Integral-type functionals for Q-processes and applications

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CONTENTS

Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

Background

Define

$$\tau_i = \inf\{t \geqslant 0 : X(t) = i\}, \ \sigma_i = \inf\{t \geqslant \text{the first jumping time} : X(t) = i\}.$$

Given $V \geqslant 0$ but not equal to 0 identically. Consider the integral-type functional:

$$\xi_i = \int_0^{\tau_i} V(X(t))\mathrm{d}t \quad \text{or} \quad \int_0^{\sigma_i} V(X(t))\mathrm{d}t.$$

When $V \equiv 1$, $\xi_i = \tau_i$.
When $V(1) = V(2) = 1$ and $V(j) = 0, j \neq 1, 2$, then ξ_0 is the total time that the process stays in the states 1 and 2 before hitting 0 firstly.

Background

- Z.K. Wang . On distribution of functions of birth and death processes and their applications in the theory of queues, Scientia Sinica, 1961, X(2): 160-170.
- D.R. McNeil. Integral functionals of birth and death processes and related limiting distributions, The Annals of Mathematical Statistics, 1970, 41(2):480-485.
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- 杨超群. 可列马氏过程的积分型泛函和双边生灭过程的边界性质, 数学进展, 1964, 7(4): 397-424.

- 侯振挺, 郭青峰. 齐次可列马尔可夫过程, 科学出版社, 1978.

$(\mathbb{E}_i \xi_{i_0}^{\ell} : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\ell V(i)}{q_i} \mathbb{E}_i \xi_{i_0}^{\ell-1}, \qquad i \in E.$$

Define $\psi_{i,i_0}(\lambda) = 1 - \mathbb{E}_i \mathrm{e}^{-\lambda \xi_{i_0}}$.

$(\psi_{i,i_0}(\lambda) : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} y_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \qquad i \in E.$$

Background

- J.K. Zhang. On the generalized birth and death processes (I)–the numeral introduction, the functional of integral type and the distributions of runs and passage times, Acta Math Sci, 1984, 4(2): 191-209.
- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.
- E. Löcherbach, O. Loukianav, D. Loukianova. Spectral condition, hitting times and Nash inequality, Annales de l'I.H.P. Probabilités et statistiques, 2014, 50(4): 1213-1230.

$$\xi_{i_0} = \int_0^{\tau_{i_0}} r(t)V(X(t))\mathrm{d}t.$$

Background

- 王梓坤. 随机过程通论(下卷), 第3版. 北京师范大学出版社, 2010.
- R. L. Dobrushin. On conditions of regularity of stationary Markov processes with a denumerable number of possible states (in Russian). Uspehi Matem. Nauk (N.S.), 1952, 7(6), 185-191.

Define $\tau_{i+} = \inf\{t \geqslant 0 : X(t) \geqslant i\}, \eta_0 = 0,$

$$\eta_n = \inf\{t > \eta_{n-1} : X(t) \neq X(\eta_{n-1})\}, \quad \eta = \lim_{n\to\infty} \eta_n.$$

$$\eta = \lim_{n\to\infty} \tau_{n+}, \text{ a.s..}$$

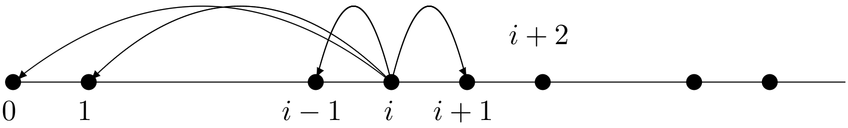
$$\xi = \lim_{n\to\infty} \int_0^{\eta_n} V(X(t))\mathrm{d}t = \int_0^{\eta} V(X(t))\mathrm{d}t = \lim_{n\to\infty} \int_0^{\tau_{n+}} V(X(t))\mathrm{d}t.$$

$$\xi = \infty \text{ a.s. } \Leftrightarrow \sum_{n\geqslant 0} V(X_n)/q_{X_n} = \infty \text{ a.s..}$$

