Single death processes

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CONTENTS

Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

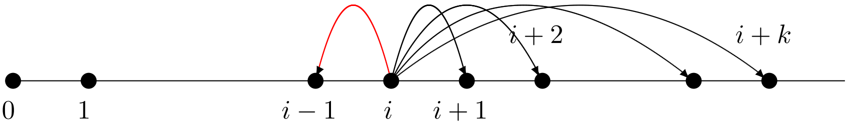
$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

Background

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0,\ \ q_{i,i-j} = 0,\ \ i \geqslant 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Background

- branching processes:

$$
\begin{aligned}
i \to\ & i-1 && \text{at rate} && \alpha i p_0 \\
\to\ & i && \text{at rate} && -\alpha i(1-p_1) \\
\to\ & i+1 && \text{at rate} && \alpha i p_2 \\
\to\ & i+2 && \text{at rate} && \alpha i p_3 \\
\to\ & \cdots
\end{aligned}
$$

Background

- single birth processes and dual approach.

Y.R. Li, A.G. Pakes, J. Li, A.H. Gu(2008). The limit behavior of dual Markov branching processes. J. Appl. Prob. 45, 176-189.

- stationary distribution. zero-entrance.
- quasi-stationary distribution: $\sup_{n\geqslant 1}\mathbb{E}_n\tau_0$.
- 2017.6.29-2017.7.1, Nanning, The 6th IMS-China International Conference on Statistics and Probability, Notes on single death processes.

Hitting times, criteria on ergodicity and strong ergodicity

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix. Then the limit

$$\lim_{t\to\infty} p_{ij}(t) =: \pi_j$$

exists and it is independent of i. Moreover, we have either $\sum_j \pi_j = 1$ or $\sum_j \pi_j = 0$.

- $P(t)$ is positive recurrent or ergodic if so is $P(h)$ for every $h > 0$. Equivalently, $\lim_{t\to\infty} p_{ii}(t) = \pi_i > 0$.
- $P(t)$ is strongly ergodic or uniformly ergodic if

$$\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Hitting times, ergodicity and strong ergodicity

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix and H be a non-empty finite subset of $\mathbb{Z}_+$. Define $\sigma_H = \inf\{t \geqslant \text{the first jump} : X(t) \in H\}$.

Theorem[Issacson & Arnold(1978)]

(1) The Q-process is ergodic iff $\mathbb{E}_i \sigma_H < \infty$ for all $i \in H$.
(2) The Q-process is strongly ergodic iff $\sup_i \mathbb{E}_i \sigma_H < \infty$.

Theorem[Tweedie(1981)]

The Q-process is ergodic (resp. strongly erergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leqslant -1, \ i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

Hitting times, ergodicity and strong ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0.$
- Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0.$
- Strong ergodicity: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$

$$\iff \lim_{t\to\infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

- Strong ergodicity $\Rightarrow$ Exponential ergodicity $\Rightarrow$ Ordinary ergodicity.

Hitting times, ergodicity and strong ergodicity

Define $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj},\ k > n \geqslant 0$.

$$G_i^{(i)} = 1, \qquad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \qquad 1 \leqslant n < i.$$

Theorem 1.

Assume that the single death Q is irreducible and the process is recurrent. Then

$$\mathbb{E}_n \sigma_0 = \sum_{1 \leqslant k \leqslant n} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}, \qquad n \geqslant 1;$$

$$\mathbb{E}_0 \sigma_0 = \frac{1}{q_0} + \frac{1}{q_0} \sum_{k \geqslant 1} q_0^{(k)} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}}.$$

Theorem 1.(continued)

Furthermore, the process is ergodic if and only if

$$D := \sum_{k \geqslant 1} q_0^{(k)} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k \geqslant 1} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

Actually, for the last conclusion, the recurrence assumption can be replaced by the uniqueness one.

