Criteria for several types of ergodicity for single birth processes

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Contents

Concepts and notations

Let $Q = (q_{ij})$ be regular on countable E, i.e. $q_{ij} \geq 0 (i \neq j)$, $q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty (i \in E)$, Q determines uniquely $P_t = (p_{ij}(t))$ (Q-process or Markov chain). Assume that Q is irreducible. $\pi = (\pi_i)$.

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- Exponential ergodicity: $\lim\limits_{t \to \infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim\limits_{t \to \infty} \sup\limits_i |p_{ij}(t) - \pi_j| = 0$

$$\Longleftrightarrow \lim_{t \to \infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Concepts and notations

Set $\sigma_j = \inf\{t \geq$ the first jumping time: $X_t = j\}$ and $m_{ij}^{(\ell)} = \mathbf{E}_i \sigma_j^\ell$.

For a positive integer ℓ, the recurrent chain P_t is said to be ℓ-ergodic if $m_{jj}^{(\ell)} < \infty$ for some (and hence for all) $j \in E$.

1-ergodic $=$ positive recurrent (ergodic),
0-ergodic $=$ null recurrent.

Concepts and notations

Discrete time:

- J.G. Kemeny, J.L. Snell & A.W. Knapp (1976)
- Y.H. Mao (2003)
- Z.T. Hou & Y.Y. Liu (2003). Queue Theory

Continuous time:

- P. Coolen-Schrijner & E.A. van Doorn (2002)
- Y.H. Mao (2004)

Concepts and notations

- ℓ-ergodicity provides an algebraic convergence rate: $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \to \infty$.

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- Criterion: P_t is ℓ-regodic iff for some (and then for all) $j \in E$, the system of inequalities

$$\begin{cases} \sum_k q_{ik} y_k \le -\ell m_{ij}^{(\ell-1)}, & i \ne j, \\ \sum_{k \ne j} q_{jk} y_k < \infty \end{cases}$$

has a nonnegative finite solution.

Concepts and notations

Given a regular birth-death (a_i, b_i). Assume that P_t is recurrent.

$$P_t \text{ is } \ell\text{-ergodic iff } \sum_{i=1}^{\infty} \mu_i m_{i0}^{(\ell-1)} < \infty,$$

where $\mu_0 = 1$, $\mu_i = b_0 \cdots b_{i-1}/a_1 \cdots a_i (i \geq 1)$,

$$m_{i0}^{(n)} = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k m_{k0}^{(n-1)}, \quad i \geq 1, \, n \geq 1.$$

Background

Single birth Q-matrix $Q = (q_{ij} : i, j \geq 0)$:
$q_{i,i+1} > 0$, $q_{ij} = 0$ if $j > i + 1$ for all $i \geq 0$.

$$\begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. irreversible; 2. single extremal point.

So the explicit criteria are expected.

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$$
\begin{array}{lll}
i \to i+1 & \text{at rate} & b_i \\
\quad \to i-1 & & a_i + c_i f_{i,i-1} \\
\quad \to i-2 & & c_i f_{i,i-2} \\
\quad \to \cdots & & \cdots \\
\quad \to 0 & & c_i f_{i0}
\end{array}
$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes f_{ij} ($b_i = b + \lambda i$, $a_i = 0$, $c_i = ci$, $i \geq 0$).

geometric: $f_{ij} = p(1-p)^{i-1-j}\,(1 \leq j < i)$; $f_{i0} = (1-p)^{i-1}$;

uniform: $f_{ij} = 1/i\,(0 \leq j < i)$;

binomial: $f_{ij} = \binom{i-1}{j} p^j (1-p)^{i-1-j}\,(0 \leq j < i)$.

Aim: extinction times and probability of extinction.

Keys: generating function of Q-resolvent.

Background

B. Cairns & P. Pollett (2004).

$i \longrightarrow i+1$	at rate	$g_i b$
$\longrightarrow i-1$		$g_i d_1$
$\longrightarrow \cdots$		$\cdots$
$\longrightarrow 1$		$g_i d_{i-1}$
$\longrightarrow 0$		$g_i \sum_{k \geq i} d_k$

where $b + \sum_{k \geq 1} d_k = 1$.

Three series of notations

$$q_n^{(k)} = \sum_{j=0}^{k} q_{nj}, \; 0 \le k < n \, (k, n \ge 0).$$

$$m_0 = \frac{1}{q_{01}}, \quad m_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k \right), \quad n \ge 1,$$

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n,$$

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k \right), \quad n \ge 1.$$

Criteria for uniq., recur. and erg.

J.K. Zhang (1984), M.F. Chen (1992).

- Uniqueness (regurlarity) $\Longleftrightarrow R := \sum_{n=0}^{\infty} m_n = \infty$.

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- Ergodicity $\iff d := \sup_{k\ge 0}(\sum_{n=0}^{k} d_n)/(\sum_{n=0}^{k} F_n^{(0)}) < \infty$.