Results: single birth process

$Q = (q_{ij})$ satisfies

$$\color{red}{q_{i,i+1} > 0},\ \ q_{i,i+j} = 0,\ \ i \geq 0, j > 1.$$



Results: single birth process

给定在 E 上定义的一个函数 c. 定义

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \qquad 0 \leqslant i < n,$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n = \sum_{j=0}^{k} q_{nj} - c_n, \qquad 0 \leqslant k < n.$$

等价地, 也可以定义为:

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \sum_{k=i+1}^{n} \frac{\widetilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, \qquad n \geqslant i+1.$$

Theorem (Wang & Z., 2020)

假定单生过程常返且对给定正整数 $\ell \geqslant 1$ 和 状态 i_0 满足: 对所有 $i \in E$ 有 $\mathbb{E}_i \xi_{i_0}^{\ell-1} < \infty$. 则

$$\mathbb{E}_n \xi_{i_0}^{\ell} = \begin{cases} \ell \sum\limits_{n \leqslant k \leqslant i_0 - 1} v_k^{(\ell)} + \left(1 - \sum\limits_{n \leqslant k \leqslant i_0 - 1} u_k\right) \mathbb{E}_{i_0} \xi_{i_0}^{\ell}, & 0 \leqslant n \leqslant i_0, \\ -\ell \sum\limits_{i_0 \leqslant k \leqslant n - 1} v_k^{(\ell)} + \left(1 + \sum\limits_{i_0 \leqslant k \leqslant n - 1} u_k\right) \mathbb{E}_{i_0} \xi_{i_0}^{\ell}, & n \geqslant i_0 + 1; \end{cases}$$

Results: single birth process

Theorem (continued)

其中

$$u_k = \begin{cases} \sum_{j=i_0-1}^{k} \dfrac{F_k^{(j)} q_{ji_0}(1-\delta_{ji_0})}{q_{j,j+1}}, & k \geqslant i_0, \\ 1, & k = i_0 - 1, \\ 0, & 0 \leqslant k \leqslant i_0 - 2; \end{cases}$$

$$v_k^{(\ell)} = \sum_{j=0}^{k} \frac{F_k^{(j)} V(j) \mathbb{E}_j \xi_{i_0}^{\ell-1}}{q_{j,j+1}}, \qquad k \geqslant 0;$$

$$\mathbb{E}_{i_0} \xi_{i_0}^{\ell} = \ell \varlimsup_{n\to\infty} \frac{\sum_{i_0 \leqslant k \leqslant n} v_k^{(\ell)}}{1 + \sum_{i_0 \leqslant k \leqslant n} u_k}$$

$$= \ell \lim_{n\to\infty} \frac{v_n^{(\ell)}}{u_n} \text{ 若此极限存在.}$$

Results: single birth process

记其分布函数 $F_{ki}(x) = \mathbb{P}_k(\xi_i \leqslant x)$, Laplace 变换记为:

$$\varphi_{ki}(\lambda) = \mathbb{E}_k \mathrm{e}^{-\lambda \xi_i} = \int_0^\infty \mathrm{e}^{-\lambda x} \mathrm{d} F_{ki}(x), \qquad \lambda > 0.$$

再记 $\psi_{ki}(\lambda) = 1 - \varphi_{ki}(\lambda)$.

Theorem (Wang & Z., 2020)

对单生过程, 给定状态 i_0, 取定 $c = -\lambda V$. 则

$$\psi_{ni_0}(\lambda) = \begin{cases} \dfrac{\lambda \sum_{n \leqslant k \leqslant i_0 - 1} \tilde{v}_k}{1 + \lambda \sum_{0 \leqslant k \leqslant i_0 - 1} \tilde{v}_k} + \psi_{i_0 i_0}(\lambda) \Bigg(1 - \displaystyle\sum_{n \leqslant k \leqslant i_0 - 1} \tilde{u}_k \Bigg), & 0 \leqslant n \leqslant i_0, \\ -\dfrac{\lambda \sum_{i_0 \leqslant k \leqslant n - 1} \tilde{v}_k}{1 + \lambda \sum_{0 \leqslant k \leqslant i_0 - 1} \tilde{v}_k} + \psi_{i_0 i_0}(\lambda) \Bigg(1 + \displaystyle\sum_{i_0 \leqslant k \leqslant n - 1} \tilde{u}_k \Bigg), & n \geqslant i_0 + 1; \end{cases}$$