Hitting times, ergodicity and strong ergodicity

$$G_i^{(i)} = 1, \qquad G_n^{(i)} = \sum_{k=n}^{i-1} \frac{G_n^{(k)} q_k^{(i)}}{q_{k,k-1}}, \qquad 1 \leqslant n \leqslant i-1.$$

- Define $\tau_i = \inf\{t \geqslant 0 : X(t) = i\}$. $\mathbb{E}_n \tau_{n-1} =?$

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \ n \geqslant 1. \qquad (*)$$

Proposition 1. Under the conditions of Theorem 1

$$\mathbb{E}_n \tau_{n-1} = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}}, \ n \geqslant 1.$$

Denote the left hand side by m_n, the right hand side by h_n.

$$h_n = \frac{1}{q_{n,n-1}} \left(1 + \sum_{k \geqslant n+1} q_n^{(k)} h_k \right).$$

Lemma 1.

(h_n) is the minimal nonnegative solution to

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \ n \geqslant 1. \qquad (*)$$

$$m_n \geqslant h_n.$$

Sketch proof of Lemma 1:
Fix $N \in \mathbb{Z}_+$, define Q-matrix $Q^{(N)} = (\tilde{q}_{ij})$ on $\{0, 1, 2, \cdots, N\}$:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + G^{(N)}) & \text{if } i = N, j = N-1; \\ -(q_N \vee N)(1 + G^{(N)}) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N-1, \end{cases}$$

where $G^{(N)} = \max_{1 \leqslant n \leqslant N} G_n^{(N)}$. Define

$$\tilde{q}_n^{(k)} = \sum_{j=k}^{N} \tilde{q}_{nj}, \ 0 \leqslant n < k \leqslant N,$$

$$\widetilde{G}_i^{(i)} = 1, \qquad \widetilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \tilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \qquad 1 \leqslant n < i \leqslant N.$$

Sketch proof of Lemma 1(continued):

$$h_n^{(N)} := \sum_{k=n}^{N} \frac{\widetilde{G}_n^{(k)}}{\tilde{q}_{k,k-1}} \quad (1 \leqslant n \leqslant N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i = \frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{\tilde{q}_i^{(\ell)}}{\tilde{q}_i} \cdot x_\ell + \frac{1}{\tilde{q}_i}, \qquad 1 \leqslant i \leqslant N.$$

$$x_N = \frac{1}{(q_N \vee N)(1 + G^{(N)})}, \quad x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{q_i^{(\ell)}}{q_i} \cdot x_\ell + \frac{1}{q_i}, \quad 1 \leqslant i < N.$$

$(h_n^{(N)})$ is increasing to the minimal non-negative solution of $(*)$ as $N \to \infty$.

$$h_n^{(N)} = \sum_{k=n}^{N-1} \frac{G_n^{(k)}}{q_{k,k-1}} + \frac{G_n^{(N)}}{(q_N \vee N)(1 + G^{(N)})} \to \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} = h_n \text{ as } N \to \infty.$$

Lemma 2. Under the conditions of Theorem 1

$(\mathbb{E}_i\tau_{i_0} : i \geqslant i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{j\neq i, j>i_0} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i}, \qquad i > i_0.$$

$\sum_{i_0+1\leqslant n\leqslant i} h_n$ satisfies the equation above.

$$\mathbb{E}_i\tau_{i_0} \leqslant \sum_{i_0+1\leqslant n\leqslant i} h_n \Rightarrow m_n \leqslant h_n.$$

Proposition 1. Under the conditions of Theorem 1

$$m_n = h_n := \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}}, \ n \geqslant 1.$$

$$\begin{aligned}\mathbb{E}_0\sigma_0 &= \frac{1}{q_0} + \sum_{j\geqslant 1} \frac{q_{0j}}{q_0}\mathbb{E}_j\tau_0 = \frac{1}{q_0} + \sum_{j\geqslant 1} \frac{q_{0j}}{q_0} \sum_{1\leqslant k\leqslant j} m_k \\ &= \frac{1}{q_0} + \frac{1}{q_0}\sum_{k\geqslant 1} q_0^{(k)} h_k =: \frac{1}{q_0} + \frac{1}{q_0} D \Leftrightarrow \text{'ergodicity iff } D < \infty\text{'}\end{aligned}$$

$$\sup_{i\geqslant 1} \mathbb{E}_i\tau_0 = \sup_{i\geqslant 1} \sum_{1\leqslant k\leqslant i} m_k = \sum_{k\geqslant 1} h_k =: S, \qquad D \leqslant q_0 S$$

