连续时间向上一致有限程马氏链的稳定性

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CONTENTS

Background

Stability:

- classical problems (uniqueness(cont.-time), recurrence, ergodicity(exp., strong ergodicity)).
- integral-type functionals: moments and distribution (Laplace transform).

$$\int_0^{\tau_0} V(X(t))\mathrm{d}t.$$

- qualitative and quantitative results.

Background

连续时间向上一致有限程马氏链?
Assume that Q is totally stable and conservative.
There exists such a positive integer $m \geqslant 1$ that

$$q_{i,i+m} > 0, q_{ij} = 0\,(j > i+m), \text{ upward.}$$

(or

$$q_{i,i-m} > 0, q_{ij} = 0\,(0 < j < i-m). \text{ downward)}$$

Generally we call them m-birth and m-death respectively.

When $m = 1$, single birth, single death.

- single birth: Yan & Chen (1986), Z.(2001). [21 papers]
- single death: Z.(2018) [10 papers]
- m-birth & m-death: Z.(2023), uniqueness and recurrence.
- 2-death: Z. & Zhao.(2024+), invariant measures & ergodicity.
- 2-birth: Yan, Z.& Zhao (2024+).

以$\mathbb{E}_i\tau_0$为例

Given a totally stable, conservative and irreducible Q. For its minimal process $X(t)$, denote

$$\tau_0 = \inf\{t > 0 : X(t) = 0\}.$$

It is well known that $(\mathbb{E}_i\tau_0 : i \geqslant 1)$ is the minimal nonnegative solution to the equations

$$x_i = \sum_{j\neq i,0} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad i \geqslant 1.$$

Approximation. Fix n, define a totally stable and conservative $Q^{(n)}$ on $\{0, 1, \cdots, n\}$:

$$q_{ij}^{(n)} = \begin{cases} q_{ij}, & \text{if } 1 \leqslant i, j \leqslant n; \\ q_{i0} + \sum_{k>n} q_{ik}, & \text{if } 1 \leqslant i \leqslant n, j = 0. \end{cases}$$

$(\mathbb{E}_i\tau_0^{(n)} : 1 \leqslant i \leqslant n)$ is the minimal nonnegative solution to the equations

$$x_i = \sum_{1\leqslant j\leqslant n, j\neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad 1 \leqslant i \leqslant n.$$

以$\mathbb{E}_i\tau_0$为例

$$x_i = \sum_{1\leqslant j\leqslant n, j\neq i} \frac{q_{ij}}{q_i}\cdot x_j + \frac{1}{q_i}, \quad 1\leqslant i\leqslant n.$$

Rewrite the above equation as $x = T^{(n)}x + f^{(n)}$, where the elements of $T^{(n)}$ and $f^{(n)}$ have the form

$$T_{ij}^{(n)} = \frac{q_{ij}}{q_i}\mathbb{1}_{\{1\leqslant i\neq j\leqslant n\}},\ f_i^{(n)} = \frac{1}{q_i}\mathbb{1}_{\{1\leqslant i\leqslant n\}},\ i,j\in\mathbb{Z}_+.$$

Then it is obvious that $T^{(n)}$ and $f^{(n)}$ are increasing in n in the element-wise sense.

$$\mathbb{E}_i\tau_0^{(n)} \uparrow \mathbb{E}_i\tau_0 \quad \text{as} \quad n\uparrow +\infty.$$

以$\mathbb{E}_i\tau_0$为例

$$x_i = \sum_{1\leqslant j\leqslant n, j\neq i} \frac{q_{ij}}{q_i}\cdot x_j + \frac{1}{q_i}, \quad 1\leqslant i\leqslant n.$$

The equations can be rewritten as

$$(I-\Pi)\mathbf{x} = \mathbf{r},$$

where I is the $n\times n$ unit matrix, $\mathbf{x} = (x_1, \cdots, x_n)^\mathsf{T}$ and $\mathbf{r} = (\frac{1}{q_1}, \cdots, \frac{1}{q_n})^\mathsf{T}$. By the irreducibility of Q, the local embedding chain is transient,

$$(I-\Pi)^{-1} = \sum_{n=0}^{\infty}\Pi^n < \infty \Rightarrow \mathbf{x} = \sum_{n=0}^{\infty}\Pi^n\mathbf{r},$$

there exists exactly a unique solution. The decomposition according to the first time arriving 0.

