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Moments of integral-type functionals downward for single death processes

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CONTENTS

Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

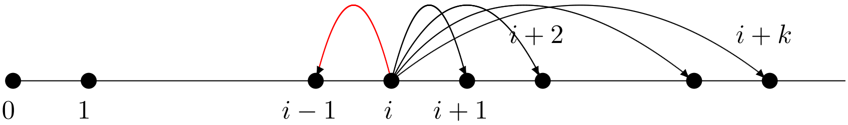
$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

Background

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0, \;\; q_{i,i-j} = 0, \;\; i \geqslant 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Background

- branching processes:

$$
\begin{aligned}
i \to\ & i-1 && \text{at rate} && \alpha i p_0 \\
\to\ & i && \text{at rate} && -\alpha i(1-p_1) \\
\to\ & i+1 && \text{at rate} && \alpha i p_2 \\
\to\ & i+2 && \text{at rate} && \alpha i p_3 \\
\to\ & \cdots
\end{aligned}
$$

- Z. Criteria on ergodicity and strong ergodicity of single death processes, Frontiers of Mathematics in China, 2018, 13(5): 1215-1243.
- Z. Moments of first hitting times for birth-death processes on trees, Frontiers of Mathematics in China, 2019, 14(4): 833-854.
- Z. & Zhou. High-order moments of the first hitting times for single death processes, Frontiers of Mathematics in China, 2019, 14(5): 1037-1061.

Background

$$Y(t) = \int_0^t f(X(s), s)\mathrm{d}s.$$

- the theory of inventories and storage, moments and distributions
- queueing, traffic jam and intersection bottlenecks
- biology, the joint distribution of $X(t)$ and $Y(t)$
- limiting behavior

Background

Define

$$\tau_i = \inf\{t \geqslant 0 : X(t) = i\},\ \sigma_i = \inf\{t \geqslant \text{the first jumping time} : X(t) = i\}.$$

Given $V \geqslant 0$ but not equal to 0 identically. Consider the integral-type functional:

$$\xi_i = \int_0^{\tau_i} V(X(t))\mathrm{d}t. \quad \left(\int_0^{\sigma_i} V(X(t))\mathrm{d}t\right)$$

When $V \equiv 1$, $\xi_i = \tau_i$.
When $V(1) = V(2) = 1$ and $V(j) = 0, j \neq 1, 2$, then ξ_0 is the total time that the process stays in the states 1 and 2 before hitting 0 firstly.

- Z.K. Wang . On distribution of functions of birth and death processes and their applications in the theory of queues, Scientia Sinica, 1961, X(2): 160-170.
- D.R. McNeil. Integral functionals of birth and death processes and related limiting distributions, The Annals of Mathematical Statistics, 1970, 41(2):480-485.
- 吴立德. 齐次可数马尔科夫过程积分型泛函的分布, 数学学报, 1963, 13(1): 86-93.
- 杨超群. 可列马氏过程的积分型泛函和双边生灭过程的边界性质, 数学进展, 1964, 7(4): 397-424.

Background

- 侯振挺, 郭青峰. 齐次可列马尔可夫过程, 科学出版社, 1978.

$(\mathbb{E}_i \xi_{i_0}^{\ell} : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\ell V(i)}{q_i} \mathbb{E}_i \xi_{i_0}^{\ell-1}, \qquad i \in E.$$

Define $\psi_{i,i_0}(\lambda) = 1 - \mathbb{E}_i \mathrm{e}^{-\lambda \xi_{i_0}}$.

$(\psi_{i,i_0}(\lambda) : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} y_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \qquad i \in E.$$

Background

- J.K. Zhang. On the generalized birth and death processes (I)–the numeral introduction, the functional of integral type and the distributions of runs and passage times, Acta Math Sci, 1984, 4(2): 191-209.
- 王婧, 张余辉. 单生过程积分型泛函的矩和分布, 2019+.
- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.

$$\xi_{i_0} = \int_0^{\tau_{i_0}} r(t)V(X(t))\mathrm{d}t.$$

Background

$V \equiv 1$: Z & Zhou. High-order moments of the first hitting times for single death processes, Frontiers of Mathematics in China, 2019, 14(5): 1037-1061.

