Integral-type functionals

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CONTENTS

Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

Define

$$\tau_i = \inf\{t \geqslant 0 : X(t) = i\},\ \sigma_i = \inf\{t \geqslant \text{the first jumping time} : X(t) = i\}.$$

Given $V \geqslant 0$ but not equal to 0 identically. Consider the integral-type functional:

$$\xi_i = \int_0^{\tau_i} V(X(t))\mathrm{d}t \quad \text{or} \quad \int_0^{\sigma_i} V(X(t))\mathrm{d}t.$$

When $V \equiv 1$, $\xi_i = \tau_i$.
When $V(1) = V(2) = 1$ and $V(j) = 0, j \neq 1, 2$, then ξ_0 is the total time that the process stays in the states 1 and 2 before hitting 0 firstly.

Background

$$Y(t) = \int_0^t f(X(s), s)\mathrm{d}s.$$

- the theory of inventories and storage, moments and distributions
- queueing, traffic jam and intersection bottlenecks
- biology, the joint distribution of $X(t)$ and $Y(t)$
- limiting behavior (including sub-exponentially ergodic theory and CLTs)

Background

- Z.K. Wang . On distribution of functions of birth and death processes and their applications in the theory of queues, Scientia Sinica, 1961, X(2): 160-170.
- D.R. McNeil. Integral functionals of birth and death processes and related limiting distributions, The Annals of Mathematical Statistics, 1970, 41(2):480-485.
- 吴立德. 齐次可数马尔科夫过程积分型泛函的分布, 数学学报, 1963, 13(1): 86-93.
- 杨超群. 可列马氏过程的积分型泛函和双边生灭过程的边界性质, 数学进展, 1964, 7(4): 397-424.

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$(\mathbb{E}_i\xi_{i_0}^{\ell} : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k\neq i,i_0} \frac{q_{ik}}{q_i} y_k + \frac{\ell V(i)}{q_i}\mathbb{E}_i\xi_{i_0}^{\ell-1}, \qquad i \in E.$$

Define $\psi_{i,i_0}(\lambda) = 1 - \mathbb{E}_i \mathrm{e}^{-\lambda\xi_{i_0}}$.

$(\psi_{i,i_0}(\lambda) : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k\neq i,i_0} \frac{q_{ik}}{\lambda V(i) + q_i} y_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \qquad i \in E.$$

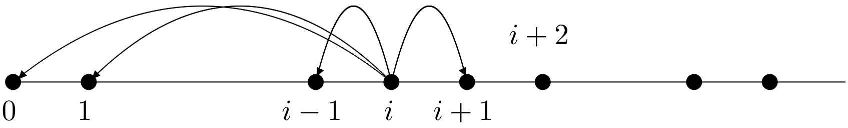
Background

- J.K. Zhang. On the generalized birth and death processes (I)–the numeral introduction, the functional of integral type and the distributions of runs and passage times, Acta Math Sci, 1984, 4(2): 191-209.
- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.
- E. Löcherbach, O. Loukianav, D. Loukianova. Spectral condition, hitting times and Nash inequality, Annales de l'I.H.P. Probabilités et statistiques, 2014, 50(4): 1213-1230.

$$\xi_{i_0} = \int_0^{\tau_{i_0}} r(t)V(X(t))\mathrm{d}t.$$

$Q = (q_{ij})$ satisfies

$$q_{i,i+1} > 0,\ \ q_{i,i+j} = 0,\ \ i \geq 0, j > 1.$$



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

给定在 E 上定义的一个函数 c. 定义

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \qquad 0 \leqslant i < n,$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n = \sum_{j=0}^{k} q_{nj} - c_n, \qquad 0 \leqslant k < n.$$

等价地, 也可以定义为:

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \sum_{k=i+1}^{n} \frac{\widetilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, \qquad n \geqslant i+1.$$

Theorem (Wang & Z., 2020)