- Z.F. Wei. Inverse problems for ergodicity of Markov chains. J. Math. Anal. Appl. 505 (2022) 125483.

By Theorem 12 (with $\ell = 1$), the minimal solution to Eq. (9) is the expectation of return time to state 0 of the $Q^{(n)}$-process and is therefore finite (by the probabilistic description of algebraic ergodicity mentioned in Section 1 and the fact that an irreducible process on finite state space must be ergodic), where $Q^{(n)}$ has the following form:

$$Q^{(n)} = \begin{pmatrix} -n & 1 & 1 & \cdots & 1 \\ q_{10} + \sum_{k=n+1}^{\infty} q_{1,k} & q_{11} & q_{12} & \cdots & q_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n0} + \sum_{k=n+1}^{\infty} q_{n,k} & q_{n1} & q_{n2} & \cdots & q_{nn} \end{pmatrix}_{(n+1)\times(n+1)}.$$

Now by Theorem 10, M_n is finite.

b) By Theorem 12, $\left(\mathbf{E}_i\, \sigma_0^{\ell+1}\right)_{i\geqslant 1}$ is the minimal solution to

$$x_i = \sum_{j\neq i} \frac{q_{ij}}{q_i} x_j + \frac{(\ell+1)}{q_i}\, \mathbf{E}_i\, \sigma_0^{\ell}, \qquad i \geqslant 1.$$

以2生, $\mathbb{E}_i\tau_0$为例

Consider the 2-birth process. Now let $q_i^{(k)} = \sum_{\ell=0}^{k} q_{i\ell}$ for all $0 \leqslant k \leqslant i$ and

$$F_n^{(n)} = 1, \qquad F_i^{(n)} = \frac{1}{q_{i,i+2}} \sum_{k=n}^{i-1} q_i^{(k+1)} F_k^{(n)}, \qquad i > n \geqslant 0;$$

$$d_0 = 0, \qquad d_i = \frac{1}{q_{i,i+2}} \left(1 + \sum_{k=0}^{i-1} q_i^{(k+1)} d_k\right), \qquad i \geqslant 1;$$

$$c_0 = 0, \qquad c_i = \frac{1}{q_{i,i+2}} \left(-q_{i1} + \sum_{k=0}^{i-1} q_i^{(k+1)} c_k\right), \qquad i \geqslant 1.$$

Theorem 1 (Regodicity & strong ergodicity)

Given a regular and irreducible 2-birth Q-matrix $Q = (q_{ij})$ on $\mathbb{Z}_+$. Then

$$\mathbb{E}_k\tau_0 = \sum_{i=0}^{k-2} c_i \cdot \mathbb{E}_1\tau_0 + \sum_{i=0}^{k-2} F_i^{(0)} \cdot \mathbb{E}_2\tau_0 - \sum_{i=0}^{k-2} d_i, \ \ k \geqslant 3;$$

and as $n \to +\infty$,

$$\mathbb{E}_1\tau_0^{(n)} = \frac{F_n^{(0)} \sum_{i=0}^{n-1} d_i - d_n \sum_{i=0}^{n-1} F_i^{(0)}}{F_n^{(0)} \sum_{i=0}^{n-1} c_i - c_n \sum_{i=0}^{n-1} F_i^{(0)}} \quad \Big\uparrow \quad D_1 =: \mathbb{E}_1\tau_0,$$

$$\mathbb{E}_2\tau_0^{(n)} = \frac{d_n \sum_{i=0}^{n-1} c_i - c_n \sum_{i=0}^{n-1} d_i}{F_n^{(0)} \sum_{i=0}^{n-1} c_i - c_n \sum_{i=0}^{n-1} F_i^{(0)}} \quad \Big\uparrow \quad D_2 =: \mathbb{E}_2\tau_0.$$

