单生过程的遍历理论

张余辉

北京师范大学数学科学学院

(武汉大学, 2007年8月3日至4日)

Contents

Background

Single birth Q-matrix $Q = (q_{ij} : i, j \geq 0)$:
$q_{i,i+1} > 0$, $q_{ij} = 0$ if $j > i + 1$ for all $i \geq 0$.

$$\begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. irreversible;
2. single extremal point, single parameter;
3. applications in other fields.
4. applications in mathematics.

So the explicit criteria are expected.

Background

Upwardly skip-free processes (W.J. Anderson, 1991);

generalized birth and death processes (J.K. Zhang, 1984);

Background

Upwardly skip-free processes (W.J. Anderson, 1991);

generalized birth and death processes (J.K. Zhang, 1984);

Population processes: birth and death processes with catastrophes; birth, death and catastrophe processes

Background

Upwardly skip-free processes (W.J. Anderson, 1991);

generalized birth and death processes (J.K. Zhang, 1984);

Population processes: birth and death processes with catastrophes; birth, death and catastrophe processes

$$
\begin{array}{lll}
i \to i+1 & \text{at rate} & b_i \\
\quad \to i-1 & & a_i + c_i f_{i,i-1} \\
\quad \to i-2 & & c_i f_{i,i-2} \\
\quad \to \cdots & & \ldots \\
\quad \to 0 & & c_i f_{i0}
\end{array}
$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes f_{ij} ($b_i = b + \lambda i$, $a_i = 0$, $c_i = ci$, $i \geq 0$).

geometric: $f_{ij} = p(1-p)^{i-1-j}\,(1 \leq j < i); f_{i0} = (1-p)^{i-1}$;

uniform: $f_{ij} = 1/i\,(0 \leq j < i)$;

binomial: $f_{ij} = \binom{i-1}{j} p^j (1-p)^{i-1-j}\,(0 \leq j < i)$.

Aim: extinction times and probability of extinction.

Keys: generating function of Q-resolvent.

Background

B. Cairns & P. Pollett (2004).

$$
\begin{aligned}
i &\rightarrow i+1 && \text{at rate} && g_i b \\
&\rightarrow i-1 && && g_i d_1 \\
&\rightarrow \cdots && && \cdots \\
&\rightarrow 1 && && g_i d_{i-1} \\
&\rightarrow 0 && && g_i \sum_{k \geq i} d_k
\end{aligned}
$$

where $b + \sum_{k \geq 1} d_k = 1$.

Background

single birth process:
Yan, S.-J. and Chen, M.-F.(1986), Multidimensional Q-processes, Chinese Ann. Math., **7(B)**(1), 90-110.

Chen, M.-F.(2004), From Markov Chains to Non-Equilibrium Particle Systems, Second Edition, World Scientific, Singapore.

陈木法, 毛永华(2007), 随机过程导论, 高等教育出版社, 北京.

Population processes:
Anderson, W. J.(1991), Continuous-Time Markov Chains, Springer-Verlag, New York.

Criteria for uniqueness

Backward Kolmogorov equation: $P'(t) = QP(t)$.
Forward Kolmogorov equation: $P'(t) = P(t)Q$.

Q-process: $P'(t)|_{t=0} = Q$.
Q-process exist? unique?

Let $Q = (q_{ij})$ be totally stable and conservative on countable E, i.e.
$q_{ij} \geq 0 (i \neq j)$, $q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty (i \in E)$.

- Every Q-process satisfies the backward Kolmogorov equation.
- Q-process always exists. The minimal process.

Criteria for uniqueness

P_t is determined uniquely if and only if for some (equivalently, all) $\lambda > 0$,

$$u_i = \sum_{j \neq i} q_{ij} u_j / (\lambda + q_i), \quad 0 \le u_i \le 1,\ i \ge 0$$

has only a trivial solution.

此判别准则的优美之处是 Q 矩阵, 而且方程的最大解也存在迭代算法. 但对单生过程, 一个更好的判别准则—显式的、完全可计算的, 是期盼的.

Criteria for uniqueness

- Zhang, J.-K.(1984), Yan, S.-J. and Chen, M.-F.(1986).

  For $0 \le k < n$, define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ and

$$m_0 = \frac{1}{q_{01}}, \qquad m_n = \frac{1}{q_{n,n+1}}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k\right), \quad n \ge 1,$$

$$F_n^{(n)} = 1, \qquad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n.$$

Criteria for uniqueness

- Zhang, J.-K.(1984), Yan, S.-J. and Chen, M.-F.(1986).

