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m 生过程和 m 死过程的稳定性

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相关结果是与赵奕, 闫艳艳合作完成.

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- 1 Background
- 2 Recurrence & uniqueness
- 3 Ergodicity, strong ergodicity & Laplace transform

Stability:

- classical problems (uniqueness(cont.-time), recurrence, ergodicity(exp., strong ergodicity)).
- integral-type functionals: moments and distribution (Laplace transform).

$$\int_0^{\tau_0} V(X(t))dt.$$

- qualitative and quantitative results.

Background

Assume that Q is totally stable and conservative.

There exists such a positive integer $m \geq 1$ that

- m 生过程

$$q_{i,i+m} > 0, q_{ij} = 0 (j > i + m), \text{ upward.}$$

向上一致有限程.

- m 死过程

$$q_{i,i-m} > 0, q_{ij} = 0 (0 < j < i - m). \text{ downward.}$$

向下一致有限程.

When $m = 1$, [single birth, single death](#).

- Mao, Yan, Z.. Criteria for three classical problems of downwardly skip-free processes, 2024+.
- 张. 向上或向下一致有限程连续时间马氏链的常返性和唯一性准则. 中国科学: 数学, 2023, 53(11): 1437-1460.

Recurrence & uniqueness: 以2生为例

Consider the 2-birth process. Now let $q_i^{(k)} = \sum_{\ell=0}^k q_{i\ell}$ for all $0 \leq k \leq i$ and

$$F_n^{(n)} = 1, \quad F_i^{(n)} = \frac{1}{q_{i,i+2}} \sum_{k=n}^{i-1} q_i^{(k+1)} F_k^{(n)}, \quad i > n \geq 0;$$

$$d_0 = 0, \quad d_i = \frac{1}{q_{i,i+2}} \left(1 + \sum_{k=0}^{i-1} q_i^{(k+1)} d_k \right), \quad i \geq 1;$$

$$c_0 = 0, \quad c_i = \frac{1}{q_{i,i+2}} \left(-q_{i1} + \sum_{k=0}^{i-1} q_i^{(k+1)} c_k \right), \quad i \geq 1.$$

Recurrence criterion

Assume that the 2-birth Q -matrix $Q = (q_{ij})$ is irreducible. Then recur. iff

$$\lim_{n \rightarrow \infty} \frac{q_{01} F_n^{(0)} - q_{02} c_n}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}} = 0.$$

Moreover the return/extinction probability can be obtained by

$$\mathbb{P}_0(\sigma_0 < \infty) = 1 - \frac{1}{q_0} \lim_{n \rightarrow \infty} \frac{q_{01} F_n^{(0)} - q_{02} c_n}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}},$$

$$\mathbb{P}_1(\sigma_0 < \infty) = 1 - \lim_{n \rightarrow \infty} \frac{F_n^{(0)}}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}},$$

$$\mathbb{P}_2(\sigma_0 < \infty) = 1 + \lim_{n \rightarrow \infty} \frac{c_n}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}},$$

$$\mathbb{P}_i(\sigma_0 < \infty) = 1 - \lim_{n \rightarrow \infty} \frac{F_n^{(0)} \sum_{k=0}^{i-2} c_k - c_n \sum_{k=0}^{i-2} F_k^{(0)}}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}}, \quad i \geq 3.$$

Recurrence & uniqueness

Fix $n \in \mathbb{Z}_+$, define the first hitting time of the set $\{n+1, n+2, \dots\}$:

$$\tau_{n+} = \inf\{t > 0 : X(t) \geq n+1\}.$$

$$\text{recur.} \Leftrightarrow \lim_{n \rightarrow +\infty} \mathbb{P}_i(\tau_{n+} < \sigma_0) = 0.$$

Denote $x_i := \mathbb{P}_i(\tau_{n+} < \sigma_0)$, $i \geq 0$. By the strong Markov property and jump as well as 2-birth property of the process, it is obtained that

$$\begin{cases} x_0 = \frac{q_{01}}{q_0} \cdot x_1 + \frac{q_{02}}{q_0} \cdot x_2, \\ x_i = \sum_{j=1}^{i-1} \frac{q_{ij}}{q_i} \cdot x_j + \frac{q_{i,i+1}}{q_i} \cdot x_{i+1} + \frac{q_{i,i+2}}{q_i} \cdot x_{i+2}, & 1 \leq i \leq n-2, \\ x_{n-1} = \sum_{j=1}^{n-2} \frac{q_{n-1,j}}{q_{n-1}} \cdot x_j + \frac{q_{n-1,n}}{q_{n-1}} \cdot x_n + \frac{q_{n-1,n+1}}{q_{n-1}}, \\ x_n = \sum_{j=1}^{n-1} \frac{q_{n,j}}{q_n} \cdot x_j + \frac{q_{n,n+1} + q_{n,n+2}}{q_n}. \end{cases}$$

