A class of multidimensional Q-processes

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Background

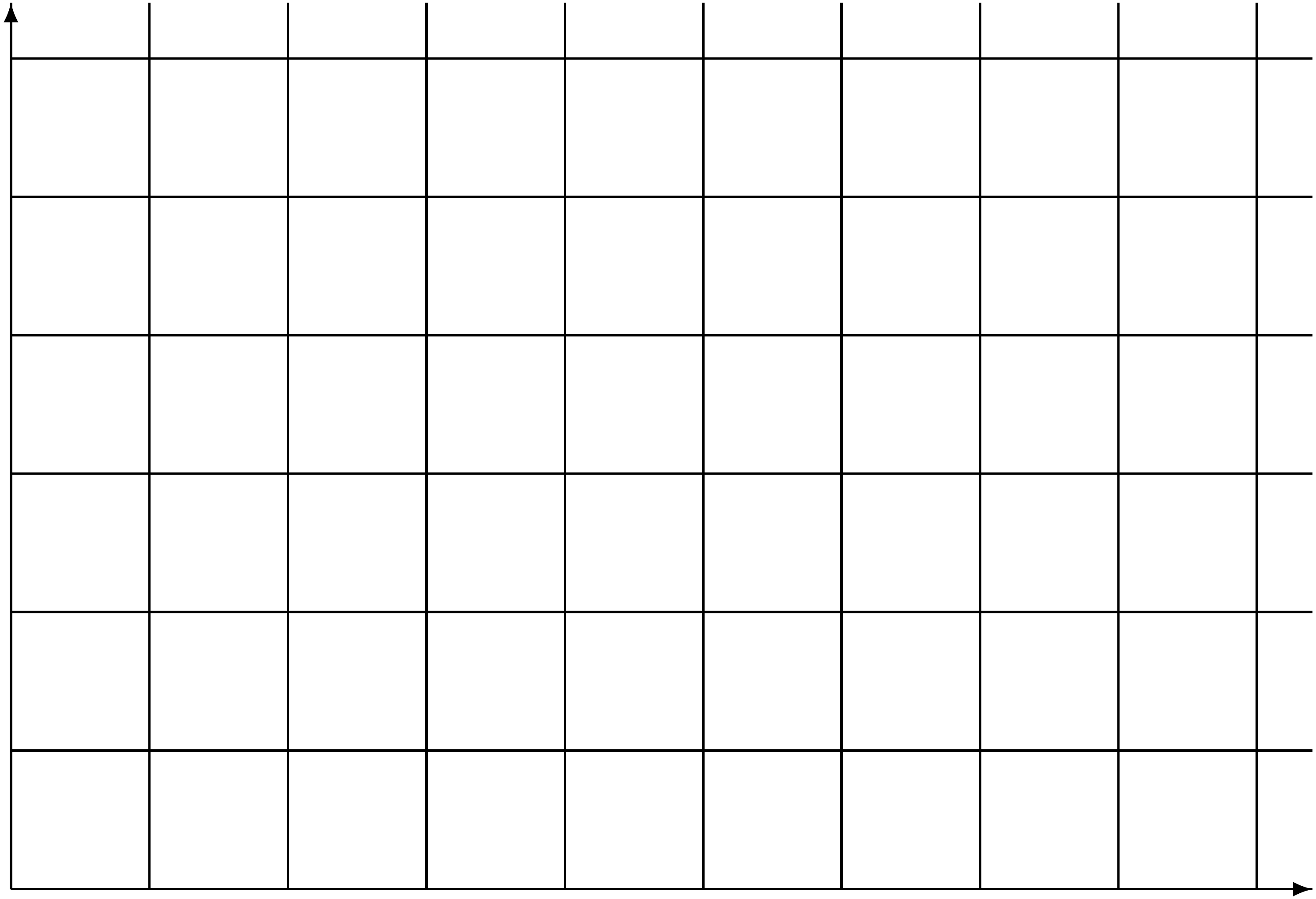
The uniqueness, recurrence and ergodicity are three classical problems in the theory of Markov processes with continuous time.

It is more difficult to study directly the properties mentioned above for multidimensional Q-processes, especially for those particle system models.