Moreover, the process is erg. iff D_1 and D_2 is finite; str. erg. iff

$$\sup_{n\geqslant 1}\left(\left(\sum_{i=0}^{n} c_i\right) D_1 + \left(\sum_{i=0}^{n} F_i^{(0)}\right) D_2 - \sum_{i=0}^{n} d_i\right) < +\infty.$$

以2生, $\mathbb{E}_i\tau_0$ 为例

Remark

Single birth case:

- Yan & Chen (1986), Z. (2003).

$$d := \sup_{n\geqslant 1} \frac{\sum_{i=0}^{n-1} d_i}{\sum_{i=0}^{n-1} F_i^{(0)}} = \mathbb{E}_1\tau_0.$$

- Chen & Z. (2014).

$$d := \varlimsup_{n\to+\infty} \frac{\sum_{i=0}^{n-1} d_i}{\sum_{i=0}^{n-1} F_i^{(0)}}.$$

- $$\mathbb{E}_1\tau_0^{(n)} = \frac{\sum_{i=0}^{n-1} d_i}{\sum_{i=0}^{n-1} F_i^{(0)}} \quad \Big\uparrow \quad \mathbb{E}_1\tau_0 = d.$$

以2生, $\mathbb{E}_i\tau_0$ 为例

$$q_{ij}^{(n)} = \begin{cases} q_{ij}, & \text{if } 0 \leqslant i \leqslant n-2, 0 \leqslant j \leqslant n, \\ q_{ij}, & \text{if } i = n-1, n, 1 \leqslant j \leqslant n, \\ q_{i0} + q_{i,n+1}, & \text{if } i = n-1, j = 0, \\ q_{i0} + q_{i,n+1} + q_{i,n+2}, & \text{if } i = n, j = 0. \end{cases}$$

$$x_i = \sum_{1 \leqslant j \leqslant n, j \neq i} \frac{q_{ij}}{q_i} \cdot x_j + \frac{1}{q_i}, \quad 1 \leqslant i \leqslant n.$$

Let $w_i = x_{i+1} - x_i$ for all $i \geqslant 1$.

$$w_{i+1} = \sum_{k=1}^{i} \frac{F_i^{(k)}(-1 + q_k^{(1)} x_2 - q_{k1} x_1)}{q_{k,k+2}}, \ 1 \leqslant i \leqslant n-2.$$

Hence for all $j \geqslant m+1$ we have

$$\begin{aligned} x_j &= \sum_{i=2}^{j-1} w_i + x_2 = x_2\left(\sum_{i=1}^{j-2} F_i^{(0)} + 1\right) + x_1 \sum_{i=1}^{j-2} c_i - \sum_{i=0}^{j-2} d_i \\ &= \sum_{i=0}^{j-2} c_i \cdot x_1 + \sum_{i=0}^{j-2} F_i^{(0)} \cdot x_2 - \sum_{i=0}^{j-2} d_i. \end{aligned}$$

- Rewrite the equation as

$$\begin{cases} x_i = \displaystyle\sum_{k=1,k\neq i}^{i+2} \frac{q_{ik}}{q_i} \cdot x_k + \frac{1}{q_i}, \qquad 1 \leqslant i \leqslant n-2, \\ x_{n-1} = \displaystyle\sum_{k=1}^{n-2} \frac{q_{n-1,k}}{q_{n-1}} \cdot x_k + \frac{q_{n-1,n}}{q_{n-1}} \cdot x_n + \frac{1}{q_{n-1}}, \\ x_n = \displaystyle\sum_{k=1}^{n-1} \frac{q_{nk}}{q_n} \cdot x_k + \frac{1}{q_n}. \end{cases}$$