假定单生过程常返且对给定正整数 $\ell \geqslant 1$ 和 状态 i_0 满足: 对所有 $i \in E$ 有 $\mathbb{E}_i \xi_{i_0}^{\ell-1} < \infty$. 则

$$\mathbb{E}_n \xi_{i_0}^{\ell} = \begin{cases} \ell \sum\limits_{n \leqslant k \leqslant i_0-1} v_k^{(\ell)} + \Big(1 - \sum\limits_{n \leqslant k \leqslant i_0-1} u_k\Big) \mathbb{E}_{i_0} \xi_{i_0}^{\ell}, & 0 \leqslant n \leqslant i_0, \\ -\ell \sum\limits_{i_0 \leqslant k \leqslant n-1} v_k^{(\ell)} + \Big(1 + \sum\limits_{i_0 \leqslant k \leqslant n-1} u_k\Big) \mathbb{E}_{i_0} \xi_{i_0}^{\ell}, & n \geqslant i_0 + 1; \end{cases}$$

Theorem (continued)

其中

$$u_k = \begin{cases} \sum_{j=i_0-1}^{k} \dfrac{F_k^{(j)} q_{ji_0}(1-\delta_{ji_0})}{q_{j,j+1}}, & k \geqslant i_0, \\ 1, & k = i_0 - 1, \\ 0, & 0 \leqslant k \leqslant i_0 - 2; \end{cases}$$

$$v_k^{(\ell)} = \sum_{j=0}^{k} \frac{F_k^{(j)} V(j) \mathbb{E}_j \xi_{i_0}^{\ell-1}}{q_{j,j+1}}, \qquad k \geqslant 0;$$

$$\mathbb{E}_{i_0} \xi_{i_0}^{\ell} = \ell \varlimsup_{n\to\infty} \frac{\sum_{i_0 \leqslant k \leqslant n} v_k^{(\ell)}}{1 + \sum_{i_0 \leqslant k \leqslant n} u_k}$$

$$= \ell \lim_{n\to\infty} \frac{v_n^{(\ell)}}{u_n} \text{ 若此极限存在.}$$

记其分布函数 $F_{ki}(x)=\mathbb{P}_k(\xi_i\leqslant x)$, Laplace 变换记为:

$$\varphi_{ki}(\lambda)=\mathbb{E}_k\mathrm{e}^{-\lambda\xi_i}=\int_0^\infty \mathrm{e}^{-\lambda x}\mathrm{d}F_{ki}(x),\qquad \lambda>0.$$

再记 $\psi_{ki}(\lambda)=1-\varphi_{ki}(\lambda)$.

Theorem (Wang & Z., 2020)

对单生过程, 给定状态 i_0, 取定 $c=-\lambda V$. 则

$$\psi_{ni_0}(\lambda)=\begin{cases}\dfrac{\lambda\sum_{n\leqslant k\leqslant i_0-1}\tilde{v}_k}{1+\lambda\sum_{0\leqslant k\leqslant i_0-1}\tilde{v}_k}+\psi_{i_0i_0}(\lambda)\Bigg(1-\displaystyle\sum_{n\leqslant k\leqslant i_0-1}\tilde{u}_k\Bigg), 0\leqslant n\leqslant i_0,\\ -\dfrac{\lambda\sum_{i_0\leqslant k\leqslant n-1}\tilde{v}_k}{1+\lambda\sum_{0\leqslant k\leqslant i_0-1}\tilde{v}_k}+\psi_{i_0i_0}(\lambda)\Bigg(1+\displaystyle\sum_{i_0\leqslant k\leqslant n-1}\tilde{u}_k\Bigg), n\geqslant i_0+1;\end{cases}$$

Theorem (continued)