Results: single birth process

Theorem (continued)

$$\tilde{u}_k = \begin{cases} \sum_{j=i_0-1}^{k} \dfrac{\widetilde{F}_k^{(j)} q_{ji_0}(1-\delta_{ji_0})}{q_{j,j+1}}, & k \geqslant i_0, \\ 1, & k = i_0 - 1, \\ 0, & 0 \leqslant k \leqslant i_0 - 2; \end{cases}$$

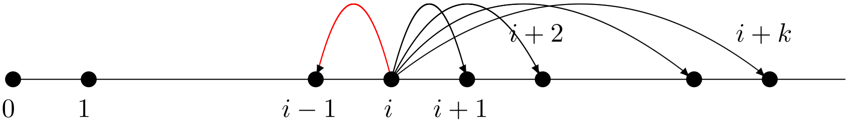
$$\tilde{v}_k = \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)} V(j)}{q_{j,j+1}}, \qquad k \geqslant 0;$$

$$\psi_{i_0 i_0}(\lambda) = \frac{\lambda}{1 + \lambda \sum_{0 \leqslant k \leqslant i_0 - 1} \tilde{v}_k} \varlimsup_{n\to\infty} \frac{\sum_{i_0 \leqslant k \leqslant n} \tilde{v}_k}{1 + \sum_{i_0 \leqslant k \leqslant n} \tilde{u}_k}.$$

Results: single death process

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0, \; q_{i,i-j} = 0, \; i \geqslant 1, j > 1.$$



- branching process:

$$\begin{aligned} i &\to i-1 \quad \text{at rate} \quad \alpha i p_0 \\ &\to i \quad \text{at rate} \quad -\alpha i(1-p_1) \\ &\to i+1 \quad \text{at rate} \quad \alpha i p_2 \\ &\to i+2 \quad \text{at rate} \quad \alpha i p_3 \\ &\to \cdots \end{aligned}$$

Results: single death process

Theorem (Wang & Z., 2020)

Assume that the single death Q is irreducible and the corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \xi_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Define

$$\kappa_0 = 1, \quad \kappa_i = \frac{1}{q_{i,i-1}} \sum_{k=1}^{i} q_0^{(k)} G_k^{(i)}, \qquad i \geqslant 1, \quad \kappa = \sum_{i \in E} \kappa_i.$$

In the ergodic case, the stationary distribution $\pi = (\pi_i)$: $\pi_i = \kappa_i / \kappa$.

Results: single death process

Theorem (Wang & Z. 2020+) (σ_i)

$$\mathbb{E}_{i_0}\xi_{i_0}^n = \frac{1}{\kappa_{i_0}}\sum_{i=0}^{i_0}\frac{\kappa_i \widetilde{H}_i^{(n,i_0)}}{q_i}, \qquad i_0 \geqslant 0,$$

$$\mathbb{E}_i\xi_{i_0}^n = \frac{D_{i_0}^{(i_0)}}{C_{i_0}^{(i_0)}} C_{i_0-1-i}^{(i_0)} - D_{i_0-1-i}^{(i_0)}, \qquad 0 \leqslant i \leqslant i_0 - 1,$$

$$H_i^{(n,i_0)} = nV(i)\mathbb{E}_i\xi_{i_0}^{n-1}, \qquad i \geqslant 0,$$

$$\widetilde{H}_i^{(n,i_0)} = \sum_{j \geqslant i_0+1} q_{ij}\mathbb{E}_j\xi_{i_0}^n + H_i^{(n,i_0)}, \qquad 0 \leqslant i \leqslant i_0.$$