以2生, $\mathbb{E}_i\tau_0$为例

- Solving the above equations we could derive that

$$D^{(n)}\mathbf{x} = \mathbf{h}^{(n)}, \quad \text{i.e.} \quad \begin{cases} d_{11}^{(n)}x_1 + d_{12}^{(n)}x_2 = h_1^{(n)}, \\ d_{21}^{(n)}x_1 + d_{22}^{(n)}x_2 = h_2^{(n)}, \end{cases}$$

where $h_1^{(n)} = \sum_{k=0}^{n-1} d_k$ and $h_2^{(n)} = d_n$.

- The assertion follows immediately by applying Cramer's rule.

- For general $i_0 \geqslant 1$, the method also works.

$$x_i = \frac{1}{q_i} \sum_{k \notin \{i_0, i\}} q_{ik}x_k + \frac{1}{q_i}, \quad i \neq i_0.$$

- For higher moments $\mathbb{E}_i\tau_0^\ell,\ \ell > 1$, it can be expressed by lower moments.

Laplace transform of τ_0, $m = 2$

- Now we turn to focus on $\mathbb{E}_i e^{-\lambda \tau_0}$ $(i \geqslant 0)$ to investigate the distribution of τ_0.
- Define $\psi_{i,i_0}(\lambda) = 1 - \mathbb{E}_i e^{-\lambda \xi_{i_0}}$.

Lemma (Z.-T. Hou and Q.-F. Guo, 1978)

$(\psi_{i,i_0}(\lambda) : i \geqslant 0)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} \cdot y_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \quad i \geqslant 0.$$

- Here $V \equiv 1$, $i_0 = 0$.

Laplace transform of τ_0, $m = 2$

Let $\tilde{q}_i^{(j)} := q_i^{(j)} + \lambda$ for $0 \leqslant j < i$ and let $\tilde{q}_i^{(i)} := q_i^{(i)}$. Correspondingly, we define the following notations.

$$\tilde{F}_n^{(n)} = 1,\ \tilde{F}_i^{(n)} = \frac{1}{q_{i,i+2}} \sum_{k=n}^{i-1} \tilde{q}_i^{(k+1)} \tilde{F}_k^{(n)},\ 0 \leqslant n < i,$$

$$\tilde{c}_0 = 0,\ \tilde{c}_i = \frac{1}{q_{i,i+2}} \bigg(- q_{i1} + \sum_{k=0}^{i-1} \tilde{q}_i^{(k+1)} \tilde{c}_k \bigg),\ i \geqslant 1,$$

$$\tilde{d}_0 = 0,\ \tilde{d}_i = \frac{1}{q_{i,i+2}} \bigg(\lambda + \sum_{k=0}^{i-1} \tilde{q}_i^{(k+1)} \tilde{d}_k \bigg),\ i \geqslant 1,$$

Laplace transform of τ_0, $m = 2$

Theorem 2 (Laplace transform)

Assume that the 2-birth Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is recurrent. Then the following relations holds:

$$\psi_{i0}(\lambda) = \sum_{k=0}^{i-2} \tilde{c}_k \cdot \psi_{10}(\lambda) + \sum_{k=0}^{i-2} \tilde{F}_k^{(0)} \cdot \psi_{20}(\lambda) - \sum_{\ell=0}^{i-2} \tilde{d}_\ell, \quad i \geqslant 3,$$

and the Laplace transform of τ_0 has the representation that

$$\mathbb{E}_1 \mathrm{e}^{-\lambda\tau_0} = 1 - \lim_{n\to\infty} \frac{\tilde{F}_n^{(0)} \sum_{i=0}^{n-1} \tilde{d}_i - \tilde{d}_n \sum_{i=0}^{n-1} \tilde{F}_i^{(0)}}{\tilde{F}_n^{(0)} \sum_{i=0}^{n-1} \tilde{c}_i - \tilde{c}_n \sum_{i=0}^{n-1} \tilde{F}_i^{(0)}},$$