Criterion for strong erg.

- Y.H. Zhang (2001). Suppose that the Q-matrix is irreducible and regular. Then

  Strong ergodicity $\iff \sup_{k\geq 0}\sum_{n=0}^{k}(F_n^{(0)}d - d_n) < \infty$.

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- H.J. Zhang, X. Lin and Z.T. Hou (1998, 2000).

  For regular birth-death (a_i, b_i),

$$\text{strong ergodicity} \Longleftrightarrow S := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} \sum_{k=n+1}^{\infty} \mu_k < \infty.$$

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- M.F. Chen (1992).

$$S < \infty \Longrightarrow \text{exponential ergodic.}$$

Uniqueness

For a totally stable and conservative $Q = (q_{ij})$, P_t is determined uniquely iff for some (equivalently, all) $\lambda > 0$,

$$u_i = \sum_{j \neq i} q_{ij} u_j / (\lambda + q_i), \quad 0 \leq u_i \leq 1,\ i \geq 0$$

has only a trivial solution.

Recurrence

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ ie recurrent iff for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} q_{ij} x_j / q_i, \qquad 0 \leq x_i \leq 1,\ i \geq 0$$

has only a trivial solution.

Erg. and Strong Erg.

Let $H \neq \emptyset$ be a finite subset of E. P_t is ergodic (resp. strong ergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, \quad i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

Probabilistic meaning.

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- Ergodicity. $d = \mathbf{E}_1 \sigma_0$.
- Strong ergodicity. $\sup_{i \geq 1} \mathbf{E}_i \sigma_0$.

Stationary distribution

Can we get the stationary distribution?

How to use $F_n^{(i)}$?

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n.$$

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Stationary distribution (Y.H. Zhang (2004)):

$$c_k := \sup_{i \ge k} \frac{\sum_{j=k}^{i} m_j}{\sum_{j=k}^{i} F_j^{(k)}}, \quad \pi_k = \frac{1}{q_{k,k+1} c_k}, \quad k \ge 0.$$

Criterion for ℓ-erg.

P_t is ℓ-ergodic iff $\sum_{i=1}^{\infty} \pi_i m_{i0}^{(\ell-1)} < \infty$.

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+1}} \Big(\ 1 \ + \sum_{k=0}^{i-1} q_i^{(k)} d_k \Big), \quad i \geq 1.$$

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$$d_0^{(\ell)} = 0, \quad d_i^{(\ell)} = \frac{1}{q_{i,i+1}}\left(m_{i0}^{(\ell-1)} + \sum_{k=0}^{i-1} q_i^{(k)} d_k^{(\ell)} \right), \quad i \geq 1.$$

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For $\ell \geq 1$, P_t is ℓ-ergodic iff $d^{(\ell)} < \infty$,

Criterion for ℓ-erg.

Key:

$$m_{10}^{(\ell)} = \ell d^{(\ell)}, \quad m_{i0}^{(\ell)} = \ell \sum_{j=0}^{i-1} (F_j^{(0)} d^{(\ell)} - d_j^{(\ell)}).$$

Exp. erg.

Y.H. Mao & Y.H. Zhang (2004).

$$\inf_i q_i > 0 \text{ and } \delta' := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty.$$

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P_t is exp. erg. iff for some $\lambda > 0$ with $\lambda < q_i$ for all i,

$$\begin{cases} \sum_j q_{ij} y_j \le -\lambda y_i - 1, & i \notin H \\ \sum_{i\in H} \sum_{j\ne i} q_{ij} y_j < \infty \end{cases}$$

has a nonnegative finite solution (y_i).

Exp. erg.

M.F. Chen (2000). For birth-death processes,

$$\text{exp. erg.} \iff \delta := \sup_{i>0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{j=i}^{\infty} \mu_j < \infty.$$

$$\lambda_1 > 0 \iff \delta < \infty.$$

$$\lambda_1 = \text{ exp. conv. rate.}$$

Exp. erg.

Mao & Zhang (2004) gives another proof.

Keys:

1. Construct a test function with double summations for sufficiency.

2. Necessity. Note that

$$m_{i0}^{(n)} \geq n\left(\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=i}^{\infty}\mu_k\right)m_{i0}^{(n-1)}$$

$$\geq \cdots \geq n!\left(\sum_{j=0}^{i-1}\frac{1}{\mu_j b_j}\sum_{k=i}^{\infty}\mu_k\right)^n.$$

Exp. erg.

Unfortunately, "$\delta' < \infty$" is not necessary for exp. erg.

Conjecture that the criterion should be

$$\delta := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \left(d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

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$$\delta := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \left(d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

New result. Suppose that

$$q_{i+1,j} = p_i q_{ij}, \quad 0 \le j \le i-1,$$

where $0 \le p_i \le c < 1 \, (i \ge 1)$. Then

$$\text{exp.erg.} \iff \delta < \infty.$$

Thanks