

ℓ -ergodicity for single birth processes

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Background

Let $Q = (q_{ij})$ be regular on countable E , i.e.
 $q_{ij} \geq 0 (i \neq j)$, $q_i := -q_{ii} = \sum_{j \neq i} q_{ij} < \infty (i \in E)$,
 Q determines uniquely $P_t = (p_{ij}(t))$ (Q -process
or Markov chain). Assume that Q is irreducible.
 $\pi = (\pi_i)$.

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- Exponential ergodicity: $\lim_{t \rightarrow \infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim_{t \rightarrow \infty} \sup_i |p_{ij}(t) - \pi_j| = 0$
 $\iff \lim_{t \rightarrow \infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0$.

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Set $\sigma_j = \inf\{t \geq \text{the first jumping time: } X_t = j\}$
and $m_{ij}^{(\ell)} = \mathbf{E}_i \sigma_j^\ell$.

Definition. For a positive integer ℓ , the recurrent chain P_t is said to be ℓ -ergodic if $m_{jj}^{(\ell)} < \infty$ for some (and hence for all) $j \in E$.

1-ergodic = positive recurrent (ergodic),
0-ergodic = null recurrent.

Background

Discrete time:

- J.G. Kemeny, J.L. Snell & A.W. Knapp (1976)
- Y.H. Mao (2003)
- Z.T. Hou & Y.Y. Liu (2003). Queue Theory

Continuous time:

- P. Coolen-Schrijner & E.A. van Doorn (2002)
- Y.H. Mao (2004)

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- ℓ -ergodicity provides an algebraic convergence rate:

$$p_{ij}(t) - \pi_j = o(t^{-(\ell-1)}) \text{ as } t \rightarrow \infty.$$



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- Criterion: P_t is ℓ -ergodic iff for some (and then for all) $j \in E$, the system of inequalities

$$\begin{cases} \sum_k q_{ik} y_k \leq -\ell m_{ij}^{(\ell-1)} & i \neq j, \\ \sum_{k \neq j} q_{jk} y_k < \infty \end{cases}$$

has a nonnegative finite solution.



Background

Given a regular birth-death (a_i, b_i) . Assume that P_t is recurrent.

P_t is ℓ -ergodic iff $\sum_{i=1}^{\infty} \mu_i m_{i0}^{(\ell-1)} < \infty$,

where $\mu_0 = 1$, $\mu_i = b_0 \cdots b_{i-1} / a_1 \cdots a_i$ ($i \geq 1$),

$$m_{i0}^{(n)} = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k m_{k0}^{(n-1)}, \quad i \geq 1, n \geq 1.$$

Background

Single birth Q -matrix $Q = (q_{ij} : i, j \geq 0)$:
 $q_{i,i+1} > 0$, $q_{ij} = 0$ if $j > i + 1$ for all $i \geq 0$.

$$\begin{pmatrix} - & + & 0 & 0 & \dots \\ * & - & + & 0 & \dots \\ * & * & - & + & \dots \\ \vdots & \ddots & \ddots & \ddots & \end{pmatrix}$$

1. **irreversible**; 2. **single extremal point**.

So the explicit criteria are expected.

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$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } b_i \\ \rightarrow i - 1 & a_i + c_i f_{i,i-1} \\ \rightarrow i - 2 & c_i f_{i,i-2} \\ \rightarrow \dots & \dots \\ \rightarrow 0 & c_i f_{i0} \end{array}$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

Background

W.J. Anderson (1991)

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for special f_{ij} , e.g. geometric, uniform, binomial catastrophes.

Keys: generating function of Q -resolvent.

Background

B. Cairns & P. Pollett (2004).

$$\begin{array}{ll} i \rightarrow i + 1 & \text{at rate } g_i b \\ \rightarrow i - 1 & g_i d_1 \\ \rightarrow \dots & \dots \\ \rightarrow 1 & g_i d_{i-1} \\ \rightarrow 0 & g_i \sum_{k \geq i} d_k \end{array}$$

where $b + \sum_{k \geq 1} d_k = 1$.

Single birth processes

$$q_n^{(k)} = \sum_{j=0}^k q_{nj}, \quad 0 \leq k < n \quad (k, n \geq 0)$$

$$m_0 = \frac{1}{q_{01}}, \quad m_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} m_k \right), \quad n \geq 1,$$

$$F_n^{(n)} = 1, \quad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \leq i < n,$$

$$d_0 = 0, \quad d_n = \frac{1}{q_{n,n+1}} \left(1 + \sum_{k=0}^{n-1} q_n^{(k)} d_k \right), \quad n \geq 1.$$

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M.F. Chen (1992,1999), J.K. Zhang (1984) and Y.H. Zhang (2001).

- Uniqueness (regularity) $\iff R := \sum_{n=0}^{\infty} m_n = \infty$.

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- Strong ergodicity $\iff \sup_{k \geq 0} \sum_{n=0}^k (F_n^{(0)} d - d_n) < \infty$.

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Single birth processes

Exp. ergo. (Y.H. Mao & Y.H. Zhang (2004)): \Leftarrow

$$\inf_i q_i > 0 \text{ and } \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1} F_j^{(0)}} < \infty.$$

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P_t is exponential ergodic iff for some $\lambda > 0$ with $\lambda < q_i$ for all i ,

$$\sum_j q_{ij} y_j \leq -\lambda y_i - 1, \quad i \geq 1$$

has a nonnegative finite solution (y_i) .

Single birth processes

Stationary distribution (Y.H. Zhang (2004)):

$$c_k := \sup_{i \geq k} \frac{\sum_{j=k}^i m_j}{\sum_{j=k}^i F_j^{(k)}}, \quad \pi_k = \frac{1}{q_{k,k+1} c_k}, \quad k \geq 0.$$

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Single birth processes

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For $\ell \geq 1$, P_t is ℓ -ergodic iff $d^{(\ell)} < \infty$,
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ℓ -ergodicity

Key:

$$m_{10}^{(\ell)} = \ell d^{(\ell)}, \quad m_{i0}^{(\ell)} = \ell \sum_{j=0}^{i-1} (F_j^{(0)} d^{(\ell)} - d_j^{(\ell)}), \quad i \geq 1.$$

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–Thanks–