  For $0 \le k < n$, define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ and

$$m_0 = \frac{1}{q_{01}}, \qquad m_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k\right), \quad n \ge 1,$$

$$F_n^{(n)} = 1, \qquad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n.$$

- Uniqueness (regurlarity) $\Longleftrightarrow R := \sum_{n=0}^{\infty} m_n = \infty$.

Criteria for uniqueness

对生灭 Q 矩阵 (a_i, b_i), 有简单的形式:

$$m_n = \frac{1}{\mu_n b_n}\mu[0,n], \qquad n \geq 0,$$

其中

$$\mu_0 = 1, \qquad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geq 1,$$

且 $\mu[i,k] = \sum_{j=i}^{k} \mu_j$.

给定生灭 Q 矩阵 (a_i, b_i). 则生灭过程唯一当且仅当 $R := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n}\mu[0,n] = \infty$.

Criteria for recurrence

- $P(t)$ is recurrent: discrete chain $P(h)$ is recurrent for all $h > 0 \Longleftrightarrow \int_0^\infty p_{ii}(t)\mathrm{d}t = \infty$.

Criteria for recurrence

- $P(t)$ is recurrent: discrete chain $P(h)$ is recurrent for all $h > 0 \iff \int_0^\infty p_{ii}(t)\mathrm{d}t = \infty$.
- For a regular and irreducible $Q = (q_{ij})$, $P(t)$ ie recurrent if and only if for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} q_{ij} x_j / q_i, \qquad 0 \leq x_i \leq 1,\ i \geq 0$$

has only a trivial solution.
regular: totally stable, conservative and Q-process is unique.

Criteria for recurrence

- $P(t)$ is recurrent: discrete chain $P(h)$ is recurrent for all $h > 0 \iff \int_0^\infty p_{ii}(t)\mathrm{d}t = \infty$.
- For a regular and irreducible $Q = (q_{ij})$, $P(t)$ ie recurrent if and only if for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} q_{ij} x_j / q_i, \qquad 0 \leq x_i \leq 1,\ i \geq 0$$

  has only a trivial solution.
  regular: totally stable, conservative and Q-process is unique.
- Suppose that the Q-matrix is regular and irreducible, then
  Recurrence $\iff \sum_{n=0}^\infty F_n^{(0)} = \infty$.

Criteria for recurrence

对生灭 Q 矩阵 (a_i, b_i), 有简单的形式:

$$F_n^{(0)} = \frac{b_0}{\mu_n b_n}, \qquad n \geq 0.$$

给定生灭 Q 矩阵 (a_i, b_i). 则生灭过程常返当且仅当 $\sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} = \infty$.

Criteria for ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0$.

Criteria for ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0$.
- Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$.

Criteria for ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0.$
- Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0.$
- Strong ergodicity: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$

$$\Longleftrightarrow \lim_{t\to\infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Criteria for ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0$.
- Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$
$$\iff \lim_{t\to\infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$
- Strong ergodicity $\Rightarrow$ Exponential ergodicity $\Rightarrow$ Ordinary ergodicity.

Criteria for ergodicity

- Let $H \neq \emptyset$ be a finite subset of E. $P(t)$ is ergodic if and only if the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite nonnegative solution.

Criteria for ergodicity

- Let $H \neq \emptyset$ be a finite subset of E. $P(t)$ is ergodic if and only if the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite nonnegative solution.

- Take $H = \{0\}$. For single birth Q-matrix,

$$\sum_{j \neq i} q_{ij}(y_j - y_i) + 1 \leq 0, \qquad i \geq 1.$$

Criteria for ergodicity

- Define

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}}\Big(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\Big), \quad n \geq 1.$$

实际上, 三组记号可以用统一表示, 因为

Criteria for ergodicity

- Define

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}}\Big(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\Big), \quad n \geq 1.$$

实际上, 三组记号可以用统一表示, 因为

- $$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \quad d_n = \sum_{k=1}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \quad n \geq 0.$$

Criteria for ergodicity

- Define

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}}\Big(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k\Big), \quad n \geq 1.$$

实际上, 三组记号可以用统一表示, 因为

- $$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \quad d_n = \sum_{k=1}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \quad n \geq 0.$$

