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单死过程首达时的矩与遍历理论

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- Q -process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies
 Q -condition: $\lim_{t \rightarrow 0} (p_{ij}(t) - \delta_{ij})/t = q_{ij}$, $i, j \in \mathbb{Z}_+$, i.e. the Q -matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

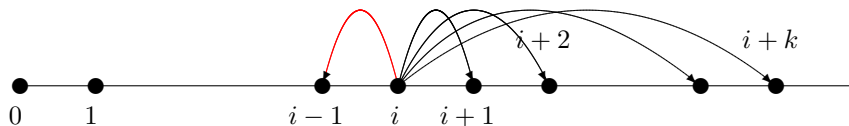
$$0 \leq q_{ij} < \infty, i \neq j; \sum_{j \neq i} q_{ij} \leq -q_{ii} =: q_i \leq \infty, i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty$, $i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j \neq i} q_{ij}$, $i \in \mathbb{Z}_+$, i.e. $\sum_{j \in \mathbb{Z}_+} q_{ij} = 0$.

Background

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0, \quad q_{i,i-j} = 0, \quad i \geq 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- branching processes:

$$\begin{aligned}i &\rightarrow i - 1 && \text{at rate } \alpha i p_0 \\&\rightarrow i && \text{at rate } -\alpha i(1 - p_1) \\&\rightarrow i + 1 && \text{at rate } \alpha i p_2 \\&\rightarrow i + 2 && \text{at rate } \alpha i p_3 \\&\rightarrow \dots\end{aligned}$$

- single birth processes and dual approach.

Y.R. Li, A.G. Pakes, J. Li, A.H. Gu(2008). The limit behavior of dual Markov branching processes. J. Appl. Prob. 45, 176-189.

- stationary distribution. zero-entrance.
- quasi-stationary distribution: $\sup_{n \geq 1} \mathbb{E}_n \tau_0$.
- 2017.6.29-2017.7.1, Nanning, The 6th IMS-China International Conference on Statistics and Probability, Notes on single death processes.

First hitting times, ergodicity and strong ergodicity

Let $Q = (q_{ij})$ be a regular irreducible Q -matrix. Then the limit

$$\lim_{t \rightarrow \infty} p_{ij}(t) =: \pi_j$$

exists and it is independent of i . Moreover, we have either $\sum_j \pi_j = 1$ or $\sum_j \pi_j = 0$.

- $P(t)$ is positive recurrent or ergodic if so is $P(h)$ for every $h > 0$.
Equivalently, $\lim_{t \rightarrow \infty} p_{ii}(t) = \pi_i > 0$.
- $P(t)$ is strongly ergodic or uniformly ergodic if

$$\lim_{t \rightarrow \infty} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

First hitting times, ergodicity and strong ergodicity

Let $Q = (q_{ij})$ be a regular irreducible Q -matrix and H be a non-empty finite subset of \mathbb{Z}_+ . Define $\sigma_H = \inf\{t \geq 0 : X(t) \in H\}$.

Theorem [Issacson & Arnold(1978)]

- (1) The Q -process is ergodic iff $\mathbb{E}_i \sigma_H < \infty$ for all $i \in H$.
- (2) The Q -process is strongly ergodic iff $\sup_i \mathbb{E}_i \sigma_H < \infty$.

Theorem [Tweedie(1981)]

The Q -process is ergodic (resp. strongly ergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

First hitting times, ergodicity and strong ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t \rightarrow \infty} |p_{ij}(t) - \pi_j| = 0$.
- Exponential ergodicity: $\lim_{t \rightarrow \infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim_{t \rightarrow \infty} \sup_i |p_{ij}(t) - \pi_j| = 0$
 $\iff \lim_{t \rightarrow \infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity \Rightarrow Exponential ergodicity \Rightarrow Ordinary ergodicity.

First hitting times, ergodicity and strong ergodicity

Define $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}$, $k > n \geq 0$.

