On classical problems about jump processes with finite skip

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CONTENTS

Background

- Q-process, jump process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  (1) Normal condition: $p_{ij}(t) \geq 0,\ \sum_j p_{ij}(t) \leq 1,\ \ i, j \in \mathbb{Z}_+, t \geq 0.$

  (2) Chapman-Kolmogorov equation: $p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s).$

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij},\ \ i, j \in \mathbb{Z}_+.$

  (4) Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ \ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

$$0 \leqslant q_{ij} < \infty,\ \ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ \ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ \ i \in \mathbb{Z}_+.$
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ \ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0.$

Background

- jump process with finite skip: $\exists m \geqslant 1$,

$$q_{i,i+m} > 0, q_{ij} = 0\,(j > i + m), i > 0; \text{ upward}$$

  or

$$q_{i,i-m} > 0, q_{ij} = 0\,(0 \leqslant j < i - m), i > m. \text{ downward}$$

- When $m = 1$, single birth, single death.
- Three classical problems: uniqueness, recurrence, ergodicity.

Background

- uniqueness: when there is only one Q-process $P(t) = (p_{ij}(t))$ for a given Q-matrix $Q = (q_{ij})$ (then the matrix Q is often called regular).
- Hou (1974, 1982), Hou & Guo (1984), Zheng (1982), Chen & Zheng (1983), Chen (1991).

Uniqueness criterion[Feller(1957), Reuter(1957)]

For a given Q-matrix $Q = (q_{ij})$, the Q-process is unique iff the equation has only the equation

$$(\lambda + q_i)u_i = \sum_{j \neq i} q_{ij} u_j, 0 \leqslant u_i \leqslant 1, i \geqslant 0$$

has only the trivial solution $u_i \equiv 0$ for some (equivalently, for all) $\lambda > 0$.

Background

For birth-death (a_i, b_i): $a_i = q_{i,i-1} > 0, i \geq 1; b_i = q_{i,i+1} > 0, i \geq 0$.

Uniqueness criterion for BDP

Given a birth-death Q matrix (a_i, b_i). The the birth-death process is unique iff

$$\sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} \sum_{k=0}^{n} \mu_k = \infty,$$

where

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geq 1.$$

Background

- For single birth: $q_{i,i+1} > 0, q_{ij} = 0\,(j > i+1), i \geqslant 0.$

Uniqueness criterion for SBP

Given a single birth Q matrix (a_i, b_i). The the single birth process is unique iff

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}} = \infty,$$

where

$$F_i^{(i)} = 1, \qquad F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \quad 0 \le i < n; \quad q_n^{(k)} = \sum_{j=0}^{k} q_{nj}.$$

Background

- For single death: $q_{i,i-1} > 0, q_{ij} = 0\,(0 \leqslant j < i-1), i \geqslant 2$. Uniqueness? Mao, Yan & Zhang (2021+): Three basic criteria for downwardly skip-free Markov processes.
- Li & Li: Down/up crossing properties of weighted Markov collision process. Front. Math. China, 2021, 16(2): 525-542.
- Recurrence: if for each $h > 0$, $P(h)$ is recurrent, $\int_0^\infty p_{ii}(t)dt = \infty$.

embedding chain (Π_{ij}): $\Pi_{ij} = I_{\{q_i \neq 0\}}(1-\delta_{ij})\dfrac{q_{ij}}{q_i} + I_{\{q_i=0\}}\delta_{ij}$.

Recurrence Theorem[Feller(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$,

$$\int_0^\infty p_{ij}^{\min}(t)dt = \sum_{n=0}^\infty \frac{\Pi_{ij}^{(n)}}{q_j}.$$

In particular, if Q is irreducible and regular, then $P(t)$ is recurrent iff so is its embedding chain.

Theorem

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ is recurrent iff

$$x_i = \sum_{j \neq 0, i} \Pi_{ij} x_j, \quad 0 \leqslant x_i \leqslant 1, \, i \geqslant 0$$

has only a trivial solution.

Recurrence criterion for BDP[Yan & Chen(1986)]

Given a regular birth-death Q-matrix (a_i, b_i). The the birth-death is recurrent iff $\sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} = \infty$.

