单生过程的遍历理论

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Contents

Criteria for ℓ-ergodicity

Set $m_{ij}^{(\ell)} = \mathbb{E}_i \sigma_j^{\ell}$.

For a positive integer ℓ, the recurrent chain $P(t)$ is said to be ℓ-ergodic if $m_{jj}^{(\ell)} < \infty$ for some (and hence for all) $j \in E$.

1-ergodic $=$ positive recurrent (ergodic),
0-ergodic $=$ null recurrent.

Criteria for ℓ-ergodicity

Discrete time:

- Kemeny, J.G., Snell, J.L. and Knapp, A.W.(1976)
- Mao, Y.-H.(2003)
- Hou, Z.-T. and Liu, Y.-Y.(2003). Queue Theory

Continuous time:

- Coolen-Schrijner, P. and van Doorn, E.A.(2002)
- Mao, Y.-H.(2004)

Criteria for ℓ-ergodicity

- ℓ-ergodicity provides an algebraic convergence rate: $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \to \infty$.

Criteria for ℓ-ergodicity

- ℓ-ergodicity provides an algebraic convergence rate: $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \to \infty$.
- Given a regular birth-death (a_i, b_i). Assume that $P(t)$ is recurrent.

  $P(t)$ is ℓ-ergodic if and only if $\sum_{i=1}^{\infty} \mu_i m_{i0}^{(\ell-1)} < \infty$,

$$m_{i0}^{(n)} = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k m_{k0}^{(n-1)}, \quad i \geq 1, \, n \geq 1.$$

Criteria for ℓ-ergodicity

Q-process is ℓ-ergodic if and only if $\sum_{i=1}^{\infty} \pi_i m_{i0}^{(\ell-1)} < \infty$.

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+1}}\Big(\, 1 \, + \sum_{k=0}^{i-1} q_i^{(k)} d_k \Big), \quad i \geq 1.$$

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For $\ell \geq 1$, the single birth process is ℓ-ergodic if and only if $d^{(\ell)} < \infty$.

Criteria for ℓ-ergodicity

Q-process is ℓ-regodic if and only if for some (and then for all) $j \in E$, the system of inequalities

$$\begin{cases} \sum_k q_{ik} y_k \leq -\ell m_{ij}^{(\ell-1)}, \quad i \neq j, \\ \sum_{k \neq j} q_{jk} y_k < \infty \end{cases}$$

has a nonnegative finite solution.

Criteria for ℓ-ergodicity

给定正则不可约 Q 过程. 则 $(m_{i0}^{(n)})$ 是下列方程的最小非负解:

$$x_0 = 0, \qquad x_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} x_j + \frac{n}{q_i} m_{i0}^{(n-1)}, \quad i \geq 1.$$

Key : $m_{10}^{(\ell)} = \ell d^{(\ell)}, \quad m_{i0}^{(\ell)} = \ell \sum_{j=0}^{i-1} (F_j^{(0)} d^{(\ell)} - d_j^{(\ell)}).$

$$m_{00}^{(\ell)} = \ell! \Big(q_{01}^{-\ell} + \sum_{k=1}^{\ell} (k! q_{01}^{\ell-k})^{-1} m_{10}^{(k)} \Big), \quad n \geq 1.$$

Exponential ergodicity

Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0$.

Mao, Y.-H. and Zhang, Y.-H.(2004).

$$\inf_i q_i > 0 \text{ and } M := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty.$$

$\Longrightarrow$ exponential ergodicity.

Example. Let $q_{n,n+1} = 1$ for all $n \geq 0$, $q_{10} = 1$, $q_{n,n-2} = 1$ for all $n \geq 2$ and $q_{ij} = 0$ for other $i \neq j$. Then the single birth process is exponentially ergodic and not strongly ergodic.

Exponential ergodicity

Chen, M.-F.(2000). For birth-death processes,

$$\text{exponentially ergodic} \iff \delta := \sup_{i>0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{j=i}^{\infty} \mu_j < \infty.$$

$$\lambda_1 > 0 \iff \delta < \infty. \quad (\delta = M)$$

$$\lambda_1 = \text{ exponential convergence rate.}$$

Key: birth-death processes are reversible!

Exponential ergodicity

$P(t)$ is exponentially ergodicity if and only if for some $\lambda > 0$ with $\lambda < q_i$ for all i,

$$\begin{cases} \sum_j q_{ij} y_j \le -\lambda y_i - 1, & i \notin H \\ \sum_{i\in H} \sum_{j\ne i} q_{ij} y_j < \infty \end{cases}$$

has a nonnegative finite solution (y_i).

"$\inf_i q_i > 0$" is necessary essentially for exponential ergodicity.

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Difficulty: two parameters!