- $$m_n = \frac{F_n^{(0)}}{q_{01}} + d_n, \quad n \geq 0.$$

Criteria for ergodicity

- $$F_n^{(i)} = \sum_{k=i+1}^{n} \frac{F_n^{(k)} q_k^{(i)}}{q_{k,k+1}}, \quad n > i \geq 0.$$

Criteria for ergodicity

- $$F_n^{(i)} = \sum_{k=i+1}^{n} \frac{F_n^{(k)} q_k^{(i)}}{q_{k,k+1}}, \quad n > i \geq 0.$$

- Suppose that the Q-matrix is regular and irreducible, then
  Ergodicity $\iff$

$$d := \sup_{k \geq 0} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}} < \infty.$$

Criteria for ergodicity

Sketch of the proof:

- 设 (u_i) 是方程的有限非负解. 取 $u_0 = 0$. 用归纳法证得

$$u_{n+1} - u_n \leq F_n^{(0)} u_1 - d_n, \qquad n \geq 0.$$

Criteria for ergodicity

Sketch of the proof:

- 设 (u_i) 是方程的有限非负解. 取 $u_0 = 0$. 用归纳法证得

$$u_{n+1} - u_n \leq F_n^{(0)} u_1 - d_n, \qquad n \geq 0.$$

-

$$0 \leq u_{n+1} \leq u_1 \sum_{k=0}^{n} F_k^{(0)} - \sum_{k=0}^{n} d_k, \qquad n \geq 0.$$

Criteria for ergodicity

Sketch of the proof:

- 设 (u_i) 是方程的有限非负解. 取 $u_0 = 0$. 用归纳法证得

$$u_{n+1} - u_n \le F_n^{(0)} u_1 - d_n, \qquad n \ge 0.$$

- $$0 \le u_{n+1} \le u_1 \sum_{k=0}^{n} F_k^{(0)} - \sum_{k=0}^{n} d_k, \qquad n \ge 0.$$

- $d \le u_1 < \infty$.

Criteria for ergodicity

Sketch of the proof:

- 设 (u_i) 是方程的有限非负解. 取 $u_0 = 0$. 用归纳法证得

$$u_{n+1} - u_n \le F_n^{(0)} u_1 - d_n, \qquad n \ge 0.$$

-

$$0 \le u_{n+1} \le u_1 \sum_{k=0}^{n} F_k^{(0)} - \sum_{k=0}^{n} d_k, \qquad n \ge 0.$$

- $d \le u_1 < \infty$.
- 反之, 假设 $d < \infty$. 令

$$u_0 = 0, \quad u_1 = d, \quad u_{n+1} = u_n + F_n^{(0)} u_1 - d_n, \qquad n \ge 0.$$

Criteria for ergodicity

Sketch of the proof:

- 设 (u_i) 是方程的有限非负解. 取 $u_0 = 0$. 用归纳法证得

$$u_{n+1} - u_n \le F_n^{(0)} u_1 - d_n, \qquad n \ge 0.$$

-

$$0 \le u_{n+1} \le u_1 \sum_{k=0}^{n} F_k^{(0)} - \sum_{k=0}^{n} d_k, \qquad n \ge 0.$$

- $d \le u_1 < \infty$.
- 反之, 假设 $d < \infty$. 令

$$u_0 = 0, \quad u_1 = d, \quad u_{n+1} = u_n + F_n^{(0)} u_1 - d_n, \qquad n \ge 0.$$

- 验证 (u_n) 是方程的一个有限非负解.

Criteria for ergodicity

例: 给定正则不可约单生 Q 矩阵 $Q = (q_{ij})$. 若 $\inf_{k\geq 1} q_{k0} > 0$, 则单生过程遍历. 因为由合分比性质及可得

$$d \leq \sup_{i\geq 1} \frac{d_i}{F_i^{(0)}} \leq \sup_{k\geq 1} \frac{1}{q_k^{(0)}} = \frac{1}{\inf_{k\geq 1} q_{k0}}.$$

所以 $d < \infty$.

实际上, 此过程是强遍历的!

Criteria for ergodicity

对生灭 Q 矩阵 (a_i, b_i), 有简单的形式:

$$d_n = \frac{1}{\mu_n b_n}\mu[1, n], \qquad n \geq 0.$$

给定正则生灭 Q 矩阵 (a_i, b_i). 则生灭过程遍历当且仅当 $\mu[0, \infty) < \infty$.

