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# Moments of integral-type functionals downward for single death processes

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- $Q$ -process: on  $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ , suppose that sub-Markov transition probability matrix  $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$  satisfies  
 $Q$ -condition:  $\lim_{t \rightarrow 0} (p_{ij}(t) - \delta_{ij})/t = q_{ij}$ ,  $i, j \in \mathbb{Z}_+$ , i.e. the  $Q$ -matrix  $Q = (q_{ij})$  is the derivative matrix at time 0 of  $P(t)$ ,

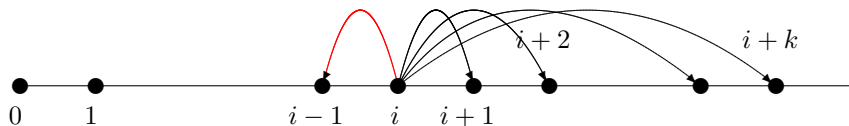
$$0 \leq q_{ij} < \infty, i \neq j; \sum_{j \neq i} q_{ij} \leq -q_{ii} =: q_i \leq \infty, i \in \mathbb{Z}_+.$$

- totally stable:  $q_i < \infty$ ,  $i \in \mathbb{Z}_+$ .
- conservative:  $q_i = \sum_{j \neq i} q_{ij}$ ,  $i \in \mathbb{Z}_+$ , i.e.  $\sum_{j \in \mathbb{Z}_+} q_{ij} = 0$ .

# Background

Single death process:  $Q = (q_{ij})$  satisfies

$$q_{i,i-1} > 0, \quad q_{i,i-j} = 0, \quad i \geq 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- branching processes:

$$\begin{aligned}i &\rightarrow i - 1 && \text{at rate } \alpha i p_0 \\&\rightarrow i && \text{at rate } -\alpha i(1 - p_1) \\&\rightarrow i + 1 && \text{at rate } \alpha i p_2 \\&\rightarrow i + 2 && \text{at rate } \alpha i p_3 \\&\rightarrow \dots\end{aligned}$$

- Z. Criteria on ergodicity and strong ergodicity of single death processes, *Frontiers of Mathematics in China*, 2018, 13(5): 1215-1243.
- Z. Moments of first hitting times for birth-death processes on trees, *Frontiers of Mathematics in China*, 2019, 14(4): 833-854.
- Z. & Zhou. High-order moments of the first hitting times for single death processes, *Frontiers of Mathematics in China*, 2019, 14(5): 1037-1061.

$$Y(t) = \int_0^t f(X(s), s) ds.$$

- the theory of inventories and storage, moments and distributions
- queueing, traffic jam and intersection bottlenecks
- biology, the joint distribution of  $X(t)$  and  $Y(t)$
- limiting behavior

Define

$$\tau_i = \inf\{t \geq 0 : X(t) = i\}, \quad \sigma_i = \inf\{t \geq \text{the first jumping time} : X(t) = i\}.$$

Given  $V \geq 0$  but not equal to 0 identically. Consider the integral-type functional:

$$\xi_i = \int_0^{\tau_i} V(X(t))dt. \quad \left( \int_0^{\sigma_i} V(X(t))dt \right)$$

When  $V \equiv 1$ ,  $\xi_i = \tau_i$ .

When  $V(1) = V(2) = 1$  and  $V(j) = 0, j \neq 1, 2$ , then  $\xi_0$  is the total time that the process stays in the states 1 and 2 before hitting 0 firstly.



- Z.K. Wang . On distribution of functions of birth and death processes and their applications in the theory of queues, *Scientia Sinica*, 1961, X(2): 160-170.
- D.R. McNeil. Integral functionals of birth and death processes and related limiting distributions, *The Annals of Mathematical Statistics*, 1970, 41(2):480-485.
- 吴立德. 齐次可数马尔科夫过程积分型泛函的分布, *数学学报*, 1963, 13(1): 86-93.
- 杨超群. 可列马氏过程的积分型泛函和双边生灭过程的边界性质, *数学进展*, 1964, 7(4): 397-424.

# Background

- 侯振挺, 郭青峰. 齐次可列马尔可夫过程, 科学出版社, 1978.

$(\mathbb{E}_i \xi_{i_0}^\ell : i \in E)$  is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\ell V(i)}{q_i} \mathbb{E}_i \xi_{i_0}^{\ell-1}, \quad i \in E.$$

Define  $\psi_{i, i_0}(\lambda) = 1 - \mathbb{E}_i e^{-\lambda \xi_{i_0}}$ .