Yan and Chen's paper: Multidimensional Q-processes, Chinese Ann. Math. 7B, 1986

Key: reduces the multidimensional problems to one-dimensional ones. Sufficient conditions.

Background: Yan-Chen's method



$\mathbf{Z}_+^2$

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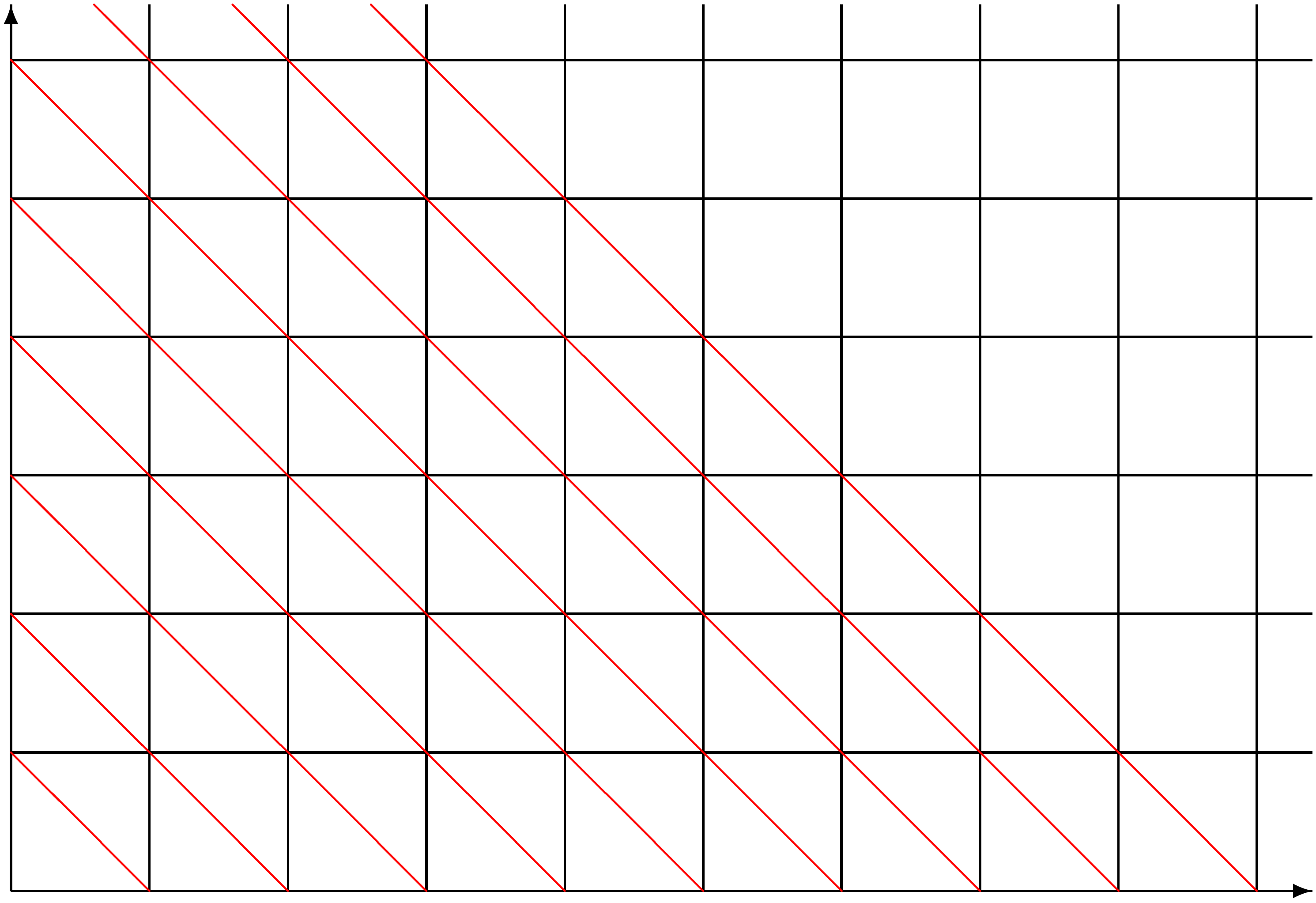