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^i q_n^{(k)} G_k^{(i)}, \quad 1 \leq n < i.$$

Theorem 1.

Assume that the single death Q is irreducible and the process is recurrent. Then

$$\mathbb{E}_n \sigma_0 = \sum_{1 \leq k \leq n} \sum_{\ell \geq k} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}}, \quad n \geq 1;$$

$$\mathbb{E}_0 \sigma_0 = \frac{1}{q_0} + \frac{1}{q_0} \sum_{k \geq 1} q_0^{(k)} \sum_{\ell \geq k} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}}.$$

Theorem 1.(continued)

Furthermore, the process is ergodic if and only if

$$D := \sum_{k \geq 1} q_0^{(k)} \sum_{\ell \geq k} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k \geq 1} \sum_{\ell \geq k} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} < \infty.$$

Actually, for the last conclusion, the recurrence assumption can be replaced by the uniqueness one.

First hitting times, ergodicity and strong ergodicity

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \sum_{k=n}^{i-1} \frac{G_n^{(k)} q_k^{(i)}}{q_{k,k-1}}, \quad 1 \leq n \leq i-1.$$

- Define $\tau_i = \inf\{t \geq 0 : X(t) = i\}$. $\mathbb{E}_n \tau_{n-1} = ?$

$$x_n = \sum_{k \geq n+1} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \quad n \geq 1. \quad (*)$$

Proposition 1. Under the conditions of Theorem 1

$$\mathbb{E}_n \tau_{n-1} = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}}, \quad n \geq 1.$$

Denote the left hand side by m_n , the right hand side by h_n .

$$h_n = \frac{1}{q_{n,n-1}} \left(1 + \sum_{k \geq n+1} q_n^{(k)} h_k \right).$$

Lemma 1.

(h_n) is the minimal nonnegative solution to

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \quad n \geq 1. \quad (*)$$

$$m_n \geq h_n.$$

First hitting times, ergodicity and strong ergodicity

Sketch proof of Lemma 1:

Fix $N \in \mathbb{Z}_+$, define Q -matrix $Q^{(N)} = (\tilde{q}_{ij})$ on $\{0, 1, 2, \dots, N\}$:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + G^{(N)}) & \text{if } i = N, j = N - 1; \\ -(q_N \vee N)(1 + G^{(N)}) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N - 1, \end{cases}$$

where $G^{(N)} = \max_{1 \leq n \leq N} G_n^{(N)}$. Define

$$\tilde{q}_n^{(k)} = \sum_{j=k}^N \tilde{q}_{nj}, \quad 0 \leq n < k \leq N,$$

$$\tilde{G}_i^{(i)} = 1, \quad \tilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^i \tilde{q}_n^{(k)} \tilde{G}_k^{(i)}, \quad 1 \leq n < i \leq N.$$

First hitting times, ergodicity and strong ergodicity

Sketch proof of Lemma 1(continued):

$$h_n^{(N)} := \sum_{k=n}^N \frac{\tilde{G}_n^{(k)}}{\tilde{q}_{k,k-1}} \quad (1 \leq n \leq N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i = \frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i} \cdot x_i + \sum_{\ell=i+1}^N \frac{\tilde{q}_i^{(\ell)}}{\tilde{q}_i} \cdot x_\ell + \frac{1}{\tilde{q}_i}, \quad 1 \leq i \leq N.$$

$$x_N = \frac{1}{(q_N \vee N)(1 + G^{(N)})}, \quad x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{\ell=i+1}^N \frac{q_i^{(\ell)}}{q_i} \cdot x_\ell + \frac{1}{q_i}, \quad 1 \leq i < N.$$

$(h_n^{(N)})$ is increasing to the minimal non-negative solution of (*) as $N \rightarrow \infty$.

$$h_n^{(N)} = \sum_{k=n}^{N-1} \frac{G_n^{(k)}}{q_{k,k-1}} + \frac{G_n^{(N)}}{(q_N \vee N)(1 + G^{(N)})} \rightarrow \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} = h_n \text{ as } N \rightarrow \infty.$$