$(\psi_{i, i_0}(\lambda) : i \in E)$  is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{\lambda V(i) + q_i} y_k + \frac{\lambda V(i)}{\lambda V(i) + q_i}, \quad i \in E.$$

- J.K. Zhang. On the generalized birth and death processes (I)—the numeral introduction, the functional of integral type and the distributions of runs and passage times, Acta Math Sci, 1984, 4(2): 191-209.
- 王婧, 张余辉. 单生过程积分型泛函的矩和分布, 2019+.
- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, Front. Math. China, 2018, 13(3): 619-632.

$$\xi_{i_0} = \int_0^{\tau_{i_0}} r(t)V(X(t))dt.$$

$V \equiv 1$ : Z & Zhou. High-order moments of the first hitting times for single death processes, *Frontiers of Mathematics in China*, 2019, 14(5): 1037-1061.

侯振挺, 郭青峰(1978). 齐次可列马尔可夫过程. 科学出版社.

最小非负解方法. 极限过渡法(王梓坤).

一般的  $V$ .

这个报告是建立在与王婧合作完成的工作上.

# Main results

Define  $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}$ ,  $k > n \geq 0$ .

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^i q_n^{(k)} G_k^{(i)}, \quad 1 \leq n < i.$$

Assume that the single death  $Q$  is irreducible and regular. Then the process is ergodic if and only if

$$D := \sum_{k \geq 1} q_0^{(k)} \sum_{\ell \geq k} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k \geq 1} \sum_{\ell \geq k} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} < \infty.$$

# Main results

Assume that the single death  $Q$  is irreducible and the corresponding process is recurrent. Given  $i_0 \in \mathbb{Z}_+$  and a positive integer  $n \geq 1$  arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell, \ell-1}}, \quad i \geq i_0 + 1.$$

## Theorem 1.

Assume that the single death  $Q$  is irreducible and the corresponding process is recurrent. Given  $i_0 \in \mathbb{Z}_+$  and a positive integer  $n \geq 1$  arbitrarily. Then

$$\mathbb{E}_i \xi_{i_0}^n = n \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell, \ell-1}}, \quad i \geq i_0 + 1.$$

# Main results

Let  $w_i^{(n)} := \mathbb{E}_i \xi_{i-1}^n$  and  $w_i := w_i^{(1)}$ .  $w_i^{(0)} = 1$ .

## Theorem 2.

Under the conditions of Theorem 1, then

$$\begin{aligned} \mathbb{E}_i \xi_{i_0}^n &= (w_{i_0+1} + \cdots + w_i)^{(n)} \\ &=: \sum_{n_{i_0+1} + \cdots + n_i = n} \frac{n!}{n_{i_0+1}! \cdots n_i!} w_{i_0+1}^{(n_{i_0+1})} \cdots w_i^{(n_i)}, \quad i > i_0, n \geq 1. \end{aligned}$$

$$\mathbb{E}_i \xi_{i_0}^n = \sum_{i_0+1 \leq \ell \leq i} \mathbb{E}_\ell \xi_{\ell-1}^n + \sum_{i_0+2 \leq \ell \leq i} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \xi_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \xi_{i_0}^s.$$

# Sketch of proof

Let

$$M_{ik}^{(n-1)} = \sum_{i+1 \leq \ell \leq k} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_\ell \xi_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \xi_{i-1}^s$$

and

$$M_i^{(n-1)} = V(i) \mathbb{E}_i \xi_{i-1}^{n-1} + \frac{1}{n} \sum_{k \geq i+1} q_{ik} M_{ik}^{(n-1)}, \quad i \geq 1.$$

## Proposition 1.

Under the conditions of Theorem 1, then

$$\mathbb{E}_i \xi_{i_0}^n = \sum_{i_0+1 \leq \ell \leq i} \mathbb{E}_\ell \xi_{\ell-1}^n + M_{i_0+1, i}^{(n-1)}, \quad i \geq i_0 + 1,$$

$$M_{ik}^{(n-1)} = M_{ij}^{(n-1)} + M_{j+1, k}^{(n-1)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_k \xi_j^{n-s} \mathbb{E}_j \xi_{i-1}^s, \quad 1 \leq i \leq j \leq k.$$



# Sketch of proof

- Single death property:  $\mathbb{E}_i \xi_{i-1}^n = ?$

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geq i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geq 1. \quad (*)$$

Theorem 3. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i-1}^n = n \sum_{k \geq i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \quad i \geq 1, n \geq 1.$$

Denote the left hand side by  $m_i$ , the right hand side by  $h_i$ .