Uniqueness criterion for SBP

Assume the single birth Q-matrix $Q = (q_{ij})$ is regular and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Background

- $$x_i = \sum_{j \neq 0, i} \Pi_{ij} x_j, \quad 0 \leqslant x_i \leqslant 1,\ i \geqslant 0$$
- uniqueness:
$$u_i = \sum_{j \neq i} \frac{q_{ij}}{\lambda + q_i} u_j, \quad 0 \leqslant u_i \leqslant 1, i \geqslant 0,$$
$$u = \Pi(\lambda) u.$$

We introduce a fictitious state Δ and define on the enlarged state space $E_\Delta = E \cup \{\Delta\}$ a new transition probability matrix

$$\Pi_{ij}^{\Delta}(\lambda) = \begin{cases} \Pi(\lambda), & i, j \in E; \\ \frac{\lambda}{\lambda + q_i}, & i \in E, j = \Delta; \\ p_j, & i = \Delta, j \in E, \end{cases}$$

where $(p_j, j \in E)$ is a positive probability measure.

Background

Theorem

The original Q-process is unique iff the $\Pi^{\Delta}(\lambda)$-chain is recurrent.

- Spieksma, F.M.: Countable state Markov processes: non-explosiveness and moment function. Probab. Eng. and Inform. Sci., 2015, 29: 623-637.
- Chen, M.-F.: Practical criterion for uniqueness of Q-processes. Chinese J. Appl. Probab Statist. 2015, 31(2): 213-224.

$$
\begin{array}{ccccccc}
 & \Delta & 0 & 1 & 2 & 3 & \cdots \\
\Delta & -1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1 & -1-q_0 & q_{01} & q_{02} & q_{03} & \cdots \\
1 & 1 & q_{10} & -1-q_1 & q_{12} & q_{13} & \cdots \\
2 & 1 & q_{20} & q_{21} & -1-q_2 & q_{23} & \cdots \\
3 & 1 & q_{30} & q_{31} & q_{32} & -1-q_3 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Suppose that Q is irreducible. Denote

$\sigma_0 = \inf\{t > \text{the first jumping time} : X(t) = 0\}, \quad \tau_{N+} = \inf\{t > 0 : X(t) \geqslant N\}.$

For recurrence, it is well known that the process is recurrent iff

$$\lim_{N\to\infty} \mathbb{P}_0(\tau_{N+} < \sigma_0) = 0.$$

Denote $p_i = \mathbb{P}_i(\tau_{N+} < \sigma_0)$ and

$$q_i^{(k)} = \sum_{j=0}^{k} q_{ij}, \quad 0 \leqslant k < i; \quad q_i^{(k)} = \sum_{j=k}^{\infty} q_{ij}, \quad k > i \geqslant 0.$$

$$p_0 = \sum_{j=1}^{N-1} \frac{q_{0j}}{q_0} \cdot p_j + \frac{q_0^{(N)}}{q_0},$$

$$p_i = \sum_{1\leqslant j\leqslant i-1} \frac{q_{ij}}{q_i} \cdot p_j + \sum_{i+1\leqslant i\leqslant N-1} \frac{q_{ij}}{q_i} \cdot p_j + \frac{q_i^{(N)}}{q_i}, \quad 1 \leqslant i \leqslant N-1, \tag{1}$$

$$p_N = 1. \tag{2}$$

Uniqueness and recurrence

Then the equations (1) can be rewritten as

$$(I-\Pi)\mathbf{p}=\mathbf{r}^{(N)},$$

where I is the unit matrix, $\mathbf{p}=(p_1,\cdots,p_{N-1})^\mathsf{T}$ and $\mathbf{r}^{(N)}=(\frac{q_1^{(N)}}{q_1},\cdots,\frac{q_{N-1}^{(N)}}{q_{N-1}})^\mathsf{T}$. By the irreducibility of Q, the local embedding chain is transient,

$$(I-\Pi)^{-1}=\sum_{n=0}^{\infty}\Pi^n<\infty\Rightarrow\mathbf{p}=\sum_{n=0}^{\infty}\Pi^n\mathbf{r}^{(N)},$$

there exists exactly a unique solution for (1). The decomposition according to the first time arriving the set $\{N,N+1,\cdots\}$ before the first return to 0 for the original embedding chain. The recurrence criterion is presented as

$$\lim_{N\to\infty}\left(\frac{q_{01}}{q_0},\cdots,\frac{q_{0,N-1}}{q_0}\right)\sum_{n=0}^{\infty}\Pi^n\mathbf{r}^{(N)}=0.$$

Let $w_k = p_k - p_{k-1}$ for all $2 \leqslant k \leqslant N$. From (1) and (2) it follows that

$$q_{i0}p_1 + \sum_{2 \leqslant k \leqslant i} q_i^{(k-1)} w_k = \sum_{i+1 \leqslant k \leqslant N} q_i^{(k)} w_k, \quad 1 \leqslant i \leqslant N-1. \tag{3}$$