Lemma 2. Under the conditions of Theorem 1

$(\mathbb{E}_i \tau_{i_0} : i \geq i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \quad x_i = \sum_{j \neq i, j > i_0} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \quad i > i_0.$$

$\sum_{i_0+1 \leq n \leq i} h_n$ satisfies the equation above.

$$\mathbb{E}_i \tau_{i_0} \leq \sum_{i_0+1 \leq n \leq i} h_n \Rightarrow m_n \leq h_n.$$

First hitting times, ergodicity and strong ergodicity

Proposition 1. Under the conditions of Theorem 1

$$m_n = h_n := \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}}, \quad n \geq 1.$$

$$\begin{aligned} \mathbb{E}_0 \sigma_0 &= \frac{1}{q_0} + \sum_{j \geq 1} \frac{q_{0j}}{q_0} \mathbb{E}_j \tau_0 = \frac{1}{q_0} + \sum_{j \geq 1} \frac{q_{0j}}{q_0} \sum_{1 \leq k \leq j} m_k \\ &= \frac{1}{q_0} + \frac{1}{q_0} \sum_{k \geq 1} q_0^{(k)} h_k =: \frac{1}{q_0} + \frac{1}{q_0} D \Leftrightarrow \text{'ergodicity iff } D < \infty \end{aligned}$$

$$\sup_{i \geq 1} \mathbb{E}_i \tau_0 = \sup_{i \geq 1} \sum_{1 \leq k \leq i} m_k = \sum_{k \geq 1} h_k =: S, \quad D \leq q_0 S$$

\Leftrightarrow 'strong ergodicity iff $S < \infty$ '

First hitting times, ergodicity and strong ergodicity

Q -matrix is zero-entrance

For some (equivalently, for all) $\lambda > 0$, the following equation has only trivial solution:

$$\lambda y_i = \sum_{j=0}^{\infty} y_j q_{ji}, \quad y_i \geq 0, \quad i \geq 0 \quad \text{and} \quad \sum_{i=0}^{\infty} y_i < \infty.$$

Proposition

Given single death Q -matrix. Then it is zero-entrance if and only if $\sum_{\ell=0}^{\infty} \bar{m}_{\ell} = \infty$, where

$$\bar{m}_0 = 0, \quad \bar{m}_{\ell} = \frac{1}{q_{\ell, \ell-1}} \left(1 + \sum_{n=0}^{\ell-1} q_n^{(\ell)} \bar{m}_n \right), \quad \ell \geq 1.$$

$$\bar{m}_\ell = \frac{1}{q_{\ell, \ell-1}} \sum_{k=1}^{\ell} G_k^{(\ell)}, \quad \ell \geq 1.$$

Corollary 1.

Assume that the single death Q is irreducible and regular. Then the process is strongly ergodic if and only if the Q -matrix is not zero-entrance.

First hitting times, ergodicity and strong ergodicity

Define

$$\kappa_0 = 1, \quad \kappa_k = \frac{1}{q_{k,k-1}} \sum_{n=0}^{k-1} q_n^{(k)} \kappa_n, \quad k \geq 1.$$

Proposition

Assume that the single death Q is irreducible and regular. Then the process is ergodic if and only if

$$\kappa := \sum_{i \geq 0} \kappa_i < \infty,$$

the distribution (π_i) :

$$\pi_i = \frac{\kappa_i}{\kappa}, \quad i \geq 0.$$

$$\kappa_k = \frac{1}{q_{k,k-1}} \sum_{\ell=1}^k q_0^{(\ell)} G_\ell^{(k)}, \quad k \geq 1.$$

$$\kappa = 1 + D.$$

First hitting times, ergodicity and strong ergodicity

Example 1.

