

Criteria for several types of ergodicity for single birth processes

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Concepts and notations

Let $Q = (q_{ij})$ be regular on countable E , i.e.

$$q_{ij} \geq 0 (i \neq j), \quad q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty (i \in E),$$

Q determines uniquely $P_t = (p_{ij}(t))$ (Q -process or Markov chain). Assume that Q is irreducible. $\pi = (\pi_i)$.

- Ordinary ergodicity: $\lim_{t \rightarrow \infty} |p_{ij}(t) - \pi_j| = 0$.

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- Exponential ergodicity: $\lim_{t \rightarrow \infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim_{t \rightarrow \infty} \sup_i |p_{ij}(t) - \pi_j| = 0$
 $\iff \lim_{t \rightarrow \infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0$.

Concepts and notations

Set $\sigma_j = \inf\{t \geq \text{the first jumping time: } X_t = j\}$ and $m_{ij}^{(\ell)} = \mathbf{E}_i \sigma_j^\ell$.

For a positive integer ℓ , the recurrent chain P_t is said to be ℓ -ergodic if $m_{jj}^{(\ell)} < \infty$ for some (and hence for all) $j \in E$.

1-ergodic = positive recurrent (ergodic),

0-ergodic = null recurrent.

Concepts and notations

Discrete time:

- J.G. Kemeny, J.L. Snell & A.W. Knapp (1976)
- Y.H. Mao (2003)
- Z.T. Hou & Y.Y. Liu (2003). Queue Theory

Continuous time:

- P. Coolen-Schrijner & E.A. van Doorn (2002)
- Y.H. Mao (2004)

Concepts and notations

- ℓ -ergodicity provides an **algebraic** convergence rate:
 $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \rightarrow \infty$.

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 $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \rightarrow \infty$.
- Given a regular birth-death (a_i, b_i) . Assume that P_t is recurrent.

P_t is ℓ -ergodic iff $\sum_{i=1}^{\infty} \mu_i m_{i0}^{(\ell-1)} < \infty$,

where $\mu_0 = 1$, $\mu_i = b_0 \cdots b_{i-1} / a_1 \cdots a_i$ ($i \geq 1$),

$$m_{i0}^{(n)} = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k m_{k0}^{(n-1)}, \quad i \geq 1, n \geq 1.$$

Background

Single birth Q -matrix $Q = (q_{ij} : i, j \geq 0)$:
 $q_{i,i+1} > 0$, $q_{ij} = 0$ if $j > i + 1$ for all $i \geq 0$.

$$\begin{pmatrix} - & + & 0 & 0 & \dots \\ * & - & + & 0 & \dots \\ * & * & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. irreversible; 2. single extremal point; 3. application.

So the explicit criteria are expected.

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$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } b_i \\ \rightarrow i - 1 & a_i + c_i f_{i,i-1} \\ \rightarrow i - 2 & c_i f_{i,i-2} \\ \rightarrow \dots & \dots \\ \rightarrow 0 & c_i f_{i0} \end{array}$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes f_{ij} ($b_i = b + \lambda i$, $a_i = 0$, $c_i = ci$, $i \geq 0$).

geometric: $f_{ij} = p(1 - p)^{i-1-j}$ ($1 \leq j < i$); $f_{i0} = (1 - p)^{i-1}$;

uniform: $f_{ij} = 1/i$ ($0 \leq j < i$);

binomial: $f_{ij} = \binom{i-1}{j} p^j (1 - p)^{i-1-j}$ ($0 \leq j < i$).

Aim: extinction times and probability of extinction.

Keys: generating function of Q -resolvent.

Background

B. Cairns & P. Pollett (2004).

$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } g_i b \\ \rightarrow i - 1 & g_i d_1 \\ \rightarrow \dots & \dots \\ \rightarrow 1 & g_i d_{i-1} \\ \rightarrow 0 & g_i \sum_{k \geq i} d_k \end{array}$$

where $b + \sum_{k \geq 1} d_k = 1$.

Three series of notations

$$q_n^{(k)} = \sum_{j=0}^k q_{nj}, \quad 0 \leq k < n \quad (k, n \geq 0).$$

$$m_0 = \frac{1}{q_{01}}, \quad m_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k \right), \quad n \geq 1,$$

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n,$$

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k \right), \quad n \geq 1.$$

Criteria for uniq., recur. and erg.

J.K. Zhang (1984), M.F. Chen (1992).

- Uniqueness (regularity) $\iff R := \sum_{n=0}^{\infty} m_n = \infty$.

Suppose that the Q -matrix is irreducible, then

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In the regular case,

- Ergodicity $\iff d := \sup_{k \geq 0} (\sum_{n=0}^k d_n) / (\sum_{n=0}^k F_n^{(0)}) < \infty$.

Criterion for strong erg.

- Y.H. Zhang (2001). Suppose that the Q -matrix is irreducible and regular. Then

$$\text{Strong ergodicity} \iff \sup_{k \geq 0} \sum_{n=0}^k (F_n^{(0)} d - d_n) < \infty.$$

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- H.J. Zhang, X. Lin and Z.T. Hou (1998, 2000).