$$C_{i_0}^{(i_0)} = \frac{q_{i_0,i_0-1}}{q_{i_0}}\kappa_{i_0}, \qquad D_{i_0}^{(i_0)} = \sum_{i=0}^{i_0-1}\frac{\kappa_i \widetilde{H}_i^{(n,i_0)}}{q_i},$$

$$C_i^{(i_0)} = \sum_{i_0-i \leqslant k \leqslant i_0} G_k^{(i_0)}, \; D_i^{(i_0)} = \sum_{i_0-i \leqslant \ell \leqslant i_0-1}\sum_{k=i_0-i}^{\ell}\frac{G_k^{(\ell)}\widetilde{H}_\ell^{(n,i_0)}}{q_{\ell,\ell-1}}, \; 0 \leqslant i \leqslant i_0 - 1.$$

Results: birth-death process on trees

The tree T is a connected graph without cycles. We fix a point on T as the root, denoted by o. For any vertex $i \in T \setminus \{o\}$, there is a unique simple path from i to the root o.

- $\mathscr{P}(i)$: the set of all the vertices on this path (the root o is excluded).
- $|i|$: the number of segments of this path is the length of i.
- $i \sim j$: two vertices i and j are called adjacent if they are joined by a segment.
- When $|j| = |i| + 1$ and $i \sim j$, j is called one offspring of i and the set of all the offsprings of i is denoted by $J(i)$.
- When $|j| = |i| - 1$ and $i \sim j$, j is called the father of i and denoted by i^*.
- Denote T_i the subtree with i as its root, including all the descendants of i.
- Assume that any vertex has finite offsprings.

We consider a birth-death process on this tree which Q-matrix satisfies: $q_{ij} > 0$ if and only if $i \sim j$, i.e., $j = i^*$, or $j \in J(i)$.

$q_i := -q_{ii} = q_{ii^*} + \sum_{j \in J(i)} q_{ij} < \infty$ for all $i \in T$.

Define a measure μ on T as follows.

$$\mu_o = 1, \qquad \mu_i = \prod_{j \in \mathscr{P}(i)} \frac{q_{j^*j}}{q_{jj^*}}, \qquad i \in T \setminus \{0\},$$

which is invariant with respect to Q. In fact, μ satisfies the so-called detailed balance equation:

$$\mu_i q_{ij} = \mu_j q_{ji}, \qquad i \sim j.$$

Results: birth-death process on trees

Theorem (Wang & Z., 2020+)

Assume that the Q-matrix on the tree T is regular and the process is recurrent. Then

$$\mathbb{E}_i \xi_k^n = n \sum_{j \in \mathscr{P}(i) \setminus \mathscr{P}(k)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell \in T_j} \mu_\ell V(\ell) \mathbb{E}_\ell \xi_k^{n-1}, \qquad k \in \mathscr{P}(i)$$

$$\mathbb{E}_i \xi_o^n = n \sum_{j \in \mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell \in T_j} \mu_\ell V(\ell) \mathbb{E}_\ell \xi_o^{n-1}.$$

Applications

- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.

$$\xi_{i_0} = \int_0^{\sigma_{i_0}} r(t)V(X(t))\mathrm{d}t.$$

Assume that $r, r' \geqslant 0$. Then $(\mathbb{E}_i \xi_{i_0} : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\overline{V}(i)}{q_i}, \qquad i \in E,$$

where

$$\overline{V} = \widetilde{V} + r(0)V, \qquad \widetilde{V}(i) = \mathbb{E}_i \int_0^{\sigma_{i_0}} r'(t)V(X(t))\mathrm{d}t.$$

Applications: polynomial ergodicity

$$\mathbb{E}_i \xi_{i_0} = \mathbb{E}_i \int_0^{\sigma_{i_0}} \overline{V}(X(t))\mathrm{d}t.$$