Recurrence & uniqueness

$$x_i = \sum_{k=0}^{i-2} c_k \cdot x_1 + \sum_{k=0}^{i-2} F_k^{(0)} \cdot x_2, \quad i \geq 3.$$

$$\begin{cases} \sum_{k=0}^{n-1} c_k x_1 + \sum_{k=0}^{n-1} F_k^{(0)} x_2 = 1, \\ c_n x_1 + F_n^{(0)} = 0, \end{cases} \Leftrightarrow \begin{cases} x_1 = \frac{F_n^{(0)}}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}} \\ x_2 = \frac{-c_n}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}}. \end{cases}$$

$$x_0 = \frac{q_{01}}{q_0} \cdot x_1 + \frac{q_{02}}{q_0} \cdot x_2.$$

Uniqueness criterion

Assume that the 2-birth Q -matrix $\hat{Q} = (\hat{q}_{ij})_{i,j \in \mathbb{Z}_+}$ is totally stable and conservative. Define an irreducible, totally stable and conservative Q -matrix Q on \mathbb{Z}_+ :

$$q_{01} = q_{02} = 1, \quad q_{0j} = 0, \quad j > 2; \quad q_{i0} = 1, \quad q_{ij} = \hat{q}_{i-1,j-1}, \quad i, j \geq 1, j \neq i.$$

Then the process determined by \hat{Q} is unique iff

$$\lim_{n \rightarrow \infty} \frac{F_n^{(0)} - c_n}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}} = 0.$$

Recurrence & uniqueness

- uniqueness:

$$u_i = \sum_{j \neq i} \frac{q_{ij}}{1 + q_i} u_j, \quad 0 \leq u_i \leq 1, i \geq 0; \quad u = \Pi(1)u.$$

We introduce a fictitious state Δ and define on the enlarged state space $E_\Delta = E \cup \{\Delta\}$ a new transition probability matrix

$$\Pi_{ij}^\Delta(1) = \begin{cases} \Pi(1), & i, j \in E; \\ \frac{1}{1+q_i}, & i \in E, j = \Delta; \\ p_j, & i = \Delta, j \in E, \end{cases}$$

where $(p_j, j \in E)$ is a positive probability measure.

Theorem

The original Q -process is unique iff the $\Pi^\Delta(1)$ -chain is recurrent.

Recurrence & uniqueness

- Chen. Practical criterion for uniqueness of Q -processes. Chinese J. Appl. Probab. Statist., 2015, 31: 213 – 224.
- Spieksma. Countable state Markov processes: Non-explosiveness and moment function. Probab. Engrg. Inform. Sci., 2015, 29: 623 – 637.

	Δ	0	1	2	3	...
Δ	-1	p_0	p_1	p_2	p_3	...
0	1	$-1 - q_0$	q_{01}	q_{02}	q_{03}	...
1	1	q_{10}	$-1 - q_1$	q_{12}	q_{13}	...
2	1	q_{20}	q_{21}	$-1 - q_2$	q_{23}	...
3	1	q_{30}	q_{31}	q_{32}	$-1 - q_3$...
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

Ergodicity & strong ergodicity

Given a totally stable, conservative and irreducible Q . For its minimal process $X(t)$, denote

$$\tau_0 = \inf\{t > 0 : X(t) = 0\}.$$

It is well known that $(\mathbb{E}_i \tau_0 : i \geq 1)$ is the minimal nonnegative solution to the equations

$$x_i = \sum_{j \neq i, 0} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad i \geq 1.$$

Approximation. Fix n , define a totally stable and conservative $Q^{(n)}$ on $\{0, 1, \dots, n\}$:

$$q_{ij}^{(n)} = \begin{cases} q_{ij}, & \text{if } 1 \leq i, j \leq n; \\ q_{i0} + \sum_{k > n} q_{ik}, & \text{if } 1 \leq i \leq n, j = 0. \end{cases}$$