侯振挺, 郭青峰(1978). 齐次可列马尔可夫过程. 科学出版社.

最小非负解方法. 极限过渡法(王梓坤).

一般的 V.

这个报告是建立在与王婧合作完成的工作上.

Main results

Define $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}, \; k > n \geqslant 0.$

$$G_i^{(i)} = 1, \qquad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \qquad 1 \leqslant n < i.$$

Assume that the single death Q is irreducible and regular. Then the process is ergodic if and only if

$$D := \sum_{k \geqslant 1} q_0^{(k)} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k \geqslant 1} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

Main results

Assume that the single death Q is irreducible and the corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Theorem 1.

Assume that the single death Q is irreducible and the corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \xi_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Main results

Let $w_i^{(n)} := \mathbb{E}_i \xi_{i-1}^n$ and $w_i := w_i^{(1)}$. $w_i^{(0)} = 1$.

Theorem 2.

Under the conditions of Theorem 1, then

$$\begin{aligned}\mathbb{E}_i \xi_{i_0}^n &= (w_{i_0+1} + \cdots + w_i)^{(n)} \\ &=: \sum_{n_{i_0+1}+\cdots+n_i=n} \frac{n!}{n_{i_0+1}! \cdots n_i!} w_{i_0+1}^{(n_{i_0+1})} \cdots w_i^{(n_i)}, \qquad i > i_0, n \geqslant 1.\end{aligned}$$

$$\mathbb{E}_i \xi_{i_0}^n = \sum_{i_0+1 \leqslant \ell \leqslant i} \mathbb{E}_\ell \xi_{\ell-1}^n + \sum_{i_0+2 \leqslant \ell \leqslant i} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \xi_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \xi_{i_0}^s.$$

Sketch of proof

Let

$$M_{ik}^{(n-1)} = \sum_{i+1\leqslant \ell \leqslant k} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \xi_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \xi_{i-1}^{s}$$

and

$$M_i^{(n-1)} = V(i)\mathbb{E}_i \xi_{i-1}^{n-1} + \frac{1}{n} \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(n-1)}, \qquad i \geqslant 1.$$

Proposition 1.

Under the conditions of Theorem 1, then

$$\mathbb{E}_i \xi_{i_0}^n = \sum_{i_0+1\leqslant \ell \leqslant i} \mathbb{E}_\ell \xi_{\ell-1}^n + M_{i_0+1,i}^{(n-1)}, \qquad i \geqslant i_0 + 1,$$

$$M_{ik}^{(n-1)} = M_{ij}^{(n-1)} + M_{j+1,k}^{(n-1)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_k \xi_j^{n-s} \mathbb{E}_j \xi_{i-1}^s, \qquad 1 \leqslant i \leqslant j \leqslant k.$$

Sketch of proof

- Single death property: $\mathbb{E}_i \xi_{i-1}^n = ?$

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \ i \geqslant 1. \qquad (*)$$

Theorem 3. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i-1}^n = n \sum_{k \geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \qquad i \geqslant 1, n \geqslant 1.$$

Denote the left hand side by m_i, the right hand side by h_i.

$$h_i = \frac{1}{q_{i,i-1}} \left(n M_i^{(n-1)} + \sum_{k \geqslant i+1} q_i^{(k)} h_k \right).$$

Sketch of proof

Lemma 4.