其中

$$\tilde{u}_k = \begin{cases} \sum_{j=i_0-1}^{k} \frac{\widetilde{F}_k^{(j)} q_{ji_0}(1-\delta_{ji_0})}{q_{j,j+1}}, & k \geqslant i_0, \\ 1, & k = i_0 - 1, \\ 0, & 0 \leqslant k \leqslant i_0 - 2; \end{cases}$$

$$\tilde{v}_k = \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)} V(j)}{q_{j,j+1}}, \qquad k \geqslant 0;$$

$$\psi_{i_0 i_0}(\lambda) = \frac{\lambda}{1 + \lambda \sum_{0 \leqslant k \leqslant i_0 - 1} \tilde{v}_k} \varlimsup_{n\to\infty} \frac{\sum_{i_0 \leqslant k \leqslant n} \tilde{v}_k}{1 + \sum_{i_0 \leqslant k \leqslant n} \tilde{u}_k}.$$

Theorem (contin.)

若还假定单生过程是常返的且极限 $\lim_{n\to\infty}\tilde{v}_n/\tilde{u}_n$ 存在, 则

$$\psi_{i_0 i_0}(\lambda)=\frac{\lambda}{1+\lambda\sum_{0\leqslant k\leqslant i_0-1}\tilde{v}_k}\lim_{n\to\infty}\frac{\tilde{v}_n}{\tilde{u}_n}.$$

定义算子 Ω: $\Omega g=Qg+cg$, 其中 $(Qg)_i=\sum_{j\in E}q_{ij}(g_j-g_i)$.
当 $c\leqslant 0$ 时, Ω 是带杀死速率 $(-c_i)$ 的单生过程对应的算子.

Theorem (Chen & Z., 2014)

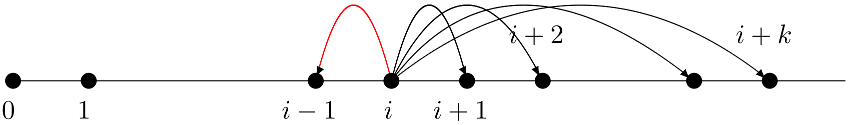
给定单生 Q 矩阵 $Q=(q_{ij})$ 和函数 c 及 f, 则 Poisson 方程 $\Omega g=f$ 的解 g 有如下表示:

$$g_n=g_0+\sum_{0\leqslant k\leqslant n-1}\sum_{0\leqslant j\leqslant k}\frac{\widetilde{F}_k^{(j)}(f_j-c_j g_0)}{q_{j,j+1}},\qquad n\geqslant 0.$$

Single death process

Single death process: $Q=(q_{ij})$ satisfies

$$q_{i,i-1}>0,\ q_{i,i-j}=0,\ i\geqslant 1, j>1.$$



$$Q=\begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Single death process

- branching process:

$$
\begin{aligned}
i \to i-1 &\quad \text{at rate} \quad \alpha i p_0 \\
\to i &\quad \text{at rate} \quad -\alpha i(1-p_1) \\
\to i+1 &\quad \text{at rate} \quad \alpha i p_2 \\
\to i+2 &\quad \text{at rate} \quad \alpha i p_3 \\
\to \cdots &
\end{aligned}
$$

Single death process

- Z. Criteria on ergodicity and strong ergodicity of single death processes, Frontiers of Mathematics in China, 2018, 13(5): 1215-1243.
- Z. & Zhou. High-order moments of the first hitting times for single death processes, Frontiers of Mathematics in China, 2019, 14(5): 1037-1061. Here, $V \equiv 1$.

一般的 V?

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最小非负解方法. 极限过渡法(生灭过程, 王梓坤).

- J. Wang & Z. Moments of integral-type downward functionals for single death processes, Frontiers of Mathematics in China, 2020, 15(4): 749-768.

Single death process

Define $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}, \ k > n \geqslant 0.$

$$G_i^{(i)} = 1, \qquad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \qquad 1 \leqslant n < i.$$

Theorem (Z., 2018) 1st moment

Assume that the single death Q is irreducible and regular. Then the process is ergodic if and only if

$$D := \sum_{k \geqslant 1} q_0^{(k)} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k \geqslant 1} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

Single death process

Theorem (Zhou & Z., 2019) high order

Assume that the single death Q is irreducible and the corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Theorem 1 (Wang & Z., 2020)

Assume that the single death Q is irreducible and the corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \xi_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Single death process

Let $w_i^{(n)} := \mathbb{E}_i \xi_{i-1}^n$ and $w_i := w_i^{(1)}$. $w_i^{(0)} = 1$.