$(\mathbb{E}_i \tau_0^{(n)} : 1 \leq i \leq n)$ is the minimal nonnegative solution to the equations

$$x_i = \sum_{1 \leq j \leq n, j \neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad 1 \leq i \leq n.$$

Ergodicity & strong ergodicity

$$x_i = \sum_{1 \leq j \leq n, j \neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad 1 \leq i \leq n.$$

Rewrite the above equation as $x = T^{(n)}x + f^{(n)}$, where the elements of $T^{(n)}$ and $f^{(n)}$ have the form

$$T_{ij}^{(n)} = \frac{q_{ij}}{q_i} \mathbb{1}_{\{1 \leq i \neq j \leq n\}}, \quad f_i^{(n)} = \frac{1}{q_i} \mathbb{1}_{\{1 \leq i \leq n\}}, \quad i, j \in \mathbb{Z}_+.$$

Then it is obvious that $T^{(n)}$ and $f^{(n)}$ are increasing in n in the element-wise sense.

$$\mathbb{E}_i \tau_0^{(n)} \uparrow \mathbb{E}_i \tau_0 \quad \text{as } n \uparrow +\infty.$$

$$\mathbb{E}_i \tau_0^{(n)} = \mathbb{E}_i (\tau_0 \wedge \tau_{n+}).$$

Ergodicity & strong ergodicity

$$x_i = \sum_{1 \leq j \leq n, j \neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad 1 \leq i \leq n.$$

The equations can be rewritten as

$$(I - \Pi)\mathbf{x} = \mathbf{r},$$

where I is the $n \times n$ unit matrix, $\mathbf{x} = (x_1, \dots, x_n)^\top$ and $\mathbf{r} = (\frac{1}{q_1}, \dots, \frac{1}{q_n})^\top$. By the irreducibility of Q , the local embedding chain is transient,

$$(I - \Pi)^{-1} = \sum_{n=0}^{\infty} \Pi^n < \infty \Rightarrow \mathbf{x} = \sum_{n=0}^{\infty} \Pi^n \mathbf{r},$$

there exists exactly a unique solution. The decomposition according to the first time arriving 0.

Ergodicity & strong ergodicity

- Z.F. Wei. Inverse problems for ergodicity of Markov chains. J. Math. Anal. Appl. 505 (2022) 125483.

By Theorem 12 (with $\ell = 1$), the minimal solution to Eq. (9) is the expectation of return time to state 0 of the $Q^{(n)}$ -process and is therefore finite (by the probabilistic description of algebraic ergodicity mentioned in Section 1 and the fact that an irreducible process on finite state space must be ergodic), where $Q^{(n)}$ has the following form:

$$Q^{(n)} = \begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ q_{10} + \sum_{k=n+1}^{\infty} q_{1,k} & q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n0} + \sum_{k=n+1}^{\infty} q_{n,k} & q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}_{(n+1) \times (n+1)}.$$

Now by Theorem 10, M_n is finite.

b) By Theorem 12, $(\mathbf{E}_i \sigma_0^{\ell+1})_{i \geq 1}$ is the minimal solution to

$$x_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} x_j + \frac{(\ell+1)}{q_i} \mathbf{E}_i \sigma_0^{\ell}, \quad i \geq 1.$$

Ergodicity & strong ergodicity

Regodicity & strong ergodicity criteria

Given a regular and irreducible 2-birth Q -matrix $Q = (q_{ij})$ on \mathbb{Z}_+ . Then

$$\mathbb{E}_k \tau_0 = \sum_{i=0}^{k-2} c_i \cdot \mathbb{E}_1 \tau_0 + \sum_{i=0}^{k-2} F_i^{(0)} \cdot \mathbb{E}_2 \tau_0 - \sum_{i=0}^{k-2} d_i, \quad k \geq 3;$$

and as $n \rightarrow +\infty$,

$$\begin{aligned} \mathbb{E}_1 \tau_0^{(n)} &= \frac{F_n^{(0)} \sum_{i=0}^{n-1} d_i - d_n \sum_{i=0}^{n-1} F_i^{(0)}}{F_n^{(0)} \sum_{i=0}^{n-1} c_i - c_n \sum_{i=0}^{n-1} F_i^{(0)}} \quad \uparrow \quad D_1 =: \mathbb{E}_1 \tau_0, \\ \mathbb{E}_2 \tau_0^{(n)} &= \frac{d_n \sum_{i=0}^{n-1} c_i - c_n \sum_{i=0}^{n-1} d_i}{F_n^{(0)} \sum_{i=0}^{n-1} c_i - c_n \sum_{i=0}^{n-1} F_i^{(0)}} \quad \uparrow \quad D_2 =: \mathbb{E}_2 \tau_0. \end{aligned}$$