$\Leftrightarrow$ 'strong ergodicity iff $S < \infty$'

Hitting times, ergodicity and strong ergodicity

Q-matrix is zero-entrance

For some (equivalently, for all) $\lambda > 0$, the following equation has only trivial solution:

$$\lambda y_i = \sum_{j=0}^{\infty} y_j q_{ji}, \qquad y_i \geqslant 0,\ i \geqslant 0 \quad \text{and} \quad \sum_{i=0}^{\infty} y_i < \infty.$$

Proposition

Given single death Q-matrix. Then it is zero-entrance if and only if $\sum_{\ell=0}^{\infty} \overline{m}_\ell = \infty$, where

$$\overline{m}_0 = 0, \qquad \overline{m}_\ell = \frac{1}{q_{\ell,\ell-1}}\left(1 + \sum_{n=0}^{\ell-1} q_n^{(\ell)} \overline{m}_n\right), \qquad \ell \geqslant 1.$$

$$\overline{m}_\ell = \frac{1}{q_{\ell,\ell-1}} \sum_{k=1}^{\ell} G_k^{(\ell)}, \qquad \ell \geqslant 1.$$

Corollary 1.

Assume that the single death Q is irreducible and regular. Then the process is strongly ergodic if and only if the Q-matrix is not zero-entrance.

Define

$$\kappa_0 = 1, \qquad \kappa_k = \frac{1}{q_{k,k-1}} \sum_{n=0}^{k-1} q_n^{(k)} \kappa_n, \qquad k \geqslant 1.$$

Proposition

Assume that the single death Q is irreducible and regular. Then the process is ergodic if and only if

$$\kappa := \sum_{i \geqslant 0} \kappa_i < \infty,$$

the distribution (π_i):

$$\pi_i = \frac{\kappa_i}{\kappa}, \qquad i \geqslant 0.$$

$$\kappa_k = \frac{1}{q_{k,k-1}} \sum_{\ell=1}^{k} q_0^{(\ell)} G_\ell^{(k)}, \qquad k \geqslant 1.$$

$$\kappa = 1 + D.$$

Example 1.

For regular birth-death chain (a_i, b_i), define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \qquad i \geqslant 1; \qquad \mu[i, +\infty) = \sum_{k \geqslant i} \mu_k, \qquad i \geqslant 0.$$

Then

$$G_n^{(i)} = \frac{\mu_i a_i}{\mu_n a_n}, \qquad 1 \leqslant n \leqslant i; \qquad m_n = \frac{\mu[n, +\infty)}{\mu_n a_n}, \qquad n \geqslant 1.$$

Therefore, the birth-death process is ergodic if and only if $D = \mu[1, +\infty) < \infty$, equivalently, $\mu := \mu[0, +\infty) < \infty$; the birth-death process is strongly ergodic if and only if

$$S = \sum_{n \geqslant 1} \frac{\mu[n, +\infty)}{\mu_n a_n} = \sum_{n \geqslant 0} \frac{\mu[n+1, +\infty)}{\mu_n b_n} < \infty.$$

Example 2.