For regular birth-death chain (a_i, b_i) , define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geq 1; \quad \mu[i, +\infty) = \sum_{k \geq i} \mu_k, \quad i \geq 0.$$

Then

$$G_n^{(i)} = \frac{\mu_i a_i}{\mu_n a_n}, \quad 1 \leq n \leq i; \quad m_n = \frac{\mu[n, +\infty)}{\mu_n a_n}, \quad n \geq 1.$$

Therefore, the birth-death process is ergodic if and only if $D = \mu[1, +\infty) < \infty$, equivalently, $\mu := \mu[0, +\infty) < \infty$; the birth-death process is strongly ergodic if and only if

$$S = \sum_{n \geq 1} \frac{\mu[n, +\infty)}{\mu_n a_n} = \sum_{n \geq 0} \frac{\mu[n+1, +\infty)}{\mu_n b_n} < \infty.$$

First hitting times, ergodicity and strong ergodicity

Example 2.

Given a constant $b > 2$ (for regularity only needed that $b > 1$). Let

$$q_{ij} = \frac{b-1}{b^{j-i+2}}, \quad j \geq i+1; \quad q_{i,i-1} = \frac{b-1}{b}, \quad q_i = -q_{ii} = \frac{b^2 - b + 1}{b^2}, \quad i \geq 1;$$

$$q_{0j} = \frac{b-1}{b^{j+1}}, \quad j \geq 1; \quad q_0 = -q_{00} = \frac{1}{b}.$$

Then $q_n^{(k)} = 1/b^{k-n+1}$ ($1 \leq n < k$), $q_0^{(k)} = 1/b^k$ ($k \geq 1$),

$$G_n^{(i)} = \frac{1}{b(b-1)^{i-n}}, \quad 1 \leq n < i; \quad m_n = \frac{b-1}{b-2}, \quad n \geq 1.$$

Therefore, by $D = 1/(b-2)$ and $S = +\infty$, we know that the process is ergodic but not strongly ergodic.

First hitting times, ergodicity and strong ergodicity

The original branching process can be described as follows. Let $\alpha > 0$ and $(p_j : j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate $\alpha ip_0 : i \rightarrow i - 1 (i \geq 1)$ and growth rate $\alpha ip_{k+1} : i \rightarrow i + k (k \geq 1, i \in \mathbb{Z}_+)$. An extended class of branching processes:

Example 3.

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i - 1, i \geq 1; \\ r_i p_{k+1}, & j = i + k, i, k \geq 1; \\ -r_i(1 - p_1), & j = i \geq 1; \\ 0, & \text{else, } i, j \in \mathbb{Z}_+. \end{cases}$$

where $r_i > 0$ for all $i \geq 1$ and $0 < q_0 := \sum_{j \geq 1} q_{0j} < \infty$.

R.R. Chen(1997). An extended class of time-continuous branching processes. J. Appl. Prob., 34(1), 14-23.

Theorem 3.

Assume that the extended branching Q is irreducible and regular. Then the process is ergodic if and only if

$$\sum_{\ell \geq 1} \frac{1}{r_\ell} \left(q_0^{(\ell)} + \sum_{1 \leq k \leq \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} \right) < \infty;$$

it is strongly ergodic if and only if

$$\sum_{\ell \geq 1} \frac{1}{r_\ell} \left(1 + \sum_{1 \leq k \leq \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{1})_{\ell-k+1}}{p_0^k} \right) < \infty.$$

where $\mathbf{f} = (\sum_{k \geq n} p_k, n \geq 2)$, $\mathbf{1} = (1, 1, \dots)$ and $\mathbf{q} = (q_0^{(n-1)}, n \geq 2)$ and

$$(\mathbf{a} * \mathbf{b})_n = \sum_{2 \leq m \leq n} a_{n+2-m} b_m, \quad n \geq 2.$$