Criteria for strong ergodicity

Strong ergodicity: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$

$$\iff \lim_{t\to\infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Let $H \neq \emptyset$ be a finite subset of E. Then $P(t)$ is strongly ergodic if and only if the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i\in H} \sum_{j\neq i} q_{ij} y_j < \infty \end{cases}$$

has a bounded nonnegative solution.

Criteria for strong ergodicity

- Zhang, Y.-H.(2001). Suppose that the Q-matrix is irreducible and regular. Then
  Strong ergodicity $\Longleftrightarrow$
  $S := \sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty$.

Criteria for strong ergodicity

- Zhang, Y.-H.(2001). Suppose that the Q-matrix is irreducible and regular. Then
  Strong ergodicity $\Longleftrightarrow$
  $S := \sup_{k\geq 0}\sum_{n=0}^{k}(F_n^{(0)}d - d_n) < \infty$.
- Zhang, H.-J., Lin, X. and Hou, Z.-T. (1998, 2000).

  For regular birth-death (a_i, b_i),

$$\text{strong ergodicity} \Longleftrightarrow S := \sum_{n=0}^{\infty}\frac{1}{\mu_n b_n}\sum_{k=n+1}^{\infty}\mu_k < \infty.$$

Criteria for strong ergodicity

- Zhang, Y.-H.(2001). Suppose that the Q-matrix is irreducible and regular. Then
  Strong ergodicity $\Longleftrightarrow$
  $S := \sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty$.
- Zhang, H.-J., Lin, X. and Hou, Z.-T. (1998, 2000).

  For regular birth-death (a_i, b_i),

  $$\text{strong ergodicity} \Longleftrightarrow S := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} \sum_{k=n+1}^{\infty} \mu_k < \infty.$$

- Chen, M.-F.(1992).

  $$S < \infty \Longrightarrow \text{exponential ergodic.}$$

Criteria for strong ergodicity

Sketch of the proof:
首先, 证明方程

$$y_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i}, \quad i \geq 1; \quad y_0 = 0 \quad (*)$$

有一有界非负解当且仅当 $S < \infty$. 此时,

$$d = \lim_{n \to \infty} \sum_{k=0}^{n} d_k \bigg/ \sum_{k=0}^{n} F_k^{(0)},$$

而且方程$(*)$的唯一解有下面的形式:

$$y_0 = 0, \quad y_1 = d, \quad y_{n+1} = y_n + F_n^{(0)} y_1 - d_n, \quad n \geq 1. \quad (**)$$

Criteria for strong ergodicity

Sketch of the proof:

- 假设单生过程强遍历, 则方程存在一有界非负解 (u_i), 即

$$u_i \geq \sum_{j \neq i} \frac{q_{ij}}{q_i} u_j + \frac{1}{q_i}, \quad i \geq 1; \quad u_0 \geq 0.$$

记 (u_i^*) 是方程(*)的最小非负解. 则由比较定理得到 $u_i \geq u_i^*$ $(i \geq 0)$. 因此, (u_i^*) 有界且方程(*)有一有界非负解 $\Rightarrow S < \infty$.

Criteria for strong ergodicity

Sketch of the proof:

- 假设单生过程强遍历, 则方程存在一有界非负解 (u_i), 即

$$u_i \geq \sum_{j\neq i} \frac{q_{ij}}{q_i} u_j + \frac{1}{q_i}, \quad i \geq 1; \quad u_0 \geq 0.$$

  记 (u_i^*) 是方程(∗)的最小非负解. 则由比较定理得到 $u_i \geq u_i^*$ $(i \geq 0)$. 因此, (u_i^*) 有界且方程(∗)有一有界非负解 $\Rightarrow S < \infty$.

- 反之, 设 $S < \infty$. 由(∗∗)定义 (y_i). 则 (y_i) 是方程(∗)的有界非负解. 显然 (y_i) 也是方程的有界非负解 $\Rightarrow$ 单生过程的强遍历性.

Criteria for strong ergodicity

命题: 给定正则不可约的 Q 矩阵$Q = (q_{ij})$. 若存在 $k \ge 0$ 满足 $c := \inf_{i \neq k} q_{ik} > 0$, 则相　的 Q 过程是强遍历的.

例: 给定正则不可约单生 Q 矩阵 $Q = (q_{ij})$. 若 $\inf_{k \ge 1} q_{k0} > 0$, 则单生过程强遍历.