Given a constant $b > 2$ (for regularity only needed that $b > 1$). Let

$$q_{ij} = \frac{b-1}{b^{j-i+2}}, \quad j \geqslant i+1; \quad q_{i,i-1} = \frac{b-1}{b}, \quad q_i = -q_{ii} = \frac{b^2-b+1}{b^2}, \quad i \geqslant 1;$$

$$q_{0j} = \frac{b-1}{b^{j+1}}, \qquad j \geqslant 1; \qquad q_0 = -q_{00} = \frac{1}{b}.$$

Then $q_n^{(k)} = 1/b^{k-n+1}(1 \leqslant n < k)$, $q_0^{(k)} = 1/b^k (k \geqslant 1)$,

$$G_n^{(i)} = \frac{1}{b(b-1)^{i-n}}, \qquad 1 \leqslant n < i; \qquad m_n = \frac{b-1}{b-2}, \qquad n \geqslant 1.$$

Therefore, by $D = 1/(b-2)$ and $S = +\infty$, we know that the process is ergodic but not strongly ergodic.

Hitting times, ergodicity and strong ergodicity

The original branching process can be described as follows. Let $\alpha > 0$ and $(p_j : j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate $\alpha i p_0 : i \to i-1\,(i \geq 1)$ and growth rate $\alpha i p_{k+1} : i \to i+k\,(k \geqslant 1, i \in \mathbb{Z}_+)$. An extended class of branching processes:

Example 3.

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i-1,\ i \geqslant 1; \\ r_i p_{k+1}, & j = i+k,\ i, k \geqslant 1; \\ -r_i(1-p_1), & j = i \geqslant 1; \\ 0, & \text{else}, \quad i, j \in \mathbb{Z}_+. \end{cases}$$

where $r_i > 0$ for all $i \geq 1$ and $0 < q_0 := \sum_{j\geq 1} q_{0j} < \infty$.

R.R. Chen(1997). An extended class of time-continuous branching processes. J. Appl. Prob., 34(1), 14-23.

Hitting times, ergodicity and strong ergodicity

Theorem 3.

Assume that the extended branching Q is irreducible and regular. Then the process is ergodic if and only if

$$\sum_{\ell\geqslant 1}\frac{1}{r_\ell}\left(q_0^{(\ell)}+\sum_{1\leqslant k\leqslant \ell-1}\frac{(\mathbf{f}^{*k}*\mathbf{q})_{\ell-k+1}}{p_0^k}\right)<\infty;$$

it is strongly ergodic if and only if

$$\sum_{\ell\geqslant 1}\frac{1}{r_\ell}\left(1+\sum_{1\leqslant k\leqslant \ell-1}\frac{(\mathbf{f}^{*k}*\mathbf{1})_{\ell-k+1}}{p_0^k}\right)<\infty.$$

where $\mathbf{f}=(\sum_{k\geqslant n}p_k, n\geqslant 2)$, $\mathbf{1}=(1,1,\cdots)$ and $\mathbf{q}=(q_0^{(n-1)}, n\geqslant 2)$ and

$$(\mathbf{a}*\mathbf{b})_n=\sum_{2\leqslant m\leqslant n}a_{n+2-m}b_m,\ \ n\geqslant 2.$$

Come back to Example 2. Fix a positive constant a such that $a < 1 - 1/(b^2 - b + 1)$. Then Example 2 is the special case of Example 3:

$$r_i = (b-1)/(ab),\ p_0 = a,\ p_j = a/b^j\,(j \geqslant 2),\ p_1 = 1 - a - a/(b^2 - b).$$

$$f_n = \frac{a}{(b-1)b^{n-1}}, \qquad n \geqslant 2; \qquad q_0^{(\ell)} = \frac{1}{b^\ell}, \qquad \ell \geqslant 1.$$

$$(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1} = \frac{a^k}{(b-1)^k b^\ell} C_{\ell-1}^k, \qquad 1 \leqslant k \leqslant \ell - 1.$$

$$\sum_{1 \leqslant k \leqslant \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} = \frac{1}{b(b-1)^{\ell-1}} - \frac{1}{b^\ell}.$$

$$\sum_{\ell \geqslant 1} \frac{1}{r_\ell} \left(q_0^{(\ell)} + \sum_{1 \leqslant k \leqslant \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} \right) = \frac{a}{b-2} < \infty.$$