$$\mathbb{E}_2 \mathrm{e}^{-\lambda\tau_0} = 1 - \lim_{n\to\infty} \frac{\tilde{d}_n \sum_{i=0}^{n-1} \tilde{c}_i - \tilde{c}_n \sum_{i=0}^{n-1} \tilde{d}_i}{\tilde{F}_n^{(0)} \sum_{i=0}^{n-1} \tilde{c}_i - \tilde{c}_n \sum_{i=0}^{n-1} \tilde{F}_i^{(0)}},$$

$$\mathbb{E}_0 \mathrm{e}^{-\lambda\sigma_0} = \frac{q_{01}}{q_0 + \lambda} \mathbb{E}_1 \mathrm{e}^{-\lambda\tau_0} + \frac{q_{02}}{q_0 + \lambda} \mathbb{E}_2 \mathrm{e}^{-\lambda\tau_0}.$$

Example

Example

Given a regular and irreducible single death Q-matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-1} = a \geqslant 0, \quad q_{i,i+1} = b \geqslant 0, \quad q_{i,i+2} = d > 0, \quad i \geqslant 1,$$

and $q_{ij} = 0$ for other $i, j \geqslant 1,\ i \neq j$. Then the process is recurrent if and only if $a \geqslant b + 2d$ and it is ergodic if and only if $a > b + 2d$. Moreover in the recurent case, starting form $j \geqslant 1$, the moment of the hitting time of 0 has the form:

$$\mathbb{E}_j \tau_0 = \frac{j}{a - b - 2d}, \quad j \geqslant 1.$$

Therefore the process could not be strongly ergodic.

- It can be regarded both as 2-birth processes and single death process.
- For single death processes, $\mathbb{E}_j \tau_0 = \sum_{k=1}^{j} \sum_{\ell=k}^{\infty} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}, \quad j \geqslant 1.$

- For general V, the method also works.

$$x_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} \cdot x_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \ i \geqslant 0.$$

Theorem 3 (Recurrence)

Assume that the 2-birth Q-matrix $Q = (q_{ij})$ is irreducible. Then recur. iff

$$\lim_{n\to\infty} \frac{q_{01}F_n^{(0)} - q_{02}c_n}{F_n^{(0)}\sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}} = 0.$$

Moreover the return/extinction probability can be obtained by

$$\mathbb{P}_0(\sigma_0 < \infty) = 1 - \frac{1}{q_0}\lim_{n\to\infty} \frac{q_{01}F_n^{(0)} - q_{02}c_n}{F_n^{(0)}\sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}},$$

$$\mathbb{P}_1(\sigma_0 < \infty) = 1 - \lim_{n\to\infty} \frac{F_n^{(0)}}{F_n^{(0)}\sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}},$$

$$\mathbb{P}_2(\sigma_0 < \infty) = 1 + \lim_{n\to\infty} \frac{c_n}{F_n^{(0)}\sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}},$$

$$\mathbb{P}_i(\sigma_0 < \infty) = 1 - \lim_{n\to\infty} \frac{F_n^{(0)}\sum_{k=0}^{i-2} c_k - c_n \sum_{k=0}^{i-2} F_k^{(0)}}{F_n^{(0)}\sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}}, \ i \geqslant 3.$$

Recurrence & uniqueness

Fix $n \in \mathbb{Z}_+$, define the first hitting time of the set $\{n+1, n+2, \cdots\}$:

$$\tau_{n+} = \inf\{t > 0 : X(t) \geqslant n+1\}.$$

$$\text{recur.} \Leftrightarrow \lim_{n \to +\infty} \mathbb{P}_i\left(\tau_{n+} < \sigma_0\right) = 0.$$