$$h_i = \frac{1}{q_{i,i-1}} \left( n M_i^{(n-1)} + \sum_{k \geq i+1} q_i^{(k)} h_k \right).$$

## Lemma 4.

$(h_i)$  is the minimal nonnegative solution to

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geq i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geq 1. \quad (*)$$

$$m_i \geq h_i.$$

# Sketch of proof

Sketch proof of Lemma 4:

Fix  $N \geq 2$ , define  $Q$ -matrix  $Q^{(N)} = (\tilde{q}_{ij})$  on  $\{0, 1, 2, \dots, N\}$ :

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + nG^{(N)}a_N) & \text{if } i = N, j = N - 1; \\ -(q_N \vee N)(1 + nG^{(N)}a_N) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N - 1, \end{cases}$$

where  $G^{(N)} = \max_{1 \leq i \leq N} G_i^{(N)}$  and

$$a_N = \begin{cases} M_N^{(n-1)} & \text{if } M_N^{(n-1)} < \infty; \\ 1 & \text{if } M_N^{(n-1)} = \infty. \end{cases}$$

Define  $\tilde{q}_n^{(k)} = \sum_{j=k}^N \tilde{q}_{nj}$ ,  $0 \leq n < k \leq N$ ,

$$\tilde{G}_i^{(i)} = 1, \quad \tilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^i \tilde{q}_n^{(k)} \tilde{G}_k^{(i)}, \quad 1 \leq n < i \leq N.$$

# Sketch of proof

Sketch proof of Lemma 4(continued):

$$h_n^{(N)} := n \sum_{k=i}^N \frac{\tilde{G}_i^{(k)} M_k^{(n-1)}}{\tilde{q}_{k,k-1}} \quad (1 \leq i \leq N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i = \frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i} \cdot x_i + \sum_{k=i+1}^N \frac{\tilde{q}_i^{(k)}}{\tilde{q}_i} \cdot x_k + \frac{n}{\tilde{q}_i} M_i^{(n-1)}, \quad 1 \leq i \leq N.$$

$$x_N = \frac{nM_N^{(n-1)}}{(q_N \vee N)(1 + nG^{(N)}a_N)}, \quad x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{k=i+1}^N \frac{q_i^{(k)}}{q_i} \cdot x_k + \frac{n}{q_i} M_i^{(n-1)}.$$

$(h_n^{(N)})$  is increasing to the minimal non-negative solution of (\*) as  $N \rightarrow \infty$ .

$$h_n^{(N)} = n \sum_{k=i}^{N-1} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}} + \frac{nG_i^{(N)} M_N^{(n-1)}}{(q_N \vee N)(1 + nG^{(N)}a_N)} \rightarrow n \sum_{k=i}^{\infty} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}} = h_i.$$

Lemma 5. Under the conditions of Theorem 1

$(\mathbb{E}_i \xi_{i_0}^n : i \geq i_0)$  is the minimal nonnegative solution to

$$x_{i_0} = 0, \quad x_i = \sum_{j \neq i, j > i_0} \frac{q_{ij}}{q_i} x_j + \frac{nV(i)}{q_i} \mathbb{E}_i \xi_{i_0}^{n-1}, \quad i > i_0.$$

$n \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell, \ell-1}} + M_{i_0+1, i}^{(n-1)}$  satisfies the equation above.

$$\mathbb{E}_i \xi_{i_0}^n \leq n \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell, \ell-1}} + M_{i_0+1, i}^{(n-1)} \Rightarrow m_i \leq h_i.$$

# Sketch of proof

By induction and Dominated Convergence Theorem, for all  $n \geq 1$ ,

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \uparrow \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell, \ell-1}},$$

$$\sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell, \ell-1}} = \mathbb{E}_i \xi_{i_0}^n, \quad i \geq i_0.$$

Lemma 6. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i_0}^n \leq n \sum_{i_0+1 \leq k \leq i} \sum_{l \geq k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell, \ell-1}}, \quad i \geq i_0, n \geq 1.$$

Lemma 7. Under the conditions of Theorem 1

$$\mathbb{E}_i \xi_{i_0} = \sum_{i_0+1 \leq k \leq i} \sum_{l \geq k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell, \ell-1}}, \quad i \geq i_0.$$

Lemma 8.

$$\mathbb{E}_i (\xi_{i_0}^{(N)})^n = n \sum_{i_0+1 \leq k \leq i} \sum_{k \leq l \leq N} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell (\xi_{i_0}^{(N)})^{n-1}}{q_{\ell, \ell-1}^{(N)}}, \quad i_0 \leq i \leq N.$$

# Sketch of proof

$n = 1$ :

$$\mathbb{E}_i \xi_{i_0}^{(N)} = \sum_{i_0+1 \leq k \leq i} \sum_{k \leq \ell \leq N} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell, \ell-1}} \quad \uparrow \quad \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell, \ell-1}},$$

$$\sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell)}{q_{\ell, \ell-1}} = \mathbb{E}_i \xi_{i_0}, \quad i \geq i_0.$$



# Sketch of proof

Assume that the assertions hold until  $n - 1$ .