$(\mathbb{E}_i \xi_{i_0} : i \geqslant i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{k \neq i, k > i_0} \frac{q_{ik}}{q_i} x_j + \frac{\overline{V}(i)}{q_i}, \qquad i > i_0.$$

For single death process,

$$\mathbb{E}_i \xi_{i_0} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \overline{V}(\ell)}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Applications: polynomial ergodicity

In particular, when $r(t) = t^n$, denote

$$\xi_{i_0} = \xi_{i_0}^{(n)} = \int_0^{\sigma_{i_0}} t^n V(X(t))\mathrm{d}t, \ \widetilde{V}(i) = n\mathbb{E}_i \xi_{i_0}^{(n-1)}.$$

$$\mathbb{E}_i \xi_{i_0}^{(n)} = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \xi_{i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

- subexponential ergodicity and polynomial ergodicity

Applications: CLT

- CLT:

$$t^{1/2}\left(\frac{1}{t}\int_0^t f(X(s))\mathrm{d}s - \pi(f)\right) \xrightarrow{D} N(0,\sigma^2(f)), \quad t \to \infty.$$

If $\pi(|f|) < \infty$, then a CLT holds iff

$$\mathbb{E}_0\left(\int_0^{\sigma_0} (f - \pi(f))(X(t))\mathrm{d}t\right)^2 < \infty$$

$$\Leftarrow \ \mathbb{E}_0\left(\int_0^{\sigma_0} |f - \pi(f)|(X(t))\mathrm{d}t\right)^2 < \infty.$$

Applications: exponential ergodicity

For any $i, j \geqslant 1$, define $\tilde{p}_{ij}(t) = \mathbb{P}_i\big(X(t) = j, \sigma_0 > t\big)$ and

$$h_{ij} = \int_0^\infty \tilde{p}_{ij}(t)\,\mathrm{d}t = \mathbb{E}_i \int_0^{\sigma_0} \mathbb{1}_j(X(t))\,\mathrm{d}t, \;\; H = (h_{ij})_{i,j\geqslant 1}.$$

Define

$$\alpha_{\max} = \sup\left\{\alpha > 0 : \mathbb{E}_k \mathrm{e}^{\alpha\sigma_0} < \infty \text{ for all } k \geqslant 1 \;\text{ and }\; \sum_{k\geqslant 1} q_{0k}\mathbb{E}_k \mathrm{e}^{\alpha\sigma_0} < \infty,\right.$$
$$\left.\text{where } \alpha < q_i \text{ for all } i \geqslant 0\right\}.$$

$$\alpha_{\max} = \sup\left\{\alpha > 0 : \mathbb{E}_0 \mathrm{e}^{\alpha\sigma_0} < \infty, \text{ where } \alpha < q_i \text{ for all } i \geqslant 0\right\}.$$

Applications: exponential ergodicity

Theorem (2020+)

Given a regular and irreducible Q-matrix Q. Assume that $\inf_{i\geqslant 0} q_i > 0$. For the corresponding Q-process, then the following variational formulae holds:

$$\frac{1}{\alpha_{\max}} = \inf_{\mathbf{x}\in\mathscr{F}} \sup_{k\geqslant 1} \left(\frac{H\mathbf{x} + H\mathbb{1}}{\mathbf{x}} \right)(k) =: \beta,$$

where the column vector $\mathbb{1} = (1, 1, 1, \cdots)^{\mathsf{T}}$ and the set of the column vectors

$$\mathscr{F} = \left\{ \mathbf{x} = (x_i, i \geqslant 1) : 0 < x_i < \infty \text{ for all } i \geqslant 1 \text{ and } \sum_{j\geqslant 1} q_{0j} x_j < \infty \right\}.$$

Furthermore, the process is exponentially ergodic if and only if $\beta < \infty$.