Denote $x_i := \mathbb{P}_i\left(\tau_{n+} < \sigma_0\right), \ i \geqslant 0$. By the strong Markov property and jump as well as 2-birth property of the process, it is obtained that

$$\begin{cases} x_0 = \dfrac{q_{01}}{q_0} \cdot x_1 + \dfrac{q_{02}}{q_0} \cdot x_2, \\ x_i = \displaystyle\sum_{j=1}^{i-1} \dfrac{q_{ij}}{q_i} \cdot x_j + \dfrac{q_{i,i+1}}{q_i} \cdot x_{i+1} + \dfrac{q_{i,i+2}}{q_i} \cdot x_{i+2}, & 1 \leqslant i \leqslant n-2, \\ x_{n-1} = \displaystyle\sum_{j=1}^{n-2} \dfrac{q_{n-1,j}}{q_{n-1}} \cdot x_j + \dfrac{q_{n-1,n}}{q_{n-1}} \cdot x_n + \dfrac{q_{n-1,n+1}}{q_{n-1}}, \\ x_n = \displaystyle\sum_{j=1}^{n-1} \dfrac{q_{n,j}}{q_n} \cdot x_j + \dfrac{q_{n,n+1} + q_{n,n+2}}{q_n}. \end{cases}$$

Recurrence & uniqueness

Theorem 4 (Uniqueness)

Assume that the 2-birth Q-matrix $\hat{Q} = (\hat{q}_{ij})_{i,j\in\mathbb{Z}_+}$ is totally stable and conservative. Define an irreducible, totally stable and conservative Q-matrix Q on $\mathbb{Z}_+$:

$$q_{01} = q_{02} = 1,\ q_{0j} = 0,\ j > 2;\ q_{i0} = 1,\ q_{ij} = \hat{q}_{i-1,j-1},\ i, j \geqslant 1, j \neq i.$$

Then the process determined by $\hat{Q}$ is unique iff

$$\lim_{n\to\infty} \frac{F_n^{(0)} - c_n}{F_n^{(0)} \sum_{k=0}^{n-1} c_k - c_n \sum_{k=0}^{n-1} F_k^{(0)}} = 0.$$

Recurrence & uniqueness

- uniqueness:

$$u_i = \sum_{j \neq i} \frac{q_{ij}}{1+q_i} u_j, \quad 0 \leqslant u_i \leqslant 1, i \geqslant 0,$$

$$u = \Pi(1)u.$$

We introduce a fictitious state Δ and define on the enlarged state space $E_\Delta = E \cup \{\Delta\}$ a new transition probability matrix

$$\Pi^{\Delta}_{ij}(1) = \begin{cases} \Pi(1), & i, j \in E; \\ \frac{1}{1+q_i}, & i \in E, j = \Delta; \\ p_j, & i = \Delta, j \in E, \end{cases}$$

where $(p_j, j \in E)$ is a positive probability measure.

Theorem

The original Q-process is unique iff the $\Pi^{\Delta}(1)$-chain is recurrent.

Recurrence & uniqueness

$$\begin{array}{ccccccc}
 & \Delta & 0 & 1 & 2 & 3 & \cdots \\
\Delta & -1 & p_0 & p_1 & p_2 & p_3 & \cdots \\
0 & 1 & -1-q_0 & q_{01} & q_{02} & q_{03} & \cdots \\
1 & 1 & q_{10} & -1-q_1 & q_{12} & q_{13} & \cdots \\
2 & 1 & q_{20} & q_{21} & -1-q_2 & q_{23} & \cdots \\
3 & 1 & q_{30} & q_{31} & q_{32} & -1-q_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Further work

- Exponential moment & exponential ergodicity: $(\mathbb{E}_i e^{\lambda\tau_0} - 1, i \geqslant 1)$ is the minimal nonnegative solution to

$$x_i = \sum_{j \neq i,0} \frac{q_{ij}}{q_i - \lambda} \cdot x_j + \frac{\lambda}{q_i - \lambda}, \ i \geqslant 1,$$

where $0 < \lambda < q_i$ for all i.