Moreover, the process is erg. iff D_1 and D_2 is finite; str. erg. iff

$$\sup_{n \geq 1} \left(\left(\sum_{i=0}^n c_i \right) D_1 + \left(\sum_{i=0}^n F_i^{(0)} \right) D_2 - \sum_{i=0}^n d_i \right) < +\infty.$$

Remark

Single birth case:

- Yan & Chen (1986), Z. (2003).

$$d := \sup_{n \geq 1} \frac{\sum_{i=0}^{n-1} d_i}{\sum_{i=0}^{n-1} F_i^{(0)}} = \mathbb{E}_1 \tau_0.$$

- Chen & Z. (2014).

$$d := \overline{\lim}_{n \rightarrow +\infty} \frac{\sum_{i=0}^{n-1} d_i}{\sum_{i=0}^{n-1} F_i^{(0)}}.$$

•

$$\mathbb{E}_1 \tau_0^{(n)} = \frac{\sum_{i=0}^{n-1} d_i}{\sum_{i=0}^{n-1} F_i^{(0)}} \quad \uparrow \quad \mathbb{E}_1 \tau_0 = d.$$

Ergodicity & strong ergodicity

$$q_{ij}^{(n)} = \begin{cases} q_{ij}, & \text{if } 0 \leq i \leq n-2, 0 \leq j \leq n, \\ q_{ij}, & \text{if } i = n-1, n, 1 \leq j \leq n, \\ q_{i0} + q_{i,n+1}, & \text{if } i = n-1, j = 0, \\ q_{i0} + q_{i,n+1} + q_{i,n+2}, & \text{if } i = n, j = 0. \end{cases}$$

$$x_i = \sum_{1 \leq j \leq n, j \neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad 1 \leq i \leq n.$$

Let $w_i = x_{i+1} - x_i$ for all $i \geq 1$.

$$w_{i+1} = \sum_{k=1}^i \frac{F_i^{(k)}(-1 + q_k^{(1)}x_2 - q_{k1}x_1)}{q_{k,k+2}}, \quad 1 \leq i \leq n-2.$$

Ergodicity & strong ergodicity

Hence for all $j \geq m + 1$ we have

$$\begin{aligned}x_j &= \sum_{i=2}^{j-1} w_i + x_2 = x_2 \left(\sum_{i=1}^{j-2} F_i^{(0)} + 1 \right) + x_1 \sum_{i=1}^{j-2} c_i - \sum_{i=0}^{j-2} d_i \\ &= \sum_{i=0}^{j-2} c_i \cdot x_1 + \sum_{i=0}^{j-2} F_i^{(0)} \cdot x_2 - \sum_{i=0}^{j-2} d_i.\end{aligned}$$

- Rewrite the equations as

$$\begin{cases}x_i = \sum_{k=1, k \neq i}^{i+2} \frac{q_{ik}}{q_i} \cdot x_k + \frac{1}{q_i}, & 1 \leq i \leq n-2, \\ x_{n-1} = \sum_{k=1}^{n-2} \frac{q_{n-1,k}}{q_{n-1}} \cdot x_k + \frac{q_{n-1,n}}{q_{n-1}} \cdot x_n + \frac{1}{q_{n-1}}, \\ x_n = \sum_{k=1}^{n-1} \frac{q_{nk}}{q_n} \cdot x_k + \frac{1}{q_n}.\end{cases}$$

Ergodicity & strong ergodicity

- Solving the above equations we could derive that

$$\begin{cases} \sum_{k=0}^{n-1} c_k x_1 + \sum_{k=0}^{n-1} F_k^{(0)} x_2 = \sum_{k=0}^{n-1} d_k, \\ c_n x_1 + F_n^{(0)} x_2 = d_n. \end{cases}$$

- The assertion follows immediately by applying Cramer's rule.
- For general $i_0 \geq 1$, the method also works.

$$x_i = \frac{1}{q_i} \sum_{k \notin \{i_0, i\}} q_{ik} x_k + \frac{1}{q_i}, \quad i \neq i_0.$$

- For higher moments $\mathbb{E}_i \tau_0^\ell$, $\ell > 1$, it can be expressed by lower moments.