Consider the double death process with possible catastrophes:

$$q_{i,i-2} > 0, \quad i \geqslant 2; \quad q_{ij} = 0, \quad 0 \leqslant j < i-2.$$

$$q_{i0} + \sum_{(i-1)\vee 2 \leqslant k \leqslant i} \left(q_i^{(k-1)} - q_{i0}\right) w_k = \sum_{i+1 \leqslant k \leqslant N} \left(q_i^{(k)} + q_{i0}\right) w_k, \quad 1 \leqslant i \leqslant N-1.$$

(w_{N-1}, w_N). For $3 \leqslant i \leqslant N-1$,

$$w_{i-1} = \frac{1}{q_{i,i-2}} \left(\sum_{i+1 \leqslant k \leqslant N} \left(q_i^{(k)} + q_{i0}\right) w_k - \left(q_i^{(i-1)} - q_{i0}\right) w_i - q_{i0} \right). \tag{4}$$

$$q_{i0} + \sum_{(i-1)\vee 2 \leqslant k \leqslant i} \left(q_i^{(k-1)} - q_{i0}\right) w_k = \sum_{i+1 \leqslant k \leqslant N} \left(q_i^{(k)} + q_{i0}\right) w_k, \quad 1 \leqslant i \leqslant 2.$$

Uniqueness and recurrence: $m = 2$ downwardly

For $2 \leqslant i \leqslant N-1$, define

$$g_N^{(N)} = 1,\ g_{N-1}^{(N)} = 0,\ g_{i-1}^{(N)} = \frac{1}{q_{i,i-2}}\left(\sum_{i+1 \leqslant k \leqslant N} q_i^{(k)} g_k^{(N)} - q_i^{(i-1)} g_i^{(N)}\right),$$

$$g_N^{(N-1)} = 0,\ g_{N-1}^{(N-1)} = 1,\ g_{i-1}^{(N-1)} = \frac{1}{q_{i,i-2}}\left(\sum_{i+1 \leqslant k \leqslant N} q_i^{(k)} g_k^{(N-1)} - q_i^{(i-1)} g_i^{(N-1)}\right).$$

Define

$$g_0^{(N)} = \sum_{2 \leqslant k \leqslant N} q_1^{(k)} g_k^{(N)} - q_1^{(0)} g_1^{(N)}, \quad g_0^{(N-1)} = \sum_{2 \leqslant k \leqslant N} q_1^{(k)} g_k^{(N-1)} - q_1^{(0)} g_1^{(N-1)}.$$

Uniqueness and recurrence: $m = 2$ downwardly

Recurrence criterion

Assume that the double death Q-matrix $Q = (q_{ij})$ without catastrophes is non-explosive and irreducible. Then the process is recurrent iff

$$\lim_{N\to\infty} \frac{\sum_{k=1}^N g_0^{(k)} \left(g_0^{(N-1)} g_k^{(N)} - g_0^{(N)} g_k^{(N-1)}\right)}{g_0^{(N-1)} \sum_{k=1}^N g_k^{(N)} - g_0^{(N)} \sum_{k=1}^N g_k^{(N-1)}} = 0. \quad (5)$$

holds. If the irreducibility is kept when only resetting $\bar{q}_{01} > 0$ and $\bar{q}_{0j} = 0$ for all $j \geqslant 2$, i.e., the Q-matrix constrained on $\mathbb{N} := \{1, 2, \cdots\}$ is irreducible, then the process is recurrent iff

$$\lim_{N\to\infty} \frac{g_0^{(N-1)} g_1^{(N)} - g_0^{(N)} g_1^{(N-1)}}{g_0^{(N-1)} \sum_{k=1}^N g_k^{(N)} - g_0^{(N)} \sum_{k=1}^N g_k^{(N-1)}} = 0.$$

Consider the double birth process:

$$q_{i,i+2} > 0, \quad i \geqslant 1; \quad q_{ij} = 0, \quad j > i+2.$$

Define

$$f_0^{(0)} = 1,\ f_1^{(0)} = 0,\ f_{i+1}^{(0)} = \frac{1}{q_{i,i+2}}\left(\sum_{k=0}^{i-1} q_i^{(k)} f_k^{(0)} - q_i^{(i+1)} f_i^{(0)}\right),\ 1 \leqslant i \leqslant N-2,$$

$$f_0^{(1)} = 0,\ f_1^{(1)} = 1,\ f_{i+1}^{(1)} = \frac{1}{q_{i,i+2}}\left(\sum_{k=0}^{i-1} q_i^{(k)} f_k^{(1)} - q_i^{(i+1)} f_k^{(1)}\right),\ 1 \leqslant i \leqslant N-2,$$

$$f_N^{(0)} = \sum_{k=0}^{N-2} q_{N-1}^{(k)} f_k^{(0)} - q_{N-1}^{(N)} f_{N-1}^{(0)},\ f_N^{(1)} = \sum_{k=0}^{N-2} q_{N-1}^{(k)} f_k^{(1)} - q_{N-1}^{(N)} f_{N-1}^{(1)}.$$