For regular birth-death (a_i, b_i) ,

$$\text{strong ergodicity} \iff S := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} \sum_{k=n+1}^{\infty} \mu_k < \infty.$$

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- M.F. Chen (1992).

$$S < \infty \implies \text{exponential ergodic.}$$

Uniqueness

For a totally stable and conservative $Q = (q_{ij})$, P_t is determined uniquely iff for some (equivalently, all) $\lambda > 0$,

$$u_i = \sum_{j \neq i} q_{ij} u_j / (\lambda + q_i), \quad 0 \leq u_i \leq 1, \quad i \geq 0$$

has only a trivial solution.

Recurrence

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ is recurrent iff for some (equivalently, all) j_0 ,

$$x_i = \sum_{j \neq j_0, i} q_{ij} x_j / q_i, \quad 0 \leq x_i \leq 1, \quad i \geq 0$$

has only a trivial solution.

Erg. and Strong Erg.

Let $H \neq \emptyset$ be a finite subset of E . P_t is ergodic (resp. strong ergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

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- Ergodicity. $d = \mathbf{E}_1 \sigma_0$.
- Strong ergodicity. $\sup_{i \geq 1} \mathbf{E}_i \sigma_0$.

Stationary distribution

Can we get the stationary distribution?

How to use $F_n^{(i)}$?

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n.$$

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Stationary distribution (Y.H. Zhang (2004)):

$$c_k := \sup_{i \geq k} \frac{\sum_{j=k}^i m_j}{\sum_{j=k}^i F_j^{(k)}}, \quad \pi_k = \frac{1}{q_{k,k+1} c_k}, \quad k \geq 0.$$

Criterion for ℓ -erg.

P_t is ℓ -ergodic iff $\sum_{i=1}^{\infty} \pi_i m_{i0}^{(\ell-1)} < \infty$.

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+1}} \left(1 + \sum_{k=0}^{i-1} q_i^{(k)} d_k \right), \quad i \geq 1.$$

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$$d_0^{(\ell)} = 0, \quad d_i^{(\ell)} = \frac{1}{q_{i,i+1}} \left(m_{i0}^{(\ell-1)} + \sum_{k=0}^{i-1} q_i^{(k)} d_k^{(\ell)} \right), \quad i \geq 1.$$

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$$d^{(\ell)} := \sup_{i \geq 0} \frac{\sum_{j=0}^i d_j^{(\ell)}}{\sum_{j=0}^i F_j^{(0)}}.$$

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For $\ell \geq 1$, P_t is ℓ -ergodic iff $d^{(\ell)} < \infty$,

ℓ -erg.

P_t is ℓ -regodic iff for some (and then for all) $j \in E$, the system of inequalities

$$\begin{cases} \sum_k q_{ik} y_k \leq -\ell m_{ij}^{(\ell-1)}, & i \neq j, \\ \sum_{k \neq j} q_{jk} y_k < \infty \end{cases}$$

has a nonnegative finite solution.

Key:

$$m_{10}^{(\ell)} = \ell d^{(\ell)}, \quad m_{i0}^{(\ell)} = \ell \sum_{j=0}^{i-1} (F_j^{(0)} d^{(\ell)} - d_j^{(\ell)}).$$

Exp. erg.

Y.H. Mao & Y.H. Zhang (2004).

$$\inf_i q_i > 0 \text{ and } \delta' := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty.$$

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P_t is exp. erg. iff for some $\lambda > 0$ with $\lambda < q_i$ for all i ,

$$\begin{cases} \sum_j q_{ij} y_j \leq -\lambda y_i - 1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a nonnegative finite solution (y_i) .

Exp. erg.

M.F. Chen (2000). For birth-death processes,

$$\text{exp. erg.} \iff \delta := \sup_{i>0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{j=i}^{\infty} \mu_j < \infty.$$

$$\lambda_1 > 0 \iff \delta < \infty.$$

$$\lambda_1 = \text{exp. conv. rate.}$$

Exp. erg.

Mao & Zhang (2004) gives another proof.

Keys:

1. Construct a test function with double summations for sufficiency.

2. Necessity. Note that

$$\begin{aligned} m_{i0}^{(n)} &\geq n \left(\sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right) m_{i0}^{(n-1)} \\ &\geq \dots \geq n! \left(\sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right)^n. \end{aligned}$$

Exp. erg.

Unfortunately, " $\delta' < \infty$ " is not necessary for exp. erg.

Conjecture that the criterion should be

$$\delta := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \left(d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

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$$\delta := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \left(d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

New result. Suppose that

$$q_{i+1,j} = p_i q_{ij}, \quad 0 \leq j \leq i-1,$$

where $0 \leq p_i \leq c < 1$ ($i \geq 1$). Then

$$\text{exp.erg.} \iff \delta < \infty.$$

Thanks