

# 单生过程的遍历性准则

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# 内容安排

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- 概念及记号
- 背景
- 单生过程的三组记号
- 唯一, 常返和遍历准则
- 强遍历准则
- 平稳分布
- $\ell$  遍历准则
- 指数遍历

# 概念及记号

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Let  $Q = (q_{ij})$  be regular on countable  $E$ , i.e.

$$q_{ij} \geq 0 (i \neq j), \quad q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty (i \in E),$$

$Q$  determines uniquely  $P_t = (p_{ij}(t))$  ( $Q$ -process or Markov chain). Assume that  $Q$  is irreducible.  $\pi = (\pi_i)$ .

- Ordinary ergodicity:  $\lim_{t \rightarrow \infty} |p_{ij}(t) - \pi_j| = 0$ .

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- Ordinary ergodicity:  $\lim_{t \rightarrow \infty} |p_{ij}(t) - \pi_j| = 0$ .
- Exponential ergodicity:  $\lim_{t \rightarrow \infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$ .
- Strong ergodicity:  $\lim_{t \rightarrow \infty} \sup_i |p_{ij}(t) - \pi_j| = 0$   
 $\iff \lim_{t \rightarrow \infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0$ .

# 概念及记号

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Set  $\sigma_j = \inf\{t \geq \text{the first jumping time: } X_t = j\}$  and  
 $m_{ij}^{(\ell)} = \mathbf{E}_i \sigma_j^\ell$ .

For a positive integer  $\ell$ , the recurrent chain  $P_t$  is said to be  $\ell$ -ergodic if  $m_{jj}^{(\ell)} < \infty$  for some (and hence for all)  $j \in E$ .

1-ergodic = positive recurrent (ergodic),  
0-ergodic = null recurrent.

# 概念及记号

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Discrete time:

- J.G. Kemeny, J.L. Snell & A.W. Knapp (1976)
- Y.H. Mao (2003)
- Z.T. Hou & Y.Y. Liu (2003). Queue Theory

Continuous time:

- P. Coolen-Schrijner & E.A. van Doorn (2002)
- Y.H. Mao (2004)

# 概念及记号

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- $\ell$ -ergodicity provides an **algebraic** convergence rate:  
 $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$  as  $t \rightarrow \infty$ .





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- $\ell$ -ergodicity provides an **algebraic** convergence rate:  
 $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$  as  $t \rightarrow \infty$ .
- Criterion:  $P_t$  is  $\ell$ -ergodic iff for some (and then for all)  $j \in E$ , the system of inequalities

$$\begin{cases} \sum_k q_{ik} y_k \leq -\ell m_{ij}^{(\ell-1)}, & i \neq j, \\ \sum_{k \neq j} q_{jk} y_k < \infty \end{cases}$$

has a nonnegative finite solution.



# 概念及记号

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Given a regular birth-death  $(a_i, b_i)$ . Assume that  $P_t$  is recurrent.

$$P_t \text{ is } \ell\text{-ergodic iff } \sum_{i=1}^{\infty} \mu_i m_{i0}^{(\ell-1)} < \infty,$$

where  $\mu_0 = 1$ ,  $\mu_i = b_0 \cdots b_{i-1} / a_1 \cdots a_i (i \geq 1)$ ,

$$m_{i0}^{(n)} = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k m_{k0}^{(n-1)}, \quad i \geq 1, n \geq 1.$$

# 背景

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Single birth  $Q$ -matrix  $Q = (q_{ij} : i, j \geq 0)$ :  
 $q_{i,i+1} > 0$ ,  $q_{ij} = 0$  if  $j > i + 1$  for all  $i \geq 0$ .

$$\begin{pmatrix} - & + & 0 & 0 & \dots \\ * & - & + & 0 & \dots \\ * & * & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. irreversible; 2. single extremal point.

So the explicit criteria are expected.

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$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } b_i \\ \rightarrow i - 1 & a_i + c_i f_{i,i-1} \\ \rightarrow i - 2 & c_i f_{i,i-2} \\ \rightarrow \dots & \dots \\ \rightarrow 0 & c_i f_{i0} \end{array}$$

where  $\sum_{j=0}^{i-1} f_{ij} = 1$ .

# 背景

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Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes  $f_{ij}$  ( $b_i = b + \lambda i$ ,  $a_i = 0$ ,  $c_i = ci$ ,  $i \geq 0$ ).

geometric:  $f_{ij} = p(1 - p)^{i-1-j}$  ( $1 \leq j < i$ );  $f_{i0} = (1 - p)^{i-1}$ ;

uniform:  $f_{ij} = 1/i$  ( $0 \leq j < i$ );

binomial:  $f_{ij} = \binom{i-1}{j} p^j (1 - p)^{i-1-j}$  ( $0 \leq j < i$ ).

Aim: extinction times and probability of extinction.

Keys: generating function of  $Q$ -resolvent.

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# 背景

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B. Cairns & P. Pollett (2004).

$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } g_i b \\ \rightarrow i - 1 & g_i d_1 \\ \rightarrow \dots & \dots \\ \rightarrow 1 & g_i d_{i-1} \\ \rightarrow 0 & g_i \sum_{k \geq i} d_k \end{array}$$

where  $b + \sum_{k \geq 1} d_k = 1$ .



# 单生过程的三组记号

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$$q_n^{(k)} = \sum_{j=0}^k q_{nj}, \quad 0 \leq k < n \quad (k, n \geq 0).$$

$$m_0 = \frac{1}{q_{01}}, \quad m_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k \right), \quad n \geq 1,$$

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n,$$

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}} \left( 1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k \right), \quad n \geq 1.$$

# 唯一, 常返和遍历准则

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J.K. Zhang (1984), M.F. Chen (1992).