Stationary distribution

Can we get the stationary distribution?

How to use $F_n^{(i)}$?

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n.$$

Stationary distribution

Can we get the stationary distribution?

How to use $F_n^{(i)}$?

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n.$$

Stationary distribution (Zhang, Y.-H.(2004)):

$$c_k := \sup_{i \ge k} \frac{\sum_{j=k}^{i} m_j}{\sum_{j=k}^{i} F_j^{(k)}}, \quad \pi_k = \frac{1}{q_{k,k+1} c_k}, \quad k \ge 0.$$

Stationary distribution

- 考虑生灭过程: 生速 $b_i = q_{i,i+1} > 0\,(i \geq 0)$, 死速 $a_i = q_{i,i-1} > 0\,(i \geq 1)$. 定义 $\mu_0 = 1$, $\mu_i = b_0 \cdots b_{i-1}/a_1 \cdots a_i\,(i \geq 1)$. 再令$\mu = \sum_{i=0}^{\infty} \mu_i$. 得到 $c_i = \mu/(\mu_i b_i)$, 平稳分布为 $\pi_i = \mu_i/\mu\,(i \geq 0)$.

Stationary distribution

- 考虑生灭过程: 生速 $b_i = q_{i,i+1} > 0\,(i \geq 0)$, 死速 $a_i = q_{i,i-1} > 0\,(i \geq 1)$. 定义 $\mu_0 = 1$, $\mu_i = b_0 \cdots b_{i-1}/a_1 \cdots a_i\,(i \geq 1)$. 再令$\mu = \sum_{i=0}^{\infty} \mu_i$. 得到 $c_i = \mu/(\mu_i b_i)$, 平稳分布为 $\pi_i = \mu_i/\mu\,(i \geq 0)$.
- 令单生 Q 矩阵为: $q_{i,i+1} = 1\,(i \geq 0)$, $q_{10} = 1$ 且 $q_{i,i-2} = 1\,(i \geq 2)$, 其它的 $q_{ij} = 0\,(i \neq j)$. 则 $\{F_j^{(k)}\}_{j=k}^{\infty}$ 是Fibonacci数列:

$$F_{k+n}^{(k)} = \frac{1}{\sqrt{5}}[A^{n+1} - (-B)^{n+1}], \qquad n \geq 0, k \geq 0,$$

其中 $A = (\sqrt{5}+1)/2, B = (\sqrt{5}-1)/2$. 得到 $c_k = A^{k+2}$, 平稳分布为 $\pi_k = B^{k+2}\,(k \geq 0)$.

Stationary distribution

- 令单生 Q 矩阵为: $q_{i,i+1} = 1\,(i \geq 0)$ 且 $q_{i0} = 1\,(i \geq 1)$, 其它的 $q_{ij} = 0\,(i \neq j)$. 该过程是强遍历的. 得到 $c_k = 2^{k+1}$, 平稳分布为 $\pi_k = 2^{-k-1}\,(k \geq 0)$.

Stationary distribution

- 令单生 Q 矩阵为: $q_{i,i+1} = 1\,(i \geq 0)$ 且 $q_{i0} = 1\,(i \geq 1)$, 其它的 $q_{ij} = 0\,(i \neq j)$. 该过程是强遍历的. 得到 $c_k = 2^{k+1}$, 平稳分布为 $\pi_k = 2^{-k-1}\,(k \geq 0)$.

- 考虑种群理论中的均匀大灾难模型,
即 $q_{i,i+1} = \lambda_i := a + \lambda i\,(i \geq 0)$, $q_{ij} = \beta\,(0 \leq j < i)$, 其它的 $q_{ij} = 0\,(i \neq j)$, 其中的 a, λ, β 均为正常数. 设该 Q 矩阵正则. 则强遍历且平稳分布为

$$\pi_0 = \frac{\beta}{a+\beta}, \quad \pi_k = \frac{(k+1)\beta}{a+\beta} \prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_{j+1} + (j+2)\beta}, \quad k \geq 1.$$

别当 $a = \lambda$ 时, 则 $\pi_k = \beta a^k/(a+\beta)^{k+1}\,(k \geq 0)$. 进一步, 当 $a = \lambda = \beta = 1$ 时, 则 $\pi_k = 2^{-k-1}\,(k \geq 0)$.