Note that

$$(\mathbf{f} * \mathbf{1})_\ell = \frac{a}{(b-1)^2}\Big(1 - \frac{1}{b^{\ell-1}}\Big).$$

From the equality above, it follows that

$$\sum_{\ell \geqslant 1} \frac{1}{r_\ell}\Big(1 + \sum_{1 \leqslant k \leqslant \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{1})_{\ell-k+1}}{p_0^k}\Big) \geqslant \sum_{\ell \geqslant 2} \frac{1}{r_\ell} \cdot \frac{(\mathbf{f} * \mathbf{1})_\ell}{p_0} = \infty.$$

So the process in Example 2 is ergodic but not strong ergodic. This is the case of $b > 2$. Note that

$$M_1 := \sum_{k=1}^{\infty} k p_k = \frac{a}{(b-1)^2} + 1 - a.$$

So $M_1 \leqslant 1$ if and only if $b \geqslant 2$. Hence, it is easy to be known that the process is unique and null recurrent when $b = 2$; the process is unique and transient when $1 < b < 2$.

High-order moments and exponential ergodicity

Theorem 4.

Assume that the single death Q is irreducible and corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Corollary 2.

Assume that the single death Q-matrix is regular and the process is exponentially ergodic. Then

$$\delta := \sup_{i \geqslant 1} \sum_{1 \leqslant k \leqslant i} \sum_{\ell \geqslant i} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

High-order moments and exponential ergodicity

ℓ-ergodicity: $\mathbb{E}_j \sigma_j^\ell < \infty$ for some j. 1-erg. $=$ posit. recur., 0-erg. $=$ null recur.

Corollary 3.

Assume that the single death Q-matrix is regular and the process is recurrent. Then the process is ℓ-ergodic if and only if

$$d_\ell := \sum_{k \geqslant 1} q_0^{(k)} \sum_{j \geqslant k} \frac{G_k^{(j)}}{q_{j,j-1}} \mathbb{E}_j \tau_0^{\ell-1} < \infty.$$

High-order moments and exponential ergodicity

- Single death property: $\mathbb{E}_i \tau_{i-1}^n = ?$

Define

$$M_{ik}^{(n-1)} = \sum_{i+1 \leqslant \ell \leqslant k} \sum_{1 \leqslant s \leqslant n-1} \binom{n}{s} \mathbb{E}_\ell \tau_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \tau_{i-1}^{s},$$

$$M_i^{(n-1)} = \mathbb{E}_i \tau_{i-1}^{n-1} + \frac{1}{n} \sum_{k \geqslant i+1} q_{ik} M_{ik}^{(n-1)}, \qquad n \geqslant 1.$$

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \ i \geqslant 1.$$

High-order moments and exponential ergodicity

Proposition. Under the conditions of Theorem 4

$$\mathbb{E}_i\tau_{i-1}^n = n\sum_{k\geqslant i}\frac{G_i^{(k)}M_k^{(n-1)}}{q_{k,k-1}},\ i\geqslant 1, n\geqslant 1.$$

Lemma A. For all $1\leqslant i\leqslant v<u$,

$$G_i^{(u)} = \sum_{i\leqslant k\leqslant v}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell=v+1}^{u} q_k^{(\ell)}G_\ell^{(u)}, \qquad 1\leqslant i\leqslant v<u.$$

Lemma B. Under the conditions of Theorem 4

For any nonnegative sequence $\{a_n\}$ and $n\geqslant 1$,

$$\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^n\sum_{\ell_2=k}^{\ell_1-1}a_{\ell_2} = n\sum_{u\geqslant i+1}\frac{G_i^{(u)}M_u^{(n-1)}}{q_{u,u-1}}\sum_{\ell_2=i}^{u-1}a_{\ell_2}.$$