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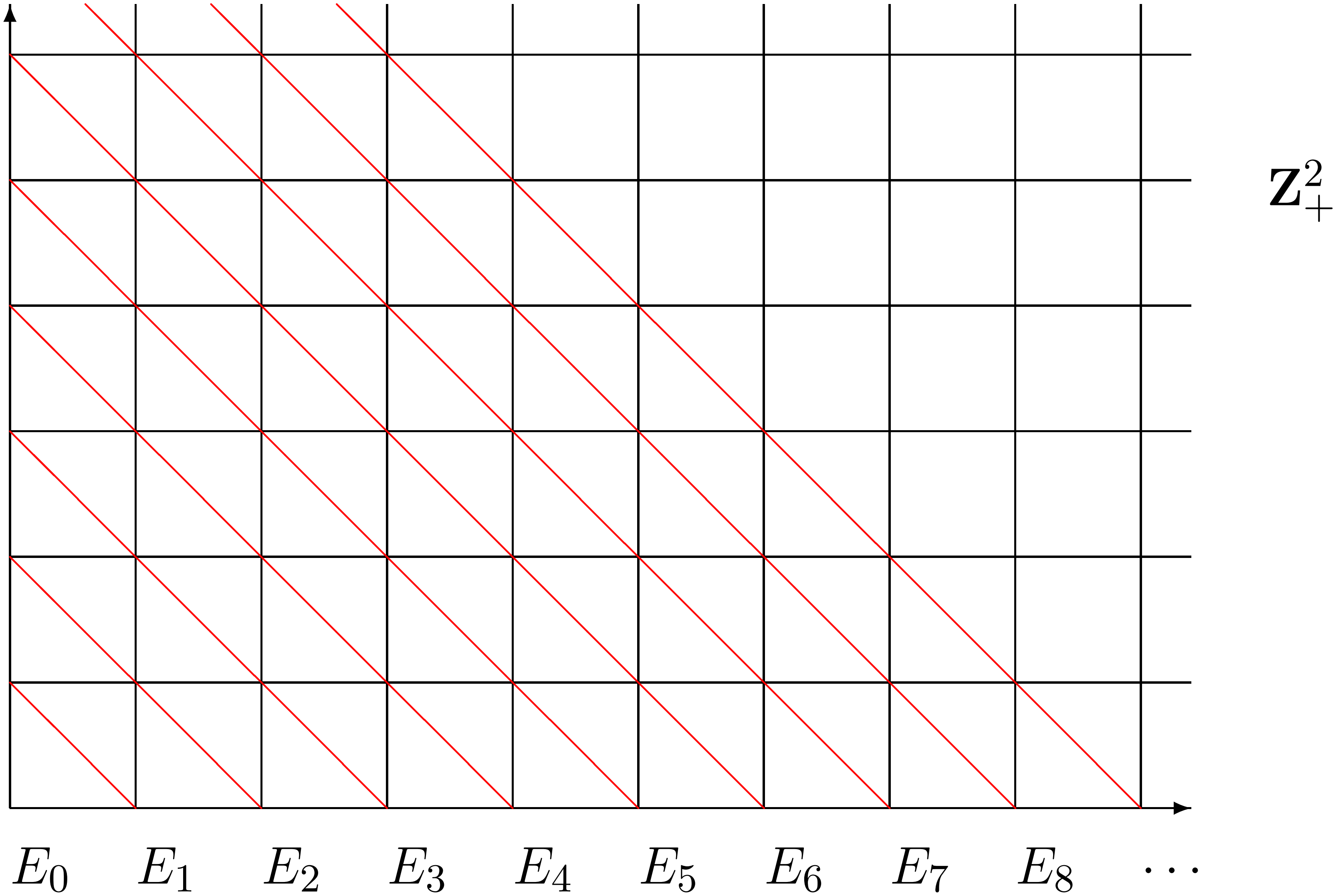