First hitting times, ergodicity and strong ergodicity

Come back to Example 2. Fix a positive constant a such that $a < 1 - 1/(b^2 - b + 1)$. Then Example 2 is the special case of Example 3:

$$r_i = (b-1)/(ab), \quad p_0 = a, \quad p_j = a/b^j (j \geq 2), \quad p_1 = 1 - a - a/(b^2 - b).$$

$$f_n = \frac{a}{(b-1)b^{n-1}}, \quad n \geq 2; \quad q_0^{(\ell)} = \frac{1}{b^\ell}, \quad \ell \geq 1.$$

$$(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1} = \frac{a^k}{(b-1)^k b^\ell} C_{\ell-1}^k, \quad 1 \leq k \leq \ell - 1.$$

$$\sum_{1 \leq k \leq \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} = \frac{1}{b(b-1)^{\ell-1}} - \frac{1}{b^\ell}.$$

$$\sum_{\ell \geq 1} \frac{1}{r_\ell} \left(q_0^{(\ell)} + \sum_{1 \leq k \leq \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} \right) = \frac{a}{b-2} < \infty.$$

First hitting times, ergodicity and strong ergodicity

Note that

$$(\mathbf{f} * \mathbf{1})_\ell = \frac{a}{(b-1)^2} \left(1 - \frac{1}{b^{\ell-1}}\right).$$

From the equality above, it follows that

$$\sum_{\ell \geq 1} \frac{1}{r_\ell} \left(1 + \sum_{1 \leq k \leq \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{1})_{\ell-k+1}}{p_0^k}\right) \geq \sum_{\ell \geq 2} \frac{1}{r_\ell} \cdot \frac{(\mathbf{f} * \mathbf{1})_\ell}{p_0} = \infty.$$

So the process in Example 2 is ergodic but not strong ergodic. This is the case of $b > 2$. Note that

$$M_1 := \sum_{k=1}^{\infty} k p_k = \frac{a}{(b-1)^2} + 1 - a.$$

So $M_1 \leq 1$ if and only if $b \geq 2$. Hence, it is easy to be known that the process is unique and null recurrent when $b = 2$; the process is unique and transient when $1 < b < 2$.

High-order moments and exponential ergodicity

Theorem 4.

Assume that the single death Q is irreducible and corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geq 1$ arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leq k \leq i} \sum_{\ell \geq k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell, \ell-1}}, \quad i \geq i_0 + 1.$$

Corollary 2.

Assume that the single death Q -matrix is regular and the process is exponentially ergodic. Then

$$\delta := \sup_{i \geq 1} \sum_{1 \leq k \leq i} \sum_{\ell \geq i} \frac{G_k^{(\ell)}}{q_{\ell, \ell-1}} < \infty.$$

High-order moments and exponential ergodicity

ℓ -ergodicity: $\mathbb{E}_j \sigma_j^\ell < \infty$ for some j . 1-erg. = posit. recur., 0-erg. = null recur.

Corollary 3.

Assume that the single death Q -matrix is regular and the process is recurrent. Then the process is ℓ -ergodic if and only if

$$d_\ell := \sum_{k \geq 1} q_0^{(k)} \sum_{j \geq k} \frac{G_k^{(j)}}{q_{j,j-1}} \mathbb{E}_j \tau_0^{\ell-1} < \infty.$$

High-order moments and exponential ergodicity

- Single death property: $\mathbb{E}_i \tau_{i-1}^n = ?$

Define

$$M_{ik}^{(n-1)} = \sum_{i+1 \leq \ell \leq k} \sum_{1 \leq s \leq n-1} \binom{n}{s} \mathbb{E}_\ell \tau_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \tau_{i-1}^s,$$
$$M_i^{(n-1)} = \mathbb{E}_i \tau_{i-1}^{n-1} + \frac{1}{n} \sum_{k \geq i+1} q_{ik} M_{ik}^{(n-1)}, \quad n \geq 1.$$

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geq i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geq 1.$$

High-order moments and exponential ergodicity

Proposition. Under the conditions of Theorem 4

$$\mathbb{E}_i \tau_{i-1}^n = n \sum_{k \geq i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \quad i \geq 1, n \geq 1.$$