High-order moments and exponential ergodicity

Lemma C. Under the conditions of Theorem 4

For any $1 \leqslant i \leqslant k < \ell_1$ and $n \geqslant 2$,

$$M_{k,\ell_1-1}^{(n-1)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-s} \mathbb{E}_{k-1}\tau_{i-1}^{s} = \sum_{\ell_2=k}^{\ell_1-1} \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s} \mathbb{E}_{\ell_2-1}\tau_{i-1}^{s}.$$

Lemma D. Under the conditions of Theorem 4

For all $1 \leqslant s_1 < n$,

$$\sum_{k\geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1} \left(M_{k,\ell_1-1}^{(s_1-1)} + \sum_{s_2=1}^{s_1-1} \binom{s_1}{s_2} \mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2} \mathbb{E}_{k-1}\tau_{i-1}^{s_2} \right)$$

$$= (n-s_1) \sum_{u\geqslant i+1} \frac{G_i^{(u)}}{q_{u,u-1}} M_u^{(n-s_1-1)} M_{i,u-1}^{(s_1-1)}.$$

High-order moments and exponential ergodicity

$$\mathbb{E}_i \tau_{i-1}^n = n \sum_{k \geqslant i} \frac{G_i^{(k)} \mathbb{E}_k \tau_{i-1}^{n-1}}{q_{k,k-1}}, \qquad i \geqslant 1, n \geqslant 1.$$

$$\mathbb{E}_i \tau_k^n = n \sum_{k+1 \leqslant j \leqslant i} \sum_{\ell \geqslant j} \frac{G_j^{(\ell)} \mathbb{E}_\ell \tau_k^{n-1}}{q_{\ell,\ell-1}}, \qquad 0 \leqslant k < i, n \geqslant 1.$$

Define $G_n = \underline{\lim}_{m\to\infty} G_n^{(m)}/G_1^{(m)},\ n \geqslant 1$. Then

$$G_n \geqslant \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{\infty} q_n^{(k)} G_k, \quad n \geqslant 1.$$

Theorem 5.

Let the single death Q-matrix be regular and irreducible. Assume that

$$\sum_{k\geqslant 1} q_0^{(k)} G_k < \infty.$$

If

$$q := \inf_{n\geqslant 0} q_n > 0 \quad \text{and} \quad \sup_{n\geqslant 1} \left[\sum_{k=1}^{n} G_k\right]\left[\sum_{j=n}^{\infty} \frac{1}{q_{j,j-1}G_j}\right] < \infty,$$

then the process is exponentially ergodic.

High-order moments and exponential ergodicity

Theorem [Tweedie(1981)]

Given a regular irreducible Q-matrix $Q = (q_{ij})$. Then the process $P(t)$ is exponentially ergodicity if and only if for some $\lambda > 0$ with $\lambda < q_i$ for all i,

$$\begin{cases} y_i \geqslant 1, \quad i \in E \\ \sum_j q_{ij} y_j \leqslant -\lambda y_i, \quad i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite solution (y_i).

For the birth-death process (a_i, b_i), $G_n^{(m)}/G_1^{(m)} = \mu_1 a_1/(\mu_n a_n)$. Thus

$$G_n = \lim_{m\to\infty} \frac{G_n^{(m)}}{G_1^{(m)}} = \frac{\mu_1 a_1}{\mu_n a_n}, \qquad n \geqslant 1$$

and $\sum_{k\geqslant 1} q_0^{(k)} G_k = b_0 < \infty$. The equality holds. Now

$$M = \sup_{n\geqslant 1} \left[\sum_{k=1}^{n} \frac{1}{\mu_k a_k}\right]\left[\sum_{j=n}^{\infty} \mu_j\right] = \sup_{n\geqslant 1} \left[\sum_{k=0}^{n-1} \frac{1}{\mu_k b_k}\right]\left[\sum_{j=n+1}^{\infty} \mu_j\right].$$

The birth-death process is exponentially ergodic if and only if $M < \infty$.