Theorem 2

Under the conditions of Theorem 1, then

$$\begin{aligned}\mathbb{E}_i \xi_{i_0}^n &= (w_{i_0+1} + \cdots + w_i)^{(n)} \\ &=: \sum_{n_{i_0+1}+\cdots+n_i=n} \frac{n!}{n_{i_0+1}!\cdots n_i!} w_{i_0+1}^{(n_{i_0+1})} \cdots w_i^{(n_i)}, \qquad i > i_0, n \geqslant 1.\end{aligned}$$

$$\mathbb{E}_i \xi_{i_0}^n = \sum_{i_0+1 \leqslant \ell \leqslant i} \mathbb{E}_\ell \xi_{\ell-1}^n + \sum_{i_0+2 \leqslant \ell \leqslant i} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \xi_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \xi_{i_0}^s.$$

Single death processes. Sketch of proof.

Let

$$M_{ik}^{(n-1)} = \sum_{i+1\leqslant \ell\leqslant k} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \xi_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \xi_{i-1}^{s}$$

and

$$M_i^{(n-1)} = V(i)\mathbb{E}_i \xi_{i-1}^{n-1} + \frac{1}{n} \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(n-1)}, \qquad i \geqslant 1.$$

Proposition 1

Under the conditions of Theorem 1, then

$$\mathbb{E}_i \xi_{i_0}^n = \sum_{i_0+1\leqslant \ell\leqslant i} \mathbb{E}_\ell \xi_{\ell-1}^n + M_{i_0+1,i}^{(n-1)}, \qquad i \geqslant i_0 + 1,$$

$$M_{ik}^{(n-1)} = M_{ij}^{(n-1)} + M_{j+1,k}^{(n-1)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_k \xi_j^{n-s} \mathbb{E}_j \xi_{i-1}^{s}, \qquad 1 \leqslant i \leqslant j \leqslant k.$$

Single death process. Sketch of proof.

- Single death property: $\mathbb{E}_i \xi_{i-1}^n = ?$

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \ i \geqslant 1. \qquad (*)$$

Theorem 3. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i-1}^n = n \sum_{k \geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \qquad i \geqslant 1, n \geqslant 1.$$

Denote the left hand side by m_i, the right hand side by h_i.

$$h_i = \frac{1}{q_{i,i-1}} \left(n M_i^{(n-1)} + \sum_{k \geqslant i+1} q_i^{(k)} h_k \right).$$

Single death process. Sketch of proof.

Lemma 1

(h_i) is the minimal nonnegative solution to

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geqslant 1. \qquad (*)$$

$$m_i \geqslant h_i.$$

Sketch proof of Lemma 1:
Fix $N \geqslant 2$, define Q-matrix $Q^{(N)} = (\tilde{q}_{ij})$ on $\{0, 1, 2, \cdots, N\}$:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + nG^{(N)} a_N) & \text{if } i = N, j = N-1; \\ -(q_N \vee N)(1 + nG^{(N)} a_N) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N-1, \end{cases}$$

where $G^{(N)} = \max_{1 \leqslant i \leqslant N} G_i^{(N)}$ and

$$a_N = \begin{cases} M_N^{(n-1)} & \text{if } M_N^{(n-1)} < \infty; \\ 1 & \text{if } M_N^{(n-1)} = \infty. \end{cases}$$

Define $\tilde{q}_n^{(k)} = \sum_{j=k}^{N} \tilde{q}_{nj},\ 0 \leqslant n < k \leqslant N$,

$$\widetilde{G}_i^{(i)} = 1, \qquad \widetilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \tilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \qquad 1 \leqslant n < i \leqslant N.$$