Ex.1: finite dimen. Schlögl model

finite dimensional Schlögl model: S be a finite set and $E = \mathbf{Z}_+^S$. Q-matrix $Q = (q(x,y) : x, y \in E)$ is defined by

$$q(x,y) = \begin{cases} \lambda_1 \binom{x(u)}{2} + \lambda_4, & \text{if } y = x + e_u, \\ \lambda_2 \binom{x(u)}{3} + \lambda_3 x(u), & \text{if } y = x - e_u, \\ x(u) p(u,v), & \text{if } y = x - e_u + e_v, \\ 0, & \text{other cases of } y \neq x, \end{cases}$$

and $q(x) = -q(x,x) = \sum_{y \neq x} q(x,y)$, where $x = (x(u) : u \in S)$, $\lambda_1, \cdots, \lambda_4$ are positive constants. $(p(u,v) : u, v \in S)$ is a transition probability matrix on S and e_u is the element in E having value 1 at u and 0 elsewhere.

Ex.1: finite dimen. Schlögl model

Theorem (Y.H. Zhang, J. Appl. Probab. 38, 2001):
The Q-process corresponding to finite dimensional Schlögl model is strongly ergodic ($\lim_{t\to\infty}\sup_i |p_{ij}(t)-\pi_j| = 0$).

Key: the following controlling birth-death process is strongly ergodic

$$\begin{cases} a_i = \inf\{\sum_{y\in E_{i-1}} q(x,y) : x \in E_i\}, & i \geq 1, \\ b_i = \sup\{\sum_{y\in E_{i+1}} q(x,y) : x \in E_i\}, & i \geq 0. \end{cases}$$

Note that

$$a_i \geq \frac{\lambda_2}{6}\Big(\frac{i^3}{|S|^2} - 3i^2\Big) + \Big(\lambda_3 + \frac{\lambda_2}{3}\Big)i, \qquad b_i = \frac{\lambda_1}{2}(i^2 - i) + \lambda_4|S|.$$

Ex.1: finite dimen. Schlögl model

Comparison method. The controlling process is a single birth (or birth-death) process. Explicit sufficient condition for uniqueness, recurrence and ergodicity. The method is efficient for most of particle systems, especially for uniqueness.

Our aims: explicit necessary conditions for uniq., recur., erg., ℓ-erg., exp. erg, and strong erg.. Comparison and coupling method.

Example 2: Brusselator model

Brusselator model is a typical model of reaction-diffusion process with several species. Activitor particles A and B. Reaction particles X_1 and X_2. The chemical reaction is:

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$2X_1 + X_2 \xrightarrow{\lambda_3} 3X_1$	$\lambda_3 \binom{x_1(u)}{2} x_2(u)$,	$y = x + e_{u1} - e_{u2}$

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$X_1 \xrightarrow{\lambda_4} C$	$\lambda_4 x_1(u)$,	$y = x - e_{u1}$

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$X_1 \xrightarrow{\lambda_4} C$	$\lambda_4 x_1(u)$,	$y = x - e_{u1}$
diffusion $k = 1, 2, v \neq u$	$x_k(u) p_k(u, v)$,	$y = x - e_{uk} + e_{vk}$

Example 2: Brusselator model

Let S be a given set and $E = (\mathbf{Z}_+^2)^S$. The Q-matrix:

$$q(x,y) = \begin{cases} \lambda_1 a(u), & \text{if } y = x + e_{u1}; \\ \lambda_2 b(u) x_1(u), & \text{if } y = x - e_{u1} + e_{u2}; \\ \lambda_3 \binom{x_1(u)}{2} x_2(u), & \text{if } y = x + e_{u1} - e_{u2}; \\ \lambda_4 x_1(u), & \text{if } y = x - e_{u1}; \\ x_k(u) p_k(u,v), & \text{if } y = x - e_{uk} + e_{vk},\ k = 1, 2,\ v \neq u; \\ 0, & \text{other cases of } y \neq x, \end{cases}$$

and $q(x) = \sum_{y \neq x} q(x,y)$, where $p_k(u,v)$ is the tr. pr. on S ($k = 1, 2$) and

$$e_{ui}(v,j) = \begin{cases} 1, & \text{if } v = u,\ j = i; \\ 0, & \text{other cases of } u, v \in S,\ i, j = 1, 2. \end{cases}$$

Example 2: Brusselator model

Yan-Chen's method is efficient for uniqueness of Brusselator model, but unefficient for recurrence.