High-order moments and exponential ergodicity

Come back to Example 2 for $b > 2$. We see that $G_n^{(m)}/G_1^{(m)} = (b-1)^{n-1}$. Then

$$G_n = \lim_{m\to\infty} \frac{G_n^{(m)}}{G_1^{(m)}} = (b-1)^{n-1}, \qquad n \geqslant 1,$$

and $\sum_{k\geqslant 1} q_0^{(k)} G_k = 1 < \infty$. The equality holds. Note that $q = 1/b > 0$ and

$$M = \frac{b(b-1)}{(b-2)^2} < \infty.$$

Hence, the process is exponentially ergodic.

High-order moments and exponential ergodicity

Theorem 6.

Let the single death Q-matrix be regular and irreducible. Assume that

$$\sum_{k\geqslant 1}\frac{q_0^{(k)}}{G_1^{(k)}}<\infty.$$

If

$$q:=\inf_{n\geqslant 0}q_n>0\quad\text{and}\quad M:=\sup_{n\geqslant 1}\left(\sum_{k=1}^{n}\frac{1}{G_1^{(k)}}\right)\left(\sum_{j=n}^{\infty}\frac{G_1^{(j)}}{q_{j,j-1}}\right)<\infty,$$

then the process is exponentially ergodic.

$$M\geqslant\delta.$$

High-order moments and exponential ergodicity

Define a positive matrix $H = (h_{ij})_{i,j\geqslant 1}$:

$$h_{ij} = \frac{1}{q_{j,j-1}} \sum_{k=1}^{i\wedge j} G_k^{(j)}, \quad H = ((-Q)|_{i\geqslant 1})^{-1} = \left(\int_0^\infty p_{ij}^{\min}(t)\mathrm{d}t \right).$$

$$\mathbb{E}_k \mathrm{e}^{\alpha \tau_0} = \sum_{n=0}^{\infty} ((\alpha H)^n \mathbb{1})(k).$$

$$\frac{1}{\alpha_{\max}} = \inf_{y\in\mathscr{F}} \sup_{k\geqslant 1} \frac{(Hy + H\mathbb{1})(k)}{y_k} < \infty \Leftrightarrow \text{exp. erg.} \Leftrightarrow \lambda_{\max} < \infty.$$

where

$$\mathscr{F} = \{y : 0 < y \upuparrows, \sum_{k\geqslant 1} q_{0k} y_k < \infty\}.$$

Recurrence and return probability

Define

$$\widetilde{G}_n = \varlimsup_{m\to\infty} \frac{G_n^{(m)}}{G_1^{(m)}}, \qquad n \geqslant 1.$$

Assume that

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} < \infty, \qquad n \geqslant 1. \qquad (**)$$

Then

$$\widetilde{G}_n \leqslant \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{\infty} q_n^{(k)} \widetilde{G}_k, \qquad n \geqslant 1.$$

Theorem 7.

Let the single death Q-matrix be regular and irreducible. If $\sum_{n\geqslant 1} G_n = \infty$, then the process is recurrent. Conversely, under the assumption $(**)$, if the process is recurrent, then $\sum_{n\geqslant 1} \widetilde{G}_n = \infty$.

Recurrence and return probability

- For the birth-death process (a_i, b_i), we have

$$\sum_{n\geqslant 1} G_n = \sum_{n\geqslant 1} \widetilde{G}_n = \sum_{n\geqslant 1} \frac{\mu_1 a_1}{\mu_n a_n} = \sum_{n\geqslant 0} \frac{b_0}{\mu_n b_n}$$

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} = \frac{b_0}{\mu_n} < \infty, \qquad n \geqslant 1.$$

  Hence the birth-death process is recurrent if and only if $\sum_{n\geqslant 0} 1/(\mu_n b_n) = \infty$.