Uniqueness and recurrence: $m = 2$ upwardly

Recurrence criterion for double birth

Assume that the 2-birth Q-matrix $Q = (q_{ij})$ is non-explosive and irreducible. Then the process is recurrent iff

$$\lim_{N\to\infty} \frac{\sum_{k=0}^{N-1} q_0^{(k+1)} (f_N^{(0)} f_k^{(1)} - f_N^{(1)} f_k^{(0)})}{f_N^{(0)} \sum_{k=0}^{N-1} f_k^{(1)} - f_N^{(1)} \sum_{k=0}^{N-1} f_k^{(0)}} = 0.$$

If the irreducibility is kept when only resetting $\bar{q}_{01} > 0$ and $\bar{q}_{0j} = 0$ for all $j \geqslant 2$, then the original process is recurrent iff

$$\lim_{N\to\infty} \frac{f_N^{(1)}}{f_N^{(0)} \sum_{k=0}^{N-1} f_k^{(1)} - f_N^{(1)} \sum_{k=0}^{N-1} f_k^{(0)}} = 0.$$

Uniqueness and recurrence

Recurrence criterion for finite skip downward

Assume that non-explosive and irreducible. Then the process is recurrent iff

$$\lim_{N\to\infty}\sum_{k=2}^{N}\left(q_0^{(1)}-q_0^{(k)}\right)\left(g_k^{(2)}+\frac{\sum_{N-m+1\leqslant j\leqslant N}\det(\mathbf{D}_j^{(N)})g_k^{(j)}}{\det(\mathbf{D}^{(N)})}\right)=q_0.$$

Recurrence criterion for finite skip upward

Assume that non-explosive and irreducible. Then the process is recurrent iff

$$\lim_{N\to\infty}\frac{\sum_{k=0}^{N-1}q_0^{(k+1)}\sum_{j=0}^{m-1}\det(\mathbf{C}_j^{(N)})f_k^{(j)}}{\det(\mathbf{C}^{(N)})}=0.$$

If the irreducibility is kept when only resetting $\bar{q}_{01}>0$ and $\bar{q}_{0j}=0$ for all $j\geqslant 2$,

$$\text{recurrence iff}\quad\lim_{N\to\infty}\frac{\det(\mathbf{C}_0^{(N)})}{\det(\mathbf{C}^{(N)})}=0.$$

Uniqueness and recurrence

Uniqueness criterion for finite skip downward

Assume that the m-death Q-matrix $\widetilde{Q} = (\tilde{q}_{ij})$ without catastrophes is irreducible. Then the process is unique iff

$$\lim_{N\to\infty} \sum_{k=2}^{N} \left(g_k^{(2)} + \frac{\sum_{N-m+1\leqslant j\leqslant N} \det(\mathbf{D}_j^{(N)}) g_k^{(j)}}{\det(\mathbf{D}^{(N)})} \right) = 1.$$

Uniqueness criterion for finite skip upward

Assume that the m-birth Q-matrix $\widetilde{Q} = (\tilde{q}_{ij})$ is irreducible. Then the process is unique iff

$$\lim_{N\to\infty} \frac{\det(\mathbf{C}_0^{(N)})}{\det(\mathbf{C}^{(N)})} = 0.$$

Examples

Example 1

Given a non-explosive, irreducible single death Q-matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-1} = a > 0, \quad q_{i,i+1} = b \geqslant 0, \quad q_{i,i+2} = d \geqslant 0, \quad i \geqslant 1,$$

where $b + d > 0$, and $q_{ij} = 0$ for other $i, j \geqslant 1$, $i \neq j$. Then the process is recurrent iff $a \geqslant b + 2d$.

Example 2

Given a non-explosive, irreducible double death Q-matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-2} = c > 0, \quad q_{i,i-1} = a \geqslant 0, \quad q_{i,i+1} = b > 0, \quad i \geqslant 1,$$

and $q_{ij} = 0$ for other $i, j \geqslant 1$, $i \neq j$. Then the process is recurrent iff $a + 2c \geqslant b$. This example with $a = 0$ and $c = b = 1$ is taken from [Mao & Z. 2004] in which it is proven that the process is exponentially ergodic but not strongly ergodic.