Lemma A. For all $1 \leq i \leq v < u$,

$$G_i^{(u)} = \sum_{i \leq k \leq v} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell=v+1}^u q_k^{(\ell)} G_\ell^{(u)}, \quad 1 \leq i \leq v < u.$$

Lemma B. Under the conditions of Theorem 4

For any nonnegative sequence $\{a_n\}$ and $n \geq 1$,

$$\sum_{k \geq i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1 \geq k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^n \sum_{\ell_2=k}^{\ell_1-1} a_{\ell_2} = n \sum_{u \geq i+1} \frac{G_i^{(u)} M_u^{(n-1)}}{q_{u,u-1}} \sum_{\ell_2=i}^{u-1} a_{\ell_2}.$$

High-order moments and exponential ergodicity

Lemma C. Under the conditions of Theorem 4

For any $1 \leq i \leq k < \ell_1$ and $n \geq 2$,

$$M_{k, \ell_1 - 1}^{(n-1)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_{\ell_1 - 1} \tau_{k-1}^{n-s} \mathbb{E}_{k-1} \tau_{i-1}^s = \sum_{\ell_2=k}^{\ell_1-1} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_{\ell_2} \tau_{\ell_2-1}^{n-s} \mathbb{E}_{\ell_2-1} \tau_{i-1}^s.$$

Lemma D. Under the conditions of Theorem 4

For all $1 \leq s_1 < n$,

$$\begin{aligned} & \sum_{k \geq i} \frac{G_i^{(k)}}{q_{k, k-1}} \sum_{\ell_1 \geq k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1-1}^{n-s_1} \left(M_{k, \ell_1-1}^{(s_1-1)} + \sum_{s_2=1}^{s_1-1} \binom{s_1}{s_2} \mathbb{E}_{\ell_1-1} \tau_{k-1}^{s_1-s_2} \mathbb{E}_{k-1} \tau_{i-1}^{s_2} \right) \\ &= (n - s_1) \sum_{u \geq i+1} \frac{G_i^{(u)}}{q_{u, u-1}} M_u^{(n-s_1-1)} M_{i, u-1}^{(s_1-1)}. \end{aligned}$$

High-order moments and exponential ergodicity

$$\mathbb{E}_i \tau_{i-1}^n = n \sum_{k \geq i} \frac{G_i^{(k)} \mathbb{E}_k \tau_{i-1}^{n-1}}{q_{k,k-1}}, \quad i \geq 1, n \geq 1.$$

$$\mathbb{E}_i \tau_k^n = n \sum_{k+1 \leq j \leq i} \sum_{\ell \geq j} \frac{G_j^{(\ell)} \mathbb{E}_\ell \tau_k^{n-1}}{q_{\ell,\ell-1}}, \quad 0 \leq k < i, n \geq 1.$$

High-order moments and exponential ergodicity

Define $G_n = \underline{\lim}_{m \rightarrow \infty} G_n^{(m)} / G_1^{(m)}$, $n \geq 1$. Then

$$G_n \geq \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{\infty} q_n^{(k)} G_k, \quad n \geq 1.$$

Theorem 5.

Let the single death Q -matrix be regular and irreducible. Assume that

$$\sum_{k \geq 1} q_0^{(k)} G_k < \infty.$$

If

$$q := \inf_{n \geq 0} q_n > 0 \quad \text{and} \quad \sup_{n \geq 1} \left[\sum_{k=1}^n G_k \right] \left[\sum_{j=n}^{\infty} \frac{1}{q_{j,j-1} G_j} \right] < \infty,$$

then the process is exponentially ergodic.

Theorem [Tweedie(1981)]

Given a regular irreducible Q -matrix $Q = (q_{ij})$. Then the process $P(t)$ is exponentially ergodic if and only if for some $\lambda > 0$ with $\lambda < q_i$ for all i ,

$$\begin{cases} y_i \geq 1, & i \in E \\ \sum_j q_{ij} y_j \leq -\lambda y_i, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite solution (y_i) .