D. Han (1991) proved the recurrence for single box ($|S| = 1$). No diffusion. Construct a finite nonnegative solution for $\sum_j q_{ij} y_j \le -1 (i \notin H)$, $\sum_{i \in H} \sum_{j \ne i} q_{ij} y_j < \infty$, where H is a finite nonempty set.

J.W. Chen (1995) proved the exponential erg. for finite boxes ($|S| < \infty$). Construct a nonnegative compact h, $C \ge 0$, $c > 0$: $\Omega h \le C - ch$.

How about strong erg.?

Example 2: Brusselator model

Theorem Assume that $a = \sum_{u \in S} a(u) < \infty$. Then the Q-process corresponding to Brusselator model on E_* is not strongly ergodic, where

$$E_* = \{x \in E : |x| = \textstyle\sum_{u \in S} (x_1(u) + x_2(u)) < \infty\}.$$

Key: the following “controlling” birth-death process is not strongly ergodic

$$\begin{cases} a_i = \sup\{\sum_{y \in E_{i-1}} q(x, y) : x \in E_i\}, & i \geq 1, \\ b_i = \inf\{\sum_{y \in E_{i+1}} q(x, y) : x \in E_i\}, & i \geq 0. \end{cases}$$

Note that

$$a_i = \lambda_4 i, \quad i \geq 1; \qquad b_i = \lambda_1 a, \quad i \geq 0.$$

Main results

Let E be a countable set and $Q = (q(x,y) : x, y \in E)$ be a totally stable and conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\sum_{k=0}^{\infty} E_k = E$ and that

(i) If $q(x,y) > 0$ and $x \in E_k$, then $y \in \sum_{j=0}^{k+1} E_j$ for all $k \geq 0$.
(ii) $\sum_{y \in E_{k+1}} q(x,y) > 0$ for all $x \in E_k$ and all $k \geq 0$.
(iii) $C_k := \sup\{q(x) : x \in E_k\} < \infty$ for all $k \geq 0$.

Define a totally stable and conservative Q-matrix $Q^* = (q_{ij} : i, j \in \mathbf{Z}_+)$ as follows:

$$q_{ij} = \begin{cases} \inf\left\{\sum_{y \in E_j} q(x,y) : x \in E_i\right\}, & \text{if } j = i+1; \\ \sup\left\{\sum_{y \in E_j} q(x,y) : x \in E_i\right\}, & \text{if } j < i; \\ 0, & \text{other cases of } j \neq i. \end{cases}$$

Main results

Denote by $X(t)$ and $X^*(t)$ the Q-processes corresponding to $(q(x,y))$ and (q_{ij}) respectively.

Theorem 1.
(1) Uniqueness. If $X(t)$ is unique, then so is $X^*(t)$.

(2) Recurrence. Suppose that $E_0 = \{\theta\}$, where $\theta \in E$ is a reference point. Moreover, suppose that both $(q(x,y))$ and (q_{ij}) are irreducible and that $(q(x,y))$ is regular.

If $X(t)$ is recurrent, then so is $X^*(t)$.

Main results

Theorem 2.
(Ergodicity) Suppose that $E_0 = \{\theta\}$, where $\theta \in E$ is a reference point, and E_k is finite for all $k \geq 1$. Moreover, suppose that both $(q(x,y))$ and (q_{ij}) are irreducible and that $(q(x,y))$ is regular. Define $\tau = \inf\{t \geq 0 : X(t) = \theta\}$.

(1) Take an $x_i \in E_i$ such that $f_i := \mathbf{E}_{x_i}\tau = \min_{x \in E_i} \mathbf{E}_x \tau$. If f is increasing and $X(t)$ is ergodic (or strongly ergodic), then so is $X^*(t)$.

(2) Take some $\lambda \in (0, \inf_{x \in E} q(x))$ and an $x_i \in E_i$ such that $g_i := \mathbf{E}_{x_i} \mathrm{e}^{\lambda\tau} = \min_{x \in E_i} \mathbf{E}_x \mathrm{e}^{\lambda\tau}$. If g is increasing and $X(t)$ is exponentially ergodic, then so is $X^*(t)$.

Thanks