Exponential ergodicity

Mao and Zhang (2004) gives another proof.

Keys:

1. Construct a test function with double summations for sufficiency.

The construction comes from the study of spectral gap essentially.

Exponential ergodicity

2. Necessity. Note that

$$m_{i0}^{(n)} \ge n\left(\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=i}^{\infty}\mu_k\right)m_{i0}^{(n-1)}$$

$$\ge \cdots \ge n!\left(\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=i}^{\infty}\mu_k\right)^n.$$

$$\mathbb{E}_i e^{\lambda\sigma_0} < \infty \Rightarrow \sum_{n=1}^{\infty}\left(\lambda\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=i}^{\infty}\mu_k\right)^n < \infty$$

$$\Rightarrow \lambda\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=i}^{\infty}\mu_k < 1$$

$$\Rightarrow \delta \le \lambda^{-1} < \infty.$$

Exponential ergodicity

Unfortunately, "$M < \infty$" is not necessary for exponential ergodicity.

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Example. Given $q_{01} > 0$ arbitrarily. Let $q_{i,i+1} = i^{\gamma}$, $q_{i0} = i^{\gamma-1}$ for all $i \geq 1$ where the constant $\gamma \in (1,2)$, and $q_{ij} = 0$ for other $i \neq j$. Obviously, the Q-matrix is irreducible. It is easily computed that $F_i^{(0)} = 1$ for all $i \geq 0$. So the single birth process is recurrent and furthermore regular. Note that $\inf_{i \geq 1} q_{i0} = 1 > 0$. It follows that the single birth process is strongly ergodic and furthermore exponentially ergodic immediately. But $M = \infty$.

Exponential ergodicity

Conjecture that the criterion should be

$$\delta' := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \left(d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

For birth-death processes, $\delta = M = \delta'$.

Exponential ergodicity

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For birth-death processes, $\delta = M = \delta'$.

Now, for the single birth Q-matrix (q_{ij}), define a conservative birth-death Q-matrix (a_i, b_i) as follows.

$$a_i = \frac{q_{i,i+1} F_i^{(0)}}{F_{i-1}^{(0)}}, \quad i \geq 1; \qquad b_i = q_{i,i+1}, \quad i \geq 0.$$

Then $\mathbb{E}_i \tau_0^n \leq \mathbb{E}_i \overline{\tau}_0^n \, (i \geq 1, n \geq 0)$, furthermore $\mathbb{E}_i \mathrm{e}^{\lambda \tau_0} \leq \mathbb{E}_i \mathrm{e}^{\lambda \overline{\tau}_0} \, (i \geq 1, \lambda > 0)$.

Exponential ergodicity

ℓ-ergodicty and exponential ergodicity:

Theorem. Given a regular and irreducible single birth Q-matrix (q_{ij}) with $\inf_i q_i > 0$. Suppose that the corresponding single birth process $X(t)$ is recurrent. Then $X(t)$ is exponentially ergodic if and only if $d^{(\ell)}/(\ell-1)! \le \gamma^\ell$ for some positive constant γ and all sufficiently large ℓ; equivalently, $\overline{\lim}_{\ell\to\infty} \sqrt[\ell]{d^{(\ell)}}/\ell < \infty$.

Exponential ergodicity

现在回到 0 的首中时的指数阶矩: $\mathbb{E}_i e^{\lambda\tau_0}$, 其中 λ是一正常数.
对所有的 $0 \le j < i$, 定义 $q_i^{(j)}(\lambda) := q_i^{(j)} - \lambda$,

$$F_n^{(n)}(\lambda) = 1, \; F_n^{(i)}(\lambda) = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)}(\lambda) F_k^{(i)}(\lambda), \quad 0 \le i < n,$$

$$d_0(\lambda) = 0, \; d_n(\lambda) = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)}(\lambda) d_k(\lambda)\right), \quad n \ge 1.$$

令

$$d(\lambda) = \sup_{i \ge 1} \frac{\sum_{j=0}^{i-1} d_j(\lambda)}{\sum_{j=0}^{i-1} F_j^{(0)}(\lambda)} = \sup_{i \ge 0} \frac{\sum_{j=0}^{i} d_j(\lambda)}{\sum_{j=0}^{i} F_j^{(0)}(\lambda)}.$$

Exponential ergodicity

类似地:

$$F_i^{(j)}(\lambda) = \sum_{k=j+1}^{i} \frac{F_i^{(k)}(\lambda) q_k^{(j)}(\lambda)}{q_{k,k+1}}, \ i > j \geq 0;$$

$$d_i(\lambda) = \sum_{k=1}^{i} \frac{F_i^{(k)}(\lambda)}{q_{k,k+1}}, \ i \geq 1.$$

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定理. 假定存在充分小的 $\lambda > 0$ 使得 $\sum_{j=0}^{i-1} F_j^{(0)}(\lambda) > 0$ 对一切 $i \geq 1$ 成立. 则单生过程指数遍历当且仅当 $d(\lambda) < \infty$, 其中 $0 < \lambda < q_i \ (i \geq 0)$.