- Uniqueness (regularity)  $\iff R := \sum_{n=0}^{\infty} m_n = \infty$ .

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Suppose that the  $Q$ -matrix is irreducible, then

- Recurrence  $\iff \sum_{n=0}^{\infty} F_n^{(0)} = \infty$ .

In the regular case,

- Ergodicity  $\iff d := \sup_{k \geq 0} (\sum_{n=0}^k d_n) / (\sum_{n=0}^k F_n^{(0)}) < \infty$ .

# 强遍历准则

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- Y.H. Zhang (2001). Suppose that the  $Q$ -matrix is irreducible and regular. Then

$$\text{Strong ergodicity} \iff \sup_{k \geq 0} \sum_{n=0}^k (F_n^{(0)} d - d_n) < \infty.$$

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- H.J. Zhang, X. Lin and Z.T. Hou (1998, 2000).

For regular birth-death  $(a_i, b_i)$ ,

$$\text{strong ergodicity} \iff S := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} \sum_{k=n+1}^{\infty} \mu_k < \infty.$$

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$$S < \infty \implies \text{exponential ergodic.}$$

# Uniqueness

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For a totally stable and conservative  $Q = (q_{ij})$ ,  $P_t$  is determined uniquely iff for some (equivalently, all)  $\lambda > 0$ ,

$$u_i = \sum_{j \neq i} q_{ij} u_j / (\lambda + q_i), \quad 0 \leq u_i \leq 1, \quad i \geq 0$$

has only a trivial solution.



# Recurrence

---

For a regular and irreducible  $Q = (q_{ij})$ ,  $P(t)$  is recurrent iff for some (equivalently, all)  $j_0$ ,

$$x_i = \sum_{j \neq j_0, i} q_{ij} x_j / q_i, \quad 0 \leq x_i \leq 1, \quad i \geq 0$$

has only a trivial solution.

# Erg. and Strong Erg.

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Let  $H \neq \emptyset$  be a finite subset of  $E$ .  $P_t$  is ergodic (resp. strong ergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

# 平穩分布

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Can we get the stationary distribution?

How to use  $F_n^{(i)}$ ?

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n.$$

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Stationary distribution (Y.H. Zhang (2004)):

$$c_k := \sup_{i \geq k} \frac{\sum_{j=k}^i m_j}{\sum_{j=k}^i F_j^{(k)}}, \quad \pi_k = \frac{1}{q_{k,k+1} c_k}, \quad k \geq 0.$$

# $\ell$ 遍历准则

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$P_t$  is  $\ell$ -ergodic iff  $\sum_{i=1}^{\infty} \pi_i m_{i0}^{(\ell-1)} < \infty$ .

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+1}} \left( 1 + \sum_{k=0}^{i-1} q_i^{(k)} d_k \right), \quad i \geq 1.$$

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$$d_0^{(\ell)} = 0, \quad d_i^{(\ell)} = \frac{1}{q_{i,i+1}} \left( m_{i0}^{(\ell-1)} + \sum_{k=0}^{i-1} q_i^{(k)} d_k^{(\ell)} \right), \quad i \geq 1.$$

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$$d^{(\ell)} := \sup_{i \geq 0} \frac{\sum_{j=0}^i d_j^{(\ell)}}{\sum_{j=0}^i F_j^{(0)}}.$$



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For  $\ell \geq 1$ ,  $P_t$  is  $\ell$ -ergodic iff  $d^{(\ell)} < \infty$ ,

# $\ell$ 遍历准则

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Key:

$$m_{10}^{(\ell)} = \ell d^{(\ell)}, \quad m_{i0}^{(\ell)} = \ell \sum_{j=0}^{i-1} (F_j^{(0)} d^{(\ell)} - d_j^{(\ell)}).$$

# 指数遍历

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Y.H. Mao & Y.H. Zhang (2004).

$$\inf_i q_i > 0 \text{ and } \delta' := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty.$$

$\implies$  exp. erg.

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$\implies$  exp. erg.

$P_t$  is exp. erg. iff for some  $\lambda > 0$  with  $\lambda < q_i$  for all  $i$ ,

$$\begin{cases} \sum_j q_{ij} y_j \leq -\lambda y_i - 1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a nonnegative finite solution  $(y_i)$ .

# 指数遍历

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M.F. Chen (2000). For birth-death processes,

$$\text{exp. erg.} \iff \delta := \sup_{i>0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{j=i}^{\infty} \mu_j < \infty.$$

$$\lambda_1 > 0 \iff \delta < \infty.$$

$$\lambda_1 = \text{exp. conv. rate.}$$

# 指数遍历

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Mao & Zhang (2004) gives another proof.

Keys:

1. Construct a test function with double summations for sufficiency.

2. Necessity. Note that

$$\begin{aligned} m_{i0}^{(n)} &\geq n \left( \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right) m_{i0}^{(n-1)} \\ &\geq \dots \geq n! \left( \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k \right)^n. \end{aligned}$$



# 指数遍历

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Unfortunately, " $\delta' < \infty$ " is not necessary for exp. erg.

Conjecture that the criterion should be

$$\delta := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \left( d - \frac{d_i}{F_i^{(0)}} \right) < \infty.$$

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**New result.** Suppose that

$$q_{i+1,j} = p_i q_{ij}, \quad 0 \leq j \leq i-1,$$

where  $0 \leq p_i \leq c < 1$  ( $i \geq 1$ ). Then

$$\text{exp.erg.} \iff \delta < \infty.$$

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Thanks