Sketch proof of Lemma 1(continued):

$$h_n^{(N)} := n\sum_{k=i}^{N}\frac{\widetilde{G}_i^{(k)}M_k^{(n-1)}}{\tilde{q}_{k,k-1}} \quad (1\leqslant i\leqslant N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i=\frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i}\cdot x_i+\sum_{k=i+1}^{N}\frac{\tilde{q}_i^{(k)}}{\tilde{q}_i}\cdot x_k+\frac{n}{\tilde{q}_i}M_i^{(n-1)}, \qquad 1\leqslant i\leqslant N.$$

$$x_N=\frac{nM_N^{(n-1)}}{(q_N\vee N)(1+nG^{(N)}a_N)},\ x_i=\frac{q_i^{(i+1)}}{q_i}\cdot x_i+\sum_{k=i+1}^{N}\frac{q_i^{(k)}}{q_i}\cdot x_k+\frac{n}{q_i}M_i^{(n-1)}.$$

$(h_n^{(N)})$ is increasing to the minimal non-negative solution of $(*)$ as $N\to\infty$.

$$h_n^{(N)}=n\sum_{k=i}^{N-1}\frac{G_i^{(k)}M_k^{(n-1)}}{q_{k,k-1}}+\frac{nG_i^{(N)}M_N^{(n-1)}}{(q_N\vee N)(1+nG^{(N)}a_N)}\to n\sum_{k=i}^{\infty}\frac{G_i^{(k)}M_k^{(n-1)}}{q_{k,k-1}}=h_i.$$

Single death process. Sketch of proof.

Lemma 2 (τ_i)

Under the conditions of Theorem 1, $(\mathbb{E}_i \xi_{i_0}^n : i \geqslant i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{j \neq i, j > i_0} \frac{q_{ij}}{q_i} x_j + \frac{nV(i)}{q_i} \mathbb{E}_i \xi_{i_0}^{n-1}, \qquad i > i_0.$$

$n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)}$ satisfies the equation above.

$$\mathbb{E}_i \xi_{i_0}^n \leqslant n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)} \Rightarrow m_i \leqslant h_i.$$

Single death process. Sketch of proof.

By induction and Dominated Convergence Theorem, for all $n \geqslant 1$,

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \quad \uparrow \quad n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell)\mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}},$$

$$n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell)\mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}} = \mathbb{E}_i \xi_{i_0}^n, \qquad i \geqslant i_0.$$

Single death processes. Sketch of proof.

Lemma 3. Under the conditions of Theorem 1

$$\mathbb{E}_i\xi_{i_0}^n \leqslant n \sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)\mathbb{E}_\ell\xi_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i\geqslant i_0, n\geqslant 1.$$

Lemma 4. Under the conditions of Theorem 1

$$\mathbb{E}_i\xi_{i_0} = \sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)}{q_{\ell,\ell-1}}, \qquad i\geqslant i_0.$$

Lemma 5

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n = n \sum_{i_0+1\leqslant k\leqslant i}\sum_{k\leqslant \ell\leqslant N}\frac{G_k^{(N,\ell)}V(\ell)\mathbb{E}_\ell(\xi_{i_0}^{(N)})^{n-1}}{q_{\ell,\ell-1}^{(N)}}, \qquad i_0\leqslant i\leqslant N.$$

Single death process. Sketch of proof.

$n = 1$:

$$\mathbb{E}_i \xi_{i_0}^{(N)} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{k \leqslant \ell \leqslant N} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}} \uparrow \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}},$$

$$\sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}} = \mathbb{E}_i \xi_{i_0}, \qquad i \geqslant i_0.$$

Single death process. Sketch of proof.

Assume that the assertions hold until $n-1$.
By Monotone Convergence Theorem,

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \quad \uparrow \quad n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}},$$

$$n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}} \geqslant \mathbb{E}_i \xi_{i_0}^n, \qquad i \geqslant i_0.$$

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \leqslant n \sum_{i_0+1\leqslant k\leqslant i} \sum_{k\leqslant \ell\leqslant N} \frac{G_k^{(\ell)} M_\ell^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} + M_{i_0+1,i}^{(N,n-1)},$$

By Dominated Convergence Theorem.