(h_i) is the minimal nonnegative solution to

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geqslant 1. \qquad (*)$$

$$m_i \geqslant h_i.$$

Sketch of proof

Sketch proof of Lemma 4:
Fix $N \geqslant 2$, define Q-matrix $Q^{(N)} = (\tilde{q}_{ij})$ on $\{0, 1, 2, \cdots, N\}$:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + nG^{(N)} a_N) & \text{if } i = N, j = N-1; \\ -(q_N \vee N)(1 + nG^{(N)} a_N) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N-1, \end{cases}$$

where $G^{(N)} = \max_{1 \leqslant i \leqslant N} G_i^{(N)}$ and

$$a_N = \begin{cases} M_N^{(n-1)} & \text{if } M_N^{(n-1)} < \infty; \\ 1 & \text{if } M_N^{(n-1)} = \infty. \end{cases}$$

Define $\tilde{q}_n^{(k)} = \sum_{j=k}^N \tilde{q}_{nj}, \ 0 \leqslant n < k \leqslant N,$

$$\widetilde{G}_i^{(i)} = 1, \qquad \widetilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \tilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \qquad 1 \leqslant n < i \leqslant N.$$

Sketch of proof

Sketch proof of Lemma 4(continued):

$$h_n^{(N)} := n \sum_{k=i}^{N} \frac{\widetilde{G}_i^{(k)} M_k^{(n-1)}}{\tilde{q}_{k,k-1}} \quad (1 \leqslant i \leqslant N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i = \frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i} \cdot x_i + \sum_{k=i+1}^{N} \frac{\tilde{q}_i^{(k)}}{\tilde{q}_i} \cdot x_k + \frac{n}{\tilde{q}_i} M_i^{(n-1)}, \qquad 1 \leqslant i \leqslant N.$$

$$x_N = \frac{n M_N^{(n-1)}}{(q_N \vee N)(1 + nG^{(N)} a_N)}, \; x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{k=i+1}^{N} \frac{q_i^{(k)}}{q_i} \cdot x_k + \frac{n}{q_i} M_i^{(n-1)}.$$

$(h_n^{(N)})$ is increasing to the minimal non-negative solution of $(*)$ as $N \to \infty$.

$$h_n^{(N)} = n \sum_{k=i}^{N-1} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}} + \frac{n G_i^{(N)} M_N^{(n-1)}}{(q_N \vee N)(1 + nG^{(N)} a_N)} \to n \sum_{k=i}^{\infty} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}} = h_i.$$

Sketch of proof

Lemma 5. Under the conditions of Theorem 1

$(\mathbb{E}_i\xi_{i_0}^n : i \geqslant i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{j\neq i, j>i_0} \frac{q_{ij}}{q_i} x_j + \frac{nV(i)}{q_i}\mathbb{E}_i\xi_{i_0}^{n-1}, \qquad i > i_0.$$

$n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)}$ satisfies the equation above.

$$\mathbb{E}_i\xi_{i_0}^n \leqslant n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)} \Rightarrow m_i \leqslant h_i.$$

Sketch of proof

By induction and Dominated Convergence Theorem, for all $n \geqslant 1$,

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \quad \uparrow \quad \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}},$$

$$\sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}} = \mathbb{E}_i \xi_{i_0}^n, \qquad i \geqslant i_0.$$

Sketch of proof

Lemma 6. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i_0}^n \leqslant n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0, n \geqslant 1.$$

Lemma 7. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i_0} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0.$$

Lemma 8.

$$\mathbb{E}_i (\xi_{i_0}^{(N)})^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{k \leqslant \ell \leqslant N} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell (\xi_{i_0}^{(N)})^{n-1}}{q_{\ell,\ell-1}^{(N)}}, \qquad i_0 \leqslant i \leqslant N.$$

Sketch of proof

$n = 1$:

$$\mathbb{E}_i \xi_{i_0}^{(N)} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{k \leqslant \ell \leqslant N} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}} \quad \uparrow \quad \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}},$$

$$\sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}} = \mathbb{E}_i \xi_{i_0}, \qquad i \geqslant i_0.$$

Sketch of proof

Assume that the assertions hold until $n-1$.
By Monotone Convergence Theorem,

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \quad \uparrow \quad \sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)\mathbb{E}_\ell\xi_{i_0}^{n-1}}{q_{\ell,\ell-1}},$$

$$\sum_{i_0+1\leqslant k\leqslant i}\sum_{\ell\geqslant k}\frac{G_k^{(\ell)}V(\ell)\mathbb{E}_\ell\xi_{i_0}^{n-1}}{q_{\ell,\ell-1}} \geqslant \mathbb{E}_i\xi_{i_0}^n, \qquad i\geqslant i_0.$$

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \leqslant n\sum_{i_0+1\leqslant k\leqslant i}\sum_{k\leqslant\ell\leqslant N}\frac{G_k^{(\ell)}M_\ell^{(N,n-1)}}{q_{\ell,\ell-1}^{(N)}} + M_{i_0+1,i}^{(N,n-1)},$$

By Dominated Convergence Theorem.