Laplace transform of τ_0

- Now we turn to focus on $\mathbb{E}_i e^{-\lambda \tau_0}$ ($i \geq 0$) to investigate the distribution of τ_0 .
- Define $\psi_{i,i_0}(\lambda) = 1 - \mathbb{E}_i e^{-\lambda \xi_{i_0}}$, $\xi_{i_0} = \int_0^{\tau_{i_0}} V(X(t)) dt$.

Lemma (Z.-T. Hou and Q.-F. Guo, 1978)

$(\psi_{i,i_0}(\lambda) : i \geq 0)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} \cdot y_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \quad i \geq 0.$$

- Here $V \equiv 1$, $i_0 = 0$.

Laplace transform of τ_0

Let $\tilde{q}_i^{(j)} := q_i^{(j)} + \lambda$ for $0 \leq j < i$ and let $\tilde{q}_i^{(i)} := q_i^{(i)}$. Correspondingly, we define the following notations.

$$\tilde{F}_n^{(n)} = 1, \quad \tilde{F}_i^{(n)} = \frac{1}{q_{i,i+2}} \sum_{k=n}^{i-1} \tilde{q}_i^{(k+1)} \tilde{F}_k^{(n)}, \quad 0 \leq n < i,$$

$$\tilde{c}_0 = 0, \quad \tilde{c}_i = \frac{1}{q_{i,i+2}} \left(-q_{i1} + \sum_{k=0}^{i-1} \tilde{q}_i^{(k+1)} \tilde{c}_k \right), \quad i \geq 1,$$

$$\tilde{d}_0 = 0, \quad \tilde{d}_i = \frac{1}{q_{i,i+2}} \left(\lambda + \sum_{k=0}^{i-1} \tilde{q}_i^{(k+1)} \tilde{d}_k \right), \quad i \geq 1,$$

Laplace transform of hitting time

Assume that the 2-birth Q -matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then the following relations holds:

$$\psi_{i0}(\lambda) = \sum_{k=0}^{i-2} \tilde{c}_k \cdot \psi_{10}(\lambda) + \sum_{k=0}^{i-2} \tilde{F}_k^{(0)} \cdot \psi_{20}(\lambda) - \sum_{\ell=0}^{i-2} \tilde{d}_\ell, \quad i \geq 3,$$

and the Laplace transform of τ_0 has the representation that

$$\mathbb{E}_1 e^{-\lambda \tau_0} = 1 - \lim_{n \rightarrow \infty} \frac{\tilde{F}_n^{(0)} \sum_{i=0}^{n-1} \tilde{d}_i - \tilde{d}_n \sum_{i=0}^{n-1} \tilde{F}_i^{(0)}}{\tilde{F}_n^{(0)} \sum_{i=0}^{n-1} \tilde{c}_i - \tilde{c}_n \sum_{i=0}^{n-1} \tilde{F}_i^{(0)}},$$

$$\mathbb{E}_2 e^{-\lambda \tau_0} = 1 - \lim_{n \rightarrow \infty} \frac{\tilde{d}_n \sum_{i=0}^{n-1} \tilde{c}_i - \tilde{c}_n \sum_{i=0}^{n-1} \tilde{d}_i}{\tilde{F}_n^{(0)} \sum_{i=0}^{n-1} \tilde{c}_i - \tilde{c}_n \sum_{i=0}^{n-1} \tilde{F}_i^{(0)}},$$

$$\mathbb{E}_0 e^{-\lambda \sigma_0} = \frac{q_{01}}{q_0 + \lambda} \mathbb{E}_1 e^{-\lambda \tau_0} + \frac{q_{02}}{q_0 + \lambda} \mathbb{E}_2 e^{-\lambda \tau_0}.$$

Example

Example

Given a regular and irreducible single death Q -matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-1} = a \geq 0, \quad q_{i,i+1} = b \geq 0, \quad q_{i,i+2} = d > 0, \quad i \geq 1,$$

and $q_{ij} = 0$ for other $i, j \geq 1, i \neq j$. Then the process is recurrent if and only if $a \geq b + 2d$ and it is ergodic if and only if $a > b + 2d$. Moreover in the recurrent case, starting from $j \geq 1$, the moment of the hitting time of 0 has the form:

$$\mathbb{E}_j \tau_0 = \frac{j}{a - b - 2d}, \quad j \geq 1.$$

Therefore the process could not be strongly ergodic.