Exponential ergodicity

Example. Let $q_{i,i+1} =: \beta_i > 0\,(i \geq 0)$, $q_{i0} = \alpha_i > 0\,(i \geq 1)$ and $q_{ij} = 0$ for other $i \neq j$. Obviously, the Q-matrix is irreducible. Suppose that the Q-matrix is regular. Now $\sum_{j=0}^{i-1} F_j^{(0)}(\lambda) > 0$ for all $\lambda > 0$ and all $i \geq 1$. Then the single birth process is exponentially ergodic if and only if

$$d(\lambda) = \sum_{i=1}^{\infty} \frac{1}{\beta_i} \prod_{j=1}^{i} \frac{\beta_j}{q_k - \lambda} < \infty$$

for some $\lambda > 0$ with $\lambda < q_i\,(i \geq 0)$.

Exponential ergodicity

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for some $\lambda > 0$ with $\lambda < q_i\,(i \geq 0)$.

Criteria for ergodicity

例: 给定正则不可约单生 Q 矩阵 $Q = (q_{ij})$.
若 $c := \inf_{k\geq 1} q_{k0} > 0$, 取 $\lambda \in (0, c)$. 则对一切 $0 \leq j < i$ 有 $q_i^{(j)}(\lambda) > 0$; 继而, 对所有 $i \geq 0$ 有 $F_i^{(0)}(\lambda) > 0$.
因此, 取定 $0 < \lambda < q_i\ (i \geq 0)$. 由合分比性质, 得到

$$d(\lambda) \leq \sup_{i\geq 1} \frac{d_i(\lambda)}{F_i^{(0)}(\lambda)} \leq \sup_{k\geq 1} \frac{1}{q_k^{(0)}(\lambda)} = \frac{1}{c-\lambda} < \infty.$$

此单生过程是指数遍历的. 已知是强遍历的!

A class of single birth processes

Given a regular and irreducible single birth Q-matrix (q_{ij}):

$$q_{i+1,j} = p_i q_{ij}, \qquad 0 \le j \le i-1,$$

where $0 \le p_i \le 1\,(i \ge 1)$. In convention $p_0 = 0$. Assume that the single birth is recurrent. Then

(1) The process is ℓ-ergodic if and only if $\sum_{k=1}^{\infty}(m_{k0}^{(\ell-1)} - p_{k-1}m_{k-1,0}^{(\ell-1)})/(q_{k,k+1}F_k^{(0)}) < \infty$. In particular, the process is ergodic if and only if $d = \sum_{i=1}^{\infty}(1-p_{i-1})/(q_{i,i+1}F_i^{(0)}) < \infty$, where $F_0^{(0)} = 1$, $F_i^{(0)} = \prod_{j=1}^{i}(q_{j,j-1} + p_{j-1}q_{j-1})/q_{j,j+1}\,(i \ge 1)$.

A class of single birth processes

(2) The process is strongly ergodic if and only if

$$\sum_{i=0}^{\infty} F_i^{(0)} \sum_{j=i+1}^{\infty} \frac{1-p_{j-1}}{q_{j,j+1} F_j^{(0)}} < \infty.$$

A class of single birth processes

(3) If the process is exponentially ergodic, then $q := \inf_{i\geq 0} q_i > 0$ and

$$\delta' := \sup_{i\geq 1} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1-p_{j-1}}{q_{j,j+1} F_j^{(0)}} < \infty.$$

(4) Moreover, assume that $\sup_{i\geq 1} p_i < 1$. If $q > 0$ and $\delta' < \infty$, then the process is exponentially ergodic.

A class of single birth processes

- When $p_i = 0\,(i \geq 1)$, it is the birth-death case: $\delta' = \delta$.

A class of single birth processes

- When $p_i = 0\,(i \geq 1)$, it is the birth-death case: $\delta' = \delta$.
- When $c := \sup_{i\geq 1} p_i < 1$, exp.erg.$\Longleftrightarrow \delta' < \infty$ and $q > 0$.

  But $(1-c)M \leq \delta' \leq M$.
  So exp.erg.$\Longleftrightarrow M < \infty$ and $q > 0$.

  Question: how to improve for some $p_i = 1$?

Applications

To study sufficient conditions for these problems of multidimensional Q-processes, we use the comparison method, i.e., compare them with a single birth Q-processes. The method reduces the multidimensional problems to one-dimensional ones.

