(q_1^{(n)}(x), q_1^{(n)}(x,dy))$ and $(q_2^{(n)}(x), q_2^{(n)}(x,dy))$ are comparable on $(E_n \cup \{b_n\}, (E_n \cup \{b_n\}) \cap \mathcal{E})$. In addition, they are bounded and totally stable q-pairs, possibly non-conservative. By [7; Proposition 8.1.4], E_n is Polish, hence $E_n \cup \{b_n\}$ is Polish. Then, by Theorem 1.3, we have $P_1^{(n)}(t) \le P_2^{(n)}(t)$. Besides, by [3; Lemma 5.14], we know that

$$\lim_{n\to\infty} P_k^{(n)}(t, x, A \cap (E_n \cup \{b_n\})) = P_k(t,x,A), \qquad x \in E,\, A \in \mathcal{E},\, t \ge 0,\, k = 1,2. \tag{2.3}$$

Obviously, if $A \in \mathcal{M}$, then $A \cap (E_n \cup \{b_n\}) \in \mathcal{M}_n$. Therefore, the assertion follows, by combining (2.3) with $P_1^{(n)}(t) \le P_2^{(n)}(t)$.

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