In the single birth case, let $\tilde{q}_i^{(k)} = q_i^{(k)} - \lambda$, $\tilde{d}_i = \cdots$, $\tilde{F}_i^{(k)} = \cdots$. Now the approximating equation is

$$x_i = \sum_{j=1}^{i-1} \frac{q_{ij}}{q_i - \lambda} \cdot x_j + \frac{q_{i,i+1}}{q_i - \lambda} \cdot x_{i+1} + \frac{\lambda}{q_i - \lambda}, \ 1 \leqslant i \leqslant n-1,$$

$$x_n = \sum_{j=1}^{n-1} \frac{q_{nj}}{q_n - \lambda} \cdot x_j + \frac{\lambda}{q_n - \lambda}.$$

Further work

Then

$$w_i = x_{i+1} - x_i = \tilde{F}_i^{(0)} x_1 - \lambda \tilde{d}_i,\ 1 \leqslant i < n.$$

Further

$$x_i = \sum_{k=0}^{i-1} (\tilde{F}_k^{(0)} x_1 - \lambda \tilde{d}_k),\ 1 \leqslant i \leqslant n.$$

Substituting it into

$$(q_n - \lambda) x_n = \sum_{j=1}^{n-1} q_{nj} x_j + \lambda,$$

we have

$$x_1 = \frac{\lambda \sum_{k=0}^{n} \tilde{d}_k}{\sum_{k=0}^{n} \tilde{F}_k^{(0)}} = \mathbb{E}_1 \mathrm{e}^{\lambda \tau_0^{(n)}} - 1.$$

exp. ergoc $\Leftrightarrow \exists 0 < \lambda < q_i \ (i \geqslant 0)$ s.t. $\dfrac{\sum_{k=0}^{n} \tilde{d}_k}{\sum_{k=0}^{n} \tilde{F}_k^{(0)}} \uparrow \tilde{d} < +\infty.$

Further work

- For m-death?

Given a regular and irreducible single death Q-matrix $Q = (q_{ij})$ on $\mathbb{Z}_+$. Then it follows that

$$\mathbb{E}_i \tau_0 = \lim_{n\to\infty} \left(\sum_{k=1}^{i} \sum_{\ell=k}^{n} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} - \sum_{k=1}^{i} G_k^{(n+1)} \cdot \frac{\sum_{k=1}^{n} \sum_{\ell=k}^{n} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}}{\sum_{k=1}^{n+1} G_k^{(n+1)}} \right), \ i \geqslant 1.$$

[Z., Front. Math. China 2018]

$$\mathbb{E}_i \tau_0 = \sum_{k=1}^{i} \sum_{\ell=k}^{+\infty} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}, \ i \geqslant 1.$$

For 2-death?

Further work

- QSD? 2-death?
- Y.-Y. Li &J.-P. Li. Down/up crossing properties of weighted Markov collision processes. Front. Math. China 2021, 16(2): 525 – 542.

$$q_{ij} = \begin{cases} w_i b_{j-i+2}, & i \geqslant 2, j \geqslant i-2, \\ 0, & \text{otherwise.} \end{cases}$$

where $b_j \geqslant 0\,(j \neq 2)$, $0 < -b_2 = \sum_{j \neq 2} b_j < +\infty$, $w_i > 0\,(i \geqslant 2)$. They consider the $(\mathbb{S}-2)$-range crossing number of the process until its extinction, i.e., the joint probability distribution of the $|\mathbb{S}|$-dimensional random vector $(Y(t) = (Y_i(t), i \in \mathbb{S})$, where $Y_i(t)$ is the $(i-2)$-step crossing number of the process until $X(t)$'s extinction.

$$\mathbb{S} = \{0,1\}, \{k\}, \{0,1,k\}\,(k \geqslant 3),\ \ (X(t), Y(t))$$

Dual method? 2-birth?

Thank you for your attention!

Homepage: http://math0.bnu.edu.cn/~zhangyh/