Stationary distribution

Set $\sigma_j = \inf\{t \geq$ the first jumping time: $X(t) = j\}$ and $\tau_j = \inf\{t \geq 0 : X(t) = j\}$.

任意给定 $i_0 \geq 0$. 单生过程的 i_0 首中时的一阶矩可由如下表达式给出:

$$\mathbb{E}_i \tau_{i_0} = \sum_{j=i}^{i_0-1} m_j, \quad i < i_0;$$

$$\mathbb{E}_i \tau_{i_0} = \sum_{j=i_0}^{i-1} (F_j^{(i_0)} c_{i_0} - m_j), \quad i \geq i_0 + 1.$$

Stationary distribution

令 $u_i = \mathbb{E}_i \tau_{i_0}$. (u_i) 是下列方程的最小非负解:

$$x_{i_0} = 0, \qquad x_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \quad i \neq i_0.$$

i 击中时一阶矩

$$\mathbb{E}_i \sigma_i = \frac{q_{i,i+1} c_i}{q_i}, \qquad i \geq 0.$$

$$\pi_i q_i \mathbb{E}_i \sigma_i = 1 \Rightarrow \pi_i = \frac{1}{q_{i,i+1} c_i}.$$

Stationary distribution

$$\mathbb{E}_1\tau_0 = d, \quad \mathbb{E}_i\tau_0 = \sum_{j=0}^{i-1}(F_j^{(0)}d - d_j), \quad i \geq 1.$$

$$\mathbb{E}_0\sigma_0 = q_{01}^{-1} + \mathbb{E}_1\sigma_0 = q_{01}^{-1} + d.$$

Probabilistic meaning.

- Uniqueness. $R = \mathbb{E}_0\sigma_\infty = \infty$.

Stationary distribution

$$\mathbb{E}_1\tau_0 = d, \quad \mathbb{E}_i\tau_0 = \sum_{j=0}^{i-1}(F_j^{(0)}d - d_j), \quad i \geq 1.$$

$$\mathbb{E}_0\sigma_0 = q_{01}^{-1} + \mathbb{E}_1\sigma_0 = q_{01}^{-1} + d.$$

Probabilistic meaning.

- Uniqueness. $R = \mathbb{E}_0\sigma_\infty = \infty$.
- Recurrence. $\mathbb{P}_0(\sigma_0 < \infty) = 1 - \left(\sum_{n=0}^{\infty} F_n^{(0)}\right)^{-1} = 1$.

Stationary distribution

$$\mathbb{E}_1\tau_0 = d, \quad \mathbb{E}_i\tau_0 = \sum_{j=0}^{i-1}(F_j^{(0)}d - d_j), \quad i \geq 1.$$

$$\mathbb{E}_0\sigma_0 = q_{01}^{-1} + \mathbb{E}_1\sigma_0 = q_{01}^{-1} + d.$$

Probabilistic meaning.

- Uniqueness. $R = \mathbb{E}_0\sigma_\infty = \infty$.
- Recurrence. $\mathbb{P}_0(\sigma_0 < \infty) = 1 - (\sum_{n=0}^{\infty} F_n^{(0)})^{-1} = 1$.
- Ergodicity. $d = \mathbb{E}_1\sigma_0 < \infty \Leftrightarrow \mathbb{E}_0\sigma_0 < \infty$.

Stationary distribution

$$\mathbb{E}_1\tau_0 = d, \quad \mathbb{E}_i\tau_0 = \sum_{j=0}^{i-1}(F_j^{(0)}d - d_j), \quad i \geq 1.$$

$$\mathbb{E}_0\sigma_0 = q_{01}^{-1} + \mathbb{E}_1\sigma_0 = q_{01}^{-1} + d.$$

Probabilistic meaning.

- Uniqueness. $R = \mathbb{E}_0\sigma_\infty = \infty$.
- Recurrence. $\mathbb{P}_0(\sigma_0 < \infty) = 1 - (\sum_{n=0}^{\infty} F_n^{(0)})^{-1} = 1$.
- Ergodicity. $d = \mathbb{E}_1\sigma_0 < \infty \Leftrightarrow \mathbb{E}_0\sigma_0 < \infty$.
- Strong ergodicity. $S = \sup_{i\geq 1} \mathbb{E}_i\sigma_0 = \mathbb{E}_\infty\sigma_0 < \infty$.

Thanks

Thanks

http://math.bnu.edu.cn/~zhangyh/