High-order moments and exponential ergodicity

For the birth-death process (a_i, b_i) , $G_n^{(m)}/G_1^{(m)} = \mu_1 a_1 / (\mu_n a_n)$. Thus

$$G_n = \lim_{m \rightarrow \infty} \frac{G_n^{(m)}}{G_1^{(m)}} = \frac{\mu_1 a_1}{\mu_n a_n}, \quad n \geq 1$$

and $\sum_{k \geq 1} q_0^{(k)} G_k = b_0 < \infty$. The equality holds. Now

$$M = \sup_{n \geq 1} \left[\sum_{k=1}^n \frac{1}{\mu_k a_k} \right] \left[\sum_{j=n}^{\infty} \mu_j \right] = \sup_{n \geq 1} \left[\sum_{k=0}^{n-1} \frac{1}{\mu_k b_k} \right] \left[\sum_{j=n}^{\infty} \mu_j \right].$$

The birth-death process is exponentially ergodic if and only if $M < \infty$.

High-order moments and exponential ergodicity

Come back to Example 2 for $b > 2$. We see that $G_n^{(m)}/G_1^{(m)} = (b-1)^{n-1}$. Then

$$G_n = \lim_{m \rightarrow \infty} \frac{G_n^{(m)}}{G_1^{(m)}} = (b-1)^{n-1}, \quad n \geq 1,$$

and $\sum_{k \geq 1} q_0^{(k)} G_k = 1 < \infty$. The equality holds. Note that $q = 1/b > 0$ and

$$M = \frac{b(b-1)}{(b-2)^2} < \infty.$$

Hence, the process is exponentially ergodic.

Theorem 6.

Let the single death Q -matrix be regular and irreducible. Assume that

$$\sum_{k \geq 1} \frac{q_0^{(k)}}{G_1^{(k)}} < \infty.$$

If

$$q := \inf_{n \geq 0} q_n > 0 \quad \text{and} \quad M := \sup_{n \geq 1} \left(\sum_{k=1}^n \frac{1}{G_1^{(k)}} \right) \left(\sum_{j=n}^{\infty} \frac{G_1^{(j)}}{q_{j,j-1}} \right) < \infty,$$

then the process is exponentially ergodic.

$$M \geq \delta.$$

High-order moments and exponential ergodicity

Define a positive matrix $H = (h_{ij})_{i,j \geq 1}$:

$$h_{ij} = \frac{1}{q_{j,j-1}} \sum_{k=1}^{i \wedge j} G_k^{(j)}, \quad H = ((-Q)|_{i \geq 1})^{-1} = \left(\int_0^\infty p_{ij}^{\min}(t) dt \right)_{i,j \geq 1}.$$

$$\mathbb{E}_k e^{\alpha \tau_0} = \sum_{n=0}^{\infty} ((\alpha H)^n \mathbf{1})(k).$$

$$\frac{1}{\alpha_{\max}} = \inf_{y \in \mathcal{F}} \sup_{k \geq 1} \frac{(Hy + H\mathbf{1})(k)}{y_k} < \infty \Leftrightarrow \text{exp. erg.} \Leftrightarrow \lambda_{\max} < \infty.$$

where

$$\mathcal{F} = \left\{ y : 0 < y \uparrow\uparrow, \sum_{k \geq 1} q_{0k} y_k < \infty \right\}.$$

High-order moments and exponential ergodicity

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定义

$$\xi_i = \int_0^{\tau_i} V(X(t)) dt, V \geq 0, V \text{不恒为} 0.$$

$$\mathbb{E}_k \xi_i^n = n \sum_{i+1 \leq j \leq k} \sum_{\ell \geq j} \frac{G_j^{(\ell)} V(\ell) \mathbb{E}_\ell \xi_i^{n-1}}{q_{\ell, \ell-1}}, \quad k \geq i \geq 0, n \geq 1.$$

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Thank you for your attention!

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