Single death process

Define

$$\kappa_0 = 1, \quad \kappa_i = \frac{1}{q_{i,i-1}} \sum_{k=1}^{i} q_0^{(k)} G_k^{(i)}, \qquad i \geqslant 1, \quad \kappa = \sum_{i \in E} \kappa_i.$$

In the ergodic case, the stationary distribution $\pi = (\pi_i)$: $\pi_i = \kappa_i / \kappa$.

Theorem 4 (Wang & Z. 2020+) (σ_i) Under the conditions of Th. 1,

$$\mathbb{E}_{i_0} \xi_{i_0}^n = \frac{1}{\kappa_{i_0}} \sum_{i=0}^{i_0} \frac{\kappa_i \widetilde{H}_i^{(n,i_0)}}{q_i}, \qquad i_0 \geqslant 0,$$

$$\mathbb{E}_i \xi_{i_0}^n = \frac{D_{i_0}^{(i_0)}}{C_{i_0}^{(i_0)}} C_{i_0-1-i}^{(i_0)} - D_{i_0-1-i}^{(i_0)}, \qquad 0 \leqslant i \leqslant i_0 - 1,$$

where

$$H_i^{(n,i_0)} = nV(i)\mathbb{E}_i \xi_{i_0}^{n-1}, \qquad i \geqslant 0,$$

$$\widetilde{H}_i^{(n,i_0)} = \sum_{j \geqslant i_0+1} q_{ij} \mathbb{E}_j \xi_{i_0}^n + H_i^{(n,i_0)}, \qquad 0 \leqslant i \leqslant i_0.$$

Theorem 4 (continued)

$$C_{i_0}^{(i_0)} = \frac{q_{i_0,i_0-1}}{q_{i_0}}\kappa_{i_0}, \qquad D_{i_0}^{(i_0)} = \sum_{i=0}^{i_0-1}\frac{\kappa_i \widetilde{H}_i^{(n,i_0)}}{q_i},$$

and

$$C_i^{(i_0)} = \sum_{i_0-i\leqslant k\leqslant i_0} G_k^{(i_0)},\ D_i^{(i_0)} = \sum_{i_0-i\leqslant \ell\leqslant i_0-1}\ \sum_{k=i_0-i}^{\ell}\frac{G_k^{(\ell)}\widetilde{H}_\ell^{(n,i_0)}}{q_{\ell,\ell-1}},\ 0\leqslant i\leqslant i_0-1.$$

Birth-death process on trees

The tree T is a connected graph without cycles. We fix a point on T as the root, denoted by o. For any vertex $i \in T \setminus \{o\}$, there is a unique simple path from i to the root o.

- $\mathscr{P}(i)$: the set of all the vertices on this path (the root o is excluded).
- $|i|$: the number of segments of this path is the length of i.
- $i \sim j$: two vertices i and j are called adjacent if they are joined by a segment.
- When $|j| = |i| + 1$ and $i \sim j$, j is called one offspring of i and the set of all the offsprings of i is denoted by $J(i)$.
- When $|j| = |i| - 1$ and $i \sim j$, j is called the father of i and denoted by i^*.
- Denote T_i the subtree with i as its root, including all the descendants of i.
- Assume that any vertex has finite offsprings.

We consider a birth-death process on this tree which Q-matrix satisfies: $q_{ij} > 0$ if and only if $i \sim j$, i.e., $j = i^*$, or $j \in J(i)$.

$q_i := -q_{ii} = q_{ii^*} + \sum_{j \in J(i)} q_{ij} < \infty$ for all $i \in T$.