The typical example is Schögl's model: a model of chemical reaction with diffusion in a container.

Applications

Let S be a finite set and $E = \mathbb{Z}_+^S$. The model is defined by the following Q-matrix $Q = (q(x, y) : x, y \in E)$:

$$q(x,y) = \begin{cases} \lambda_1 \binom{x(u)}{2} + \lambda_4, & \text{if } y = x + e_u, \\ \lambda_2 \binom{x(u)}{3} + \lambda_3 x(u), & \text{if } y = x - e_u, \\ x(u)p(u,v), & \text{if } y = x - e_u + e_v, \\ 0, & \text{other } y \neq x, \end{cases}$$

$q(x) = -q(x, x) = \sum_{y \neq x} q(x, y)$, where $x = (x(u) : u \in S)$, $\binom{n}{k}$ is the usual combination, $\lambda_1, \cdots, \lambda_4$ are positive constants. $(p(u, v) : u, v \in S)$ is a transition probability matrix on S and e_u is the element in E having value 1 at u and 0 elsewhere.

Applications

Let E be a countable set. $Q = (q(x,y) : x, y \in E)$ be a conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\sum_{k=0}^{\infty} E_k = E$ and

(i) If $q(x,y) > 0$ and $x \in E_k$, then $y \in \sum_{j=0}^{k+1} E_j$ for all $k \geq 0$.

(ii) $\sum_{y \in E_{k+1}} q(x,y) > 0$ for all $x \in E_k$ and all $k \geq 0$.

(iii) $C_k := \sup\{q(x) : x \in E_k\} < \infty$ for all $k \geq 0$.

Define a conservative single birth Q-matrix $Q = (q_{ij})$:

$$q_{ij} = \begin{cases} \sup\{\sum_{y \in E_j} q(x,y) : x \in E_i\}, & \text{if } j = i+1; \\ \inf\{\sum_{y \in E_j} q(x,y) : x \in E_i\}, & \text{if } j < i; \\ 0, & \text{other cases of } j \neq i. \end{cases}$$

Applications

(0) If the (q_{ij})-process is unique (i.e. $R = \infty$), then so is the $(q(x,y))$-process.

Now suppose that $E_0 = \{\theta\}$ where $\theta \in E$ is a reference point, and that both $(q(x,y))$ and (q_{ij}) are irreducible and (q_{ij}) is regular.

(1) Moreover, assume that E_k is finite for all $k \geq 1$. If the (q_{ij})-process is recurrent (i.e. $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$), then so is the $(q(x,y))$-process.

Applications

(2) If $\widehat{d} := \sup_{k\geq 0} d_k/F_k^{(0)} < \infty$, then both processes are ergodic.

(3) If $\inf_i q_i > 0$ and $M = \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} (q_{j,j+1} F_j^{(0)})^{-1} < \infty$, then both processes are exponentially ergodic.

(4) If $\sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} \widehat{d} - d_n) < \infty$, then both processes are strongly ergodic.

Keys: increasing solution of equations.

Applications

The Q-process corresponding to finite dimensional Schlögl model is strongly ergodic.

The method is efficient for necessary condition.

The Q-process corresponding to Brusselator model is not strongly ergodic.

Related topics

- 允许 $N := \max\{i+1 : q_{i,i+1} = 0\} < \infty$.
  当 $N = \max\emptyset = 0$ 时, 回到单生 Q 矩阵. 当 $N \geq 1$ 时, 称之为单生型 Q 矩阵. 有吸收态.

  Chen, M.-F.(1999), Single birth processes, Chin. Ann. Math., **20B**, 77-82.
  统一处理唯一性, $\mathbb{P}_i(\tau_N < \infty) = 1$, $\mathbb{E}_i\tau_N < \infty\,(i \geq N)$, 这里 $\tau_N = \inf\{t \geq 0 : X(t) \leq N-1\}$.

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- 允许 $q_{0j} \geq 0\,(j \geq 1)$, 称之为带移民的单生 Q 矩阵.

  张余辉, 赵倩倩, 带移民的单生过程, 预印本, 2007.

  处理这些问题: 唯一性, 常返性, 各种遍历性

Related topics

- 广义生灭过程

  吴群英, 张汉君(2003), 广义生灭过程, 系统科学与数学, **23**(4), 517-528.

  吴群英(2004), 广义生灭过程, 科学出版社, 北京.

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- 遍历速度: 指数, 强遍历

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- 遍历速度: 指数, 强遍历
- 谱空隙
- Lipschitz norm

  Ma, Y.-T., Lipschitz norm for Q-matrix of single birth processes, preprint, 2007

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- 用: 粒子系统

Thanks

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http://math.bnu.edu.cn/~zhangyh/