- For Example 2 when $b > 1$. Then

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} = (b-1)^n < \infty, \qquad n \geqslant 1$$

$$\sum_{n\geqslant 1} G_n = \sum_{n\geqslant 1} \widetilde{G}_n = \sum_{n\geqslant 1} (b-1)^{n-1},$$

  which is infinite when $b \geqslant 2$ and finite when $1 < b < 2$. So the process is recurrent if and only if $b \geqslant 2$.

Recurrence and return probability

Define $G_0 = \frac{1}{q_0} \sum_{n \geqslant 1} q_0^{(n)} G_n$, $G = \sum_{n \geqslant 1} G_n$, $\widetilde{G}_0 = \frac{1}{q_0} \sum_{n \geqslant 1} q_0^{(n)} \widetilde{G}_n$, $\widetilde{G} = \sum_{n \geqslant 1} \widetilde{G}_n$.

Theorem 8.

Let the single death Q-matrix be regular and irreducible. Assume that $(**)$ holds. Then

$$\mathbb{P}_0(\sigma_0 < \infty) \leqslant 1 - \frac{\widetilde{G}_0}{\widetilde{G}}, \ \mathbb{P}_n(\sigma_0 < \infty) \leqslant 1 - \frac{1}{\widetilde{G}} \sum_{1 \leqslant k \leqslant n} \widetilde{G}_k, \ n \geqslant 1.$$

In addition, assume that $\displaystyle \varliminf_{m \to \infty} \frac{1}{G_1^{(m)}} \sum_{1 \leqslant j \leqslant k} G_j^{(m)} = \sum_{1 \leqslant j \leqslant k} G_j, \ k \geqslant 1. \qquad (***)$

Then

$$\mathbb{P}_0(\sigma_0 < \infty) \geqslant 1 - \frac{G_0}{G}, \ \mathbb{P}_n(\sigma_0 < \infty) \geqslant 1 - \frac{1}{G} \sum_{1 \leqslant k \leqslant n} G_k, \ n \geqslant 1.$$

Recurrence and return probability

Lemma 3.

$$x_0 = 1 - \frac{\widetilde{G}_0}{\widetilde{G}}, \qquad x_k = 1 - \frac{1}{\widetilde{G}} \sum_{1 \leqslant n \leqslant k} \widetilde{G}_n$$

satisfies the inequality $x_i \geqslant \sum_{k \neq 0, i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i}(1 - \delta_{i0}), \quad i \geqslant 0.$

$(\mathbb{P}_i(\sigma_0 < \infty))$ is the minimal nonnegative solution to the equation above. Thus

$$\mathbb{P}_i(\sigma_0 < \infty) \leqslant x_i, \qquad i \geqslant 0.$$

Lemma 4.

$$\mathbb{P}_k(\tau_0 < \tau_m) = \frac{\sum_{k+1 \leqslant n \leqslant m} G_n^{(m)}}{\sum_{1 \leqslant n \leqslant m} G_n^{(m)}}, \qquad k \geqslant 0.$$

$$\mathbb{P}_k(\sigma_0 < \infty) \geqslant 1 - \frac{1}{G} \sum_{1 \leqslant n \leqslant k} G_n, \qquad k \geqslant 1.$$

Remark 1.

If there exist the limits

$$\lim_{m\to\infty}\frac{G_n^{(m)}}{G_1^{(m)}}$$

and the assumption $(**)$ holds, then $G_n=\widetilde{G}_n\,(n\geqslant 0)$ and $(***)$ holds. Hence

$$\mathbb{P}_0(\sigma_0<\infty)=1-\frac{G_0}{G},\qquad \mathbb{P}_n(\sigma_0<\infty)=1-\frac{1}{G}\sum_{1\leqslant n\leqslant k}G_n,\qquad n\geqslant 1.$$

Thus, $\mathbb{P}_i(\sigma_0<\infty)=1$ for all $i\geqslant 1$ if and only if $\mathbb{P}_0(\sigma_0<\infty)=1$, equivalently, if and only if $G=\infty\,(G_0<\infty)$. See Example 1 and 2.

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Thank you for your attention!

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