By Monotone Convergence Theorem,

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \uparrow \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell, \ell-1}},$$

$$\sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_{i_0}^{n-1}}{q_{\ell, \ell-1}} \geq \mathbb{E}_i \xi_{i_0}^n, \quad i \geq i_0.$$

$$\mathbb{E}_i(\xi_{i_0}^{(N)})^n \leq n \sum_{i_0+1 \leq k \leq i} \sum_{k \leq \ell \leq N} \frac{G_k^{(\ell)} M_\ell^{(N, n-1)}}{q_{\ell, \ell-1}^{(N)}} + M_{i_0+1, i}^{(N, n-1)},$$

By Dominated Convergence Theorem.

# Sketch of proof

$$\mathbb{E}_i e^{-\lambda \xi_{i_0}^{(N)}} \rightarrow \mathbb{E}_i e^{-\lambda \xi_{i_0}}, \quad i > i_0?$$

Define  $\tilde{q}_n^{(k)} := \sum_{j=k}^{\infty} q_{nj} + \lambda V(n)$ ,  $k > n \geq 0$ .

$$\tilde{G}_i^{(i)} = 1, \quad \tilde{G}_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^i \tilde{q}_n^{(k)} \tilde{G}_k^{(i)}, \quad 1 \leq n < i.$$

$$1 - \mathbb{E}_i e^{-\lambda \xi_{i_0}^{(N)}} = \frac{\lambda \sum_{i_0+1 \leq k \leq i} \sum_{k \leq \ell \leq N} \frac{\tilde{G}_k^{(\ell)} V(\ell)}{q_{\ell, \ell-1}}}{1 + \lambda \sum_{i_0+1 \leq k \leq N} \sum_{k \leq \ell \leq N} \frac{\tilde{G}_k^{(\ell)} V(\ell)}{q_{\ell, \ell-1}}}, \quad i \geq i_0 + 1.$$

- Y.Y. Liu & Y.H. Song. Integral-type functionals of first hitting times for continuous-time Markov chains, *Front. Math. China*, 2018, 13(3): 619-632.

$$\xi_{i_0} = \int_0^{\sigma_{i_0}} r(t)V(X(t))dt.$$

Assume that  $r(0) = 0$  and  $r'(t) \geq 0$ . Then  $(\mathbb{E}_i \xi_{i_0} : i \in E)$  is the minimal nonnegative solution to:

$$y_i = \sum_{k \neq i, i_0} \frac{q_{ik}}{q_i} y_k + \frac{\tilde{V}(i)}{q_i}, \quad i \in E,$$

where

$$\tilde{V}(i) = \mathbb{E}_i \int_0^{\sigma_{i_0}} r'(t)V(X(t))dt.$$

# Application

$$\mathbb{E}_i \xi_{i_0} = \mathbb{E}_i \int_0^{\sigma_{i_0}} \tilde{V}(X(t)) dt.$$

$(\mathbb{E}_i \xi_{i_0} : i \geq i_0)$  is the minimal nonnegative solution to

$$x_{i_0} = 0, \quad x_i = \sum_{k \neq i, k > i_0} \frac{q_{ik}}{q_i} x_k + \frac{\tilde{V}(i)}{q_i}, \quad i > i_0.$$

$$\mathbb{E}_i \xi_{i_0} = \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} \tilde{V}(\ell)}{q_{\ell, \ell-1}}, \quad i \geq i_0 + 1.$$

In particular, when  $r(t) = t^n$ , denote

$$\xi_{i_0} = \xi_{i_0}^{(n)} = \int_0^{\sigma_{i_0}} t^n V(X(t)) dt, \quad \tilde{V}(i) = n \mathbb{E}_i \xi_{i_0}^{(n-1)}.$$

$$\mathbb{E}_i \xi_{i_0}^{(n)} = n \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \xi_{i_0}^{(n-1)}}{q_{\ell, \ell-1}}, \quad i \geq i_0 + 1.$$

**Thank you for your attention!**

Homepage: <http://math0.bnu.edu.cn/~zhangyh/>