Define a measure μ on T as follows.

$$\mu_o = 1, \qquad \mu_i = \prod_{j \in \mathscr{P}(i)} \frac{q_{j^*j}}{q_{jj^*}}, \qquad i \in T \setminus \{0\},$$

which is invariant with respect to Q. In fact, μ satisfies the so-called detailed balance equation:

$$\mu_i q_{ij} = \mu_j q_{ji}, \qquad i \sim j.$$

- Z. Moments of first hitting times for birth-death processes on trees, Frontiers of Mathematics in China, 2019, 14(4): 833-854.

Theorem (Z., 2019)

Assume that the Q-matrix on T is regular and the process is recurrent. Then

$$\mathbb{E}_i\tau_k^n = n\sum_{j\in\mathscr{P}(i)\setminus\mathscr{P}(k)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell\mathbb{E}_\ell\xi_k^{n-1}, \qquad k\in\mathscr{P}(i),$$

$$\mathbb{E}_i\tau_o^n = n\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell\mathbb{E}_\ell\xi_o^{n-1}.$$

Corollary. ergodic $\Leftrightarrow \mu<\infty$; strongly ergodic $\Leftrightarrow$

$$\sup_{i\in T\setminus\{o\}}\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell<\infty.$$

Theorem (Wang & Z., 2020+)

Assume that the Q-matrix on the tree T is regular and the process is recurrent. Then

$$\mathbb{E}_i \xi_k^n = n \sum_{j \in \mathscr{P}(i) \setminus \mathscr{P}(k)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell \in T_j} \mu_\ell V(\ell) \mathbb{E}_\ell \xi_k^{n-1}, \qquad k \in \mathscr{P}(i)$$

$$\mathbb{E}_i \xi_o^n = n \sum_{j \in \mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell \in T_j} \mu_\ell V(\ell) \mathbb{E}_\ell \xi_o^{n-1}.$$

- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.

$$\xi_{i_0} = \int_0^{\sigma_{i_0}} r(t)V(X(t))\mathrm{d}t.$$

Assume that $r, r' \geqslant 0$. Then $(\mathbb{E}_i\xi_{i_0} : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\overline{V}(i)}{q_i}, \qquad i \in E,$$

where

$$\overline{V} = \widetilde{V} + r(0)V, \qquad \widetilde{V}(i) = \mathbb{E}_i \int_0^{\sigma_{i_0}} r'(t)V(X(t))\mathrm{d}t.$$

Application. Single death process for example.

$$\mathbb{E}_i \xi_{i_0} = \mathbb{E}_i \int_0^{\sigma_{i_0}} \overline{V}(X(t))\mathrm{d}t.$$

$(\mathbb{E}_i \xi_{i_0} : i \geqslant i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{k \neq i, k > i_0} \frac{q_{ik}}{q_i} x_j + \frac{\overline{V}(i)}{q_i}, \qquad i > i_0.$$

For single death process,

$$\mathbb{E}_i \xi_{i_0} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \overline{V}(\ell)}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Application. Single death process for example.

In particular, when $r(t) = t^n$, denote

$$\xi_{i_0} = \xi_{i_0}^{(n)} = \int_0^{\sigma_{i_0}} t^n V(X(t))\mathrm{d}t, \ \widetilde{V}(i) = n\mathbb{E}_i \xi_{i_0}^{(n-1)}.$$

$$\mathbb{E}_i \xi_{i_0}^{(n)} = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \xi_{i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

- subexponential ergodicity and polynomial ergodicity
- CLT:

$$t^{1/2}\left(\frac{1}{t}\int_0^t f(X(s))\mathrm{d}s - \pi(f)\right) \xrightarrow{D} N(0, \sigma^2(f)), \quad t \to \infty.$$

If $\pi(|f|) < \infty$, then a CLT holds iff

$$\mathbb{E}_0\left(\int_0^{\sigma_0} (f-\pi(f))(X(t))\mathrm{d}t\right)^2 < \infty \Leftarrow \mathbb{E}_0\left(\int_0^{\sigma_0} |f-\pi(f)|(X(t))\mathrm{d}t\right)^2 < \infty.$$

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Thank you for your attention!

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