Examples

Example 3

Given a non-explosive, irreducible double death Q-matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-2} = c > 0, \quad i \geqslant 2; \quad q_{i,i+2} = d > 0, \quad i \geqslant 0; \quad q_{01} = b > 0, \quad q_{10} = a > 0,$$

and $q_{ij} = 0$ for other $i \neq j$. Then the process is recurrent iff $c \geqslant d$.

Example 4

Given a non-explosive, irreducible double death Q-matrix $Q = (q_{ij})$ satisfying:

$$q_{i,i-2} = q_{i,i-1} = 1, \quad i \geqslant 2; \quad q_{i,i+2} = q_{i,i+1} = 1, \quad i \geqslant 0; \quad q_{10} = 1,$$

and $q_{ij} = 0$ for other $i \neq j$. Then the process is recurrent.

Examples

Example 5

Given a irreducible single death Q-matrix $Q = (q_{ij})$ without catastrophes satisfying:

$$q_{i,i-1} > 0, \quad q_{ij} = 0, \quad i \geqslant 1, j \neq i, i-1; \quad q_{0j} > 0, \quad j \geqslant 1, \quad \sum_{j \geqslant 1} q_{0j} < \infty.$$

Then the process is unique.

Ergodicity

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix. Then the limit

$$\lim_{t\to\infty} p_{ij}(t) =: \pi_j$$

exists and it is independent of i. Moreover, we have either $\sum_j \pi_j = 1$ or $\sum_j \pi_j = 0$.

- $P(t)$ is positive recurrent or ergodic if so is $P(h)$ for every $h > 0$. Equivalently, $\lim_{t\to\infty} p_{ii}(t) = \pi_i > 0$.

Theorem[Issacson & Arnold(1978)]

The Q-process is ergodic iff $\mathbb{E}_0\sigma_0 < \infty$.

Assume that the process is recurrent.

$$\lim_{N\to\infty} \mathbb{E}_i\sigma_0\mathbb{1}_{\{\sigma_0<\tau_{N+}\}} = \mathbb{E}_i\sigma_0\mathbb{1}_{\{\sigma_0<\infty\}} = \mathbb{E}_i\sigma_0, \qquad i \geqslant 0.$$

Define $h_i = \mathbb{E}_i\sigma_0\mathbb{1}_{\{\sigma_0<\tau_{N+}\}}$. Then for all $i \geqslant N$, $h_i = 0$, and for $0 \leqslant i < N$, So for $0 \leqslant i < N$,

$$\begin{aligned} h_0 &= \frac{1}{q_0}\sum_{0<j<N}\frac{q_{0j}}{q_0}\mathbb{P}_j(\sigma_0<\tau_{N+}) + \sum_{0<j<N}\frac{q_{0j}}{q_0}h_j \\ &=: s_0^{(N)} + \sum_{0<j<N}\frac{q_{0j}}{q_0}h_j, \end{aligned} \tag{6}$$

$$\begin{aligned} h_i &= \frac{1}{q_i}\sum_{i\neq j<N}\frac{q_{ij}}{q_i}\mathbb{P}_j(\sigma_0<\tau_{N+}) + \sum_{j\neq i, 0<j<N}\frac{q_{ij}}{q_i}h_j \\ &=: s_i^{(N)} + \sum_{j\neq i, 0<j<N}\frac{q_{ij}}{q_i}h_j. \end{aligned} \tag{7}$$

Ergodicity

Set $\mathbf{h} = (h_1, \cdots, h_{N-1})^\mathsf{T}$ and $\mathbf{s}^{(N)} = (s_1^{(N)}, \cdots, s_{N-1}^{(N)})^\mathsf{T}$. Then

$$(I - \Pi)\mathbf{h} = \mathbf{s}^{(N)}.$$

Hence

$$\mathbf{h} = (I - \Pi)^{-1}\mathbf{s}^{(N)} = \sum_{n=0}^{\infty} \Pi^n \mathbf{s}^{(N)},$$

and

$$h_0 = s_0^{(N)} + \left(\frac{q_{01}}{q_0}, \cdots, \frac{q_{0,N-1}}{q_0}\right) \sum_{n=0}^{\infty} \Pi^n \mathbf{s}^{(N)}.$$

Assume that $\mathbf{h}$ is finite and let $w_k = h_k - h_{k-1}$ for all $1 \leqslant k \leqslant N$. From (7) it follows that

$$q_{i0}h_1 + \sum_{2 \leqslant k \leqslant i} q_i^{(k-1)} w_k = q_i s_i^{(N)} + \sum_{i+1 \leqslant k \leqslant N} q_i^{(k)} w_k, \quad 1 \leqslant i < N. \tag{8}$$

Thank you for your attention!

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