Sketch of proof

$$\mathbb{E}_i \mathrm{e}^{-\lambda \xi_{i_0}^{(N)}} \to \mathbb{E}_i \mathrm{e}^{-\lambda \xi_{i_0}}, \quad i > i_0?$$

Define $\tilde{q}_n^{(k)} := \sum_{j=k}^{\infty} q_{nj} + \lambda V(n),\ k > n \geqslant 0.$

$$\widetilde{G}_i^{(i)} = 1, \qquad \widetilde{G}_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} \tilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \qquad 1 \leqslant n < i.$$

$$1 - \mathbb{E}_i \mathrm{e}^{-\lambda \xi_{i_0}^{(N)}} = \frac{\lambda \sum_{i_0+1 \leqslant k \leqslant i} \sum_{k \leqslant \ell \leqslant N} \frac{\widetilde{G}_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}}}{1 + \lambda \sum_{i_0+1 \leqslant k \leqslant N} \sum_{k \leqslant \ell \leqslant N} \frac{\widetilde{G}_k^{(\ell)} V(\ell)}{q_{\ell,\ell-1}}}, \qquad i \geqslant i_0 + 1.$$

Application

- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.

$$\xi_{i_0} = \int_0^{\sigma_{i_0}} r(t)V(X(t))\mathrm{d}t.$$

Assume that $r, r' \geqslant 0$. Then $(\mathbb{E}_i \xi_{i_0} : i \in E)$ is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\overline{V}(i)}{q_i}, \qquad i \in E,$$

where

$$\overline{V} = \widetilde{V} + r(0)V, \qquad \widetilde{V}(i) = \mathbb{E}_i \int_0^{\sigma_{i_0}} r'(t)V(X(t))\mathrm{d}t.$$

Application

$$\mathbb{E}_i \xi_{i_0} = \mathbb{E}_i \int_0^{\sigma_{i_0}} \overline{V}(X(t))\mathrm{d}t.$$

$(\mathbb{E}_i \xi_{i_0} : i \geqslant i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{k \neq i, k > i_0} \frac{q_{ik}}{q_i} x_j + \frac{\overline{V}(i)}{q_i}, \qquad i > i_0.$$

$$\mathbb{E}_i \xi_{i_0} = \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \overline{V}(\ell)}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Application

In particular, when $r(t) = t^n$, denote

$$\xi_{i_0} = \xi_{i_0}^{(n)} = \int_0^{\sigma_{i_0}} t^n V(X(t))\mathrm{d}t, \ \widetilde{V}(i) = n\mathbb{E}_i \xi_{i_0}^{(n-1)}.$$

$$\mathbb{E}_i \xi_{i_0}^{(n)} = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \xi_{i_0}^{(n-1)}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

- subexponential ergodicity and polynomial ergodicity
- CLT:

$$t^{1/2}\left(\frac{1}{t}\int_0^t f(X(s))\mathrm{d}s - \pi(f)\right) \xrightarrow{D} N(0, \sigma^2(f)), \quad t \to \infty.$$

If $\pi(|f|) < \infty$, then a CLT holds iff

$$\mathbb{E}_0\left(\int_0^{\sigma_0} (f-\pi(f))(X(t))\mathrm{d}t\right)^2 < \infty \Leftarrow \mathbb{E}_0\left(\int_0^{\sigma_0} |f-\pi(f)|(X(t))\mathrm{d}t\right)^2 < \infty.$$

Thank you for your attention!

Homepage: http://math0.bnu.edu.cn/~zhangyh/