- It can be regarded both as 2-birth processes and single death process.

- For single death processes, $\mathbb{E}_j \tau_0 = \sum_{k=1}^j \sum_{\ell=k}^{\infty} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}}, \quad j \geq 1.$

Laplace transform of integral-type functionals

- For general V , the method also works.

$$x_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} \cdot x_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \quad i \geq 0.$$

Given a regular and irreducible single death Q -matrix $Q = (q_{ij})$ on \mathbb{Z}_+ . Then it follows that

$$\mathbb{E}_i \tau_0 = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^i \sum_{\ell=k}^n \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} - \sum_{k=1}^n \sum_{\ell=k}^n \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} \cdot \frac{\sum_{k=1}^i G_k^{(n+1)}}{\sum_{k=1}^{n+1} G_k^{(n+1)}} \right), \quad i \geq 1.$$

[Z., Front. Math. China 2018]

$$\mathbb{E}_i \tau_0 = \sum_{k=1}^i \sum_{\ell=k}^{+\infty} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}}, \quad i \geq 1.$$

$$\begin{aligned} \mathbb{E}_i \tau_0 &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^i \sum_{\ell=k}^n \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} - \sum_{k=1}^n \sum_{\ell=k}^n \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} \cdot \frac{\sum_{k=1}^i G_k^{(n+1)}}{\sum_{k=1}^{n+1} G_k^{(n+1)}} \right) \\ &= \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \sum_{\ell=k}^n \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} \cdot \frac{\sum_{k=i+1}^{n+1} G_k^{(n+1)}}{\sum_{k=1}^{n+1} G_k^{(n+1)}} - \sum_{k=i+1}^n \sum_{\ell=k}^n \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} \right), \quad i \geq 1. \end{aligned}$$

Exponential moment

- Exponential moment & exponential ergodicity: $(\mathbb{E}_i e^{\lambda \tau_0} - 1, i \geq 1)$ is the minimal nonnegative solution to

$$x_i = \sum_{j \neq i, 0} \frac{q_{ij}}{q_i - \lambda} \cdot x_j + \frac{\lambda}{q_i - \lambda}, \quad i \geq 1,$$

where $0 < \lambda < q_i$ for all i .

In the single birth case, let $\tilde{q}_i^{(k)} = q_i^{(k)} - \lambda$, $\tilde{d}_i = \dots$, $\tilde{F}_i^{(k)} = \dots$. Now the approximating equation is

$$x_i = \sum_{j=1}^{i-1} \frac{q_{ij}}{q_i - \lambda} \cdot x_j + \frac{q_{i,i+1}}{q_i - \lambda} \cdot x_{i+1} + \frac{\lambda}{q_i - \lambda}, \quad 1 \leq i \leq n-1,$$

$$x_n = \sum_{j=1}^{n-1} \frac{q_{nj}}{q_n - \lambda} \cdot x_j + \frac{\lambda}{q_n - \lambda}.$$

Exponential moment

Then

$$w_i = x_{i+1} - x_i = \tilde{F}_i^{(0)} x_1 - \lambda \tilde{d}_i, \quad 1 \leq i < n.$$

Further

$$x_i = \sum_{k=0}^{i-1} (\tilde{F}_k^{(0)} x_1 - \lambda \tilde{d}_k), \quad 1 \leq i \leq n.$$

Substituting it into

$$(q_n - \lambda)x_n = \sum_{j=1}^{n-1} q_{nj} x_j + \lambda,$$

we have

$$x_1 = \frac{\lambda \sum_{k=0}^n \tilde{d}_k}{\sum_{k=0}^n \tilde{F}_k^{(0)}} = \mathbb{E}_1 e^{\lambda \tau_0^{(n)}} - 1.$$

$$\text{exp. ergoc} \Leftrightarrow \exists 0 < \lambda < q_i \ (i \geq 0) \text{ s.t. } \frac{\sum_{k=0}^n \tilde{d}_k}{\sum_{k=0}^n \tilde{F}_k^{(0)}} \uparrow \tilde{d} < +\infty.$$

Thank you for your attention!

Homepage: <http://math0.bnu.edu.cn/~zhangyh/>