Applications: exponential ergodicity

Birth-death process:

$$\beta = \inf_{f\in\mathscr{F}} \sup_{i\geqslant 1} \frac{1}{f_i} \sum_{j=1}^{i} \frac{1}{\mu_j a_j} \sum_{k=j}^{\infty} \mu_k (f_k + 1),$$

$$\delta := \sup_{i\geqslant 1} \sum_{j=1}^{i} \frac{1}{\mu_j a_j} \sum_{j=i}^{\infty} \mu_j, \qquad \delta \leqslant \beta \leqslant 4(1+b_0)\delta.$$

Single death process:

$$\beta = \inf_{f\in\mathscr{F}} \sup_{i\geqslant 1} \frac{1}{f_i} \sum_{j=1}^{i} \sum_{k=j}^{\infty} \frac{G_j^{(k)}(f_k+1)}{q_{k,k-1}}.$$

Birth-death process on tree:

$$\beta = \inf_{f\in\mathscr{F}} \sup_{i\neq o} \frac{1}{f_i} \sum_{j\in\mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{k\in T_j} \mu_k (f_k + 1).$$

Applications: exponential ergodicity

Example(Chen, Example 9.14)

Let $q_k = q_{k,k-1} = 1$, $q_{0k} = \theta^k (k \geqslant 1)$ for some $0 < \theta < 1$ and $q_{ij} = 0$ for all other $j \neq i$. Then gap(Q) $\geqslant \{(1-\sqrt{\theta})^{-2} + \theta/(1-\theta+\theta^2)\}^{-1}$.

$$\pi_0 = \{1+\theta/(1-\theta)^2\}^{-1}, \ \pi_n = \pi_0\theta^n/(1-\theta), \ n \geqslant 1.$$

Example

Let $q_k = q_{k,k-1}$, $q_{0k} = \theta^k (k \geqslant 1)$ for some $0 < \theta < 1$ and $q_{ij} = 0$ for all other $j \neq i$. Then exp. erg. iff $c := \inf_{i\geqslant 1} q_{i,i-1} > 0$.

$$\sum_{n\geqslant 1} \frac{1}{q_{Y_n}} = \infty \Leftrightarrow \text{recur.}(\Rightarrow \ \text{uniqueness}), \ \sum_{n\geqslant 1} \frac{1}{q_{n,n-1}} < \infty \Leftrightarrow \text{strong. erg.}$$

$$\sum_{n\geqslant 1} \frac{q_0^{(n)}}{q_{n,n-1}} = \frac{1}{1-\theta} \sum_{n\geqslant 1} \frac{\theta^n}{q_{n,n-1}} < \infty \Leftrightarrow \text{erg.}$$

Applications: exponential ergodicity

$$\beta = \inf_{f\in\mathscr{F}} \sup_{i\geqslant 1} \frac{1}{f_i}\sum_{j=1}^{i}\frac{f_j+1}{q_{j,j-1}} \geqslant \Big(\inf_{i\geqslant 1} q_{i,i-1}\Big)^{-1} = \frac{1}{c},$$

$$\beta \leqslant \frac{1}{c}\inf_{f\in\mathscr{F}} \sup_{i\geqslant 1} \frac{1}{f_i}\sum_{j=1}^{i}(f_j+1) \leqslant \frac{1}{c}\inf_{f\in\mathscr{F}''} \sup_{i\geqslant 1} \frac{f_i}{f_i - f_{i-1}},$$

$$\mathscr{F}'' = \Big\{ f : f_0 = 1, f \uparrow\uparrow, \sum_{j\geqslant 1} q_{0j} f_j < \infty \Big\}.$$

Take $\tilde\theta : 0 < \theta < \tilde\theta < 1$, let $\lambda = \ln\tilde\theta - \ln\theta > 0$ and $f_i = \exp(\lambda i)$. Then $\beta \leqslant (1-\exp(-\lambda))^{-1} < \infty$.

Thank you for your attention!

Homepage: http://math0.bnu.edu.cn/~zhangyh/