单死过程首中时的高阶矩

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Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

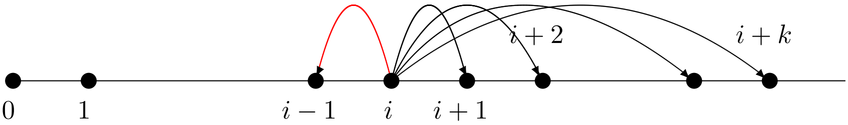
$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

Background

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0,\ q_{i,i-j} = 0,\ i \geqslant 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Background

- branching processes:

$$
\begin{aligned}
i \to\ & i-1 \quad \text{at rate} \quad \alpha i p_0 \\
\to\ & i \quad\quad\ \ \text{at rate} \quad -\alpha i(1-p_1) \\
\to\ & i+1 \quad \text{at rate} \quad \alpha i p_2 \\
\to\ & i+2 \quad \text{at rate} \quad \alpha i p_3 \\
\to\ & \cdots
\end{aligned}
$$

Background

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这个报告的工作是与周晓凤合作完成的.

Main results

Define

$$\tau_i = \inf\{t \geqslant 0 : X(t) = i\},\ \ \sigma_i = \inf\{t \geqslant \text{the first jumping time} : X(t) = i\}.$$

Define $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj},\ \ k > n \geqslant 0.$

$$G_i^{(i)} = 1, \qquad G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \qquad 1 \leqslant n < i.$$

Theorem 1.

Assume that the single death Q is irreducible and the corresponding process is recurrent. Given $i_0 \in \mathbb{Z}_+$ and a positive integer $n \geqslant 1$ arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0}^n = n \sum_{i_0+1 \leqslant k \leqslant i} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)} \mathbb{E}_\ell \tau_{i_0}^{n-1}}{q_{\ell,\ell-1}}, \qquad i \geqslant i_0 + 1.$$

Main results

Corollary 2.

Assume that the single death Q-matrix is regular and the proc. is exp. erg.. Then

$$\delta := \sup_{i \geqslant 1} \sum_{1 \leqslant k \leqslant i} \sum_{\ell \geqslant i} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

ℓ-ergodicity: $\mathbb{E}_j \sigma_j^{\ell} < \infty$ for some j. 1-erg. $=$ posit. recur., 0-erg. $=$ null recur.

Corollary 3.

Assume that the single death Q-matrix is regular and the process is recurrent. Then the process is ℓ-ergodic if and only if

$$d_\ell := \sum_{k \geqslant 1} q_0^{(k)} \sum_{j \geqslant k} \frac{G_k^{(j)}}{q_{j,j-1}} \mathbb{E}_j \tau_0^{\ell-1} < \infty.$$

Main results

Proposition 4

Let the single death Q-matrix be regular and irreducible. Assume that

$$\sum_{k\geqslant 1}\frac{q_0^{(k)}}{G_1^{(k)}}<\infty.$$

If

$$q:=\inf_{n\geqslant 0} q_n>0 \quad \text{and} \quad M:=\sup_{n\geqslant 1}\left(\sum_{k=1}^{n}\frac{1}{G_1^{(k)}}\right)\left(\sum_{j=n}^{\infty}\frac{G_1^{(j)}}{q_{j,j-1}}\right)<\infty,$$

then the process is exponentially ergodic.

$$M\geqslant\delta.$$

Sketch of proof

Define

$$M_{ik}^{(n-1)} = \sum_{i+1\leqslant \ell \leqslant k} \sum_{1\leqslant s\leqslant n-1} \binom{n}{s} \mathbb{E}_\ell \tau_{\ell-1}^{n-s} \mathbb{E}_{\ell-1} \tau_{i-1}^{s},$$

$$M_i^{(n-1)} = \mathbb{E}_i \tau_{i-1}^{n-1} + \frac{1}{n} \sum_{k\geqslant i+1} q_{ik} M_{ik}^{(n-1)}, \qquad n \geqslant 1.$$

- Single death property: $\mathbb{E}_i \tau_{i-1}^{n} = ?$

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k\geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \ i \geqslant 1. \qquad (*)$$

Sketch of proof

Proposition 5. Under the conditions of Theorem 1

$$\mathbb{E}_i \tau_{i-1}^n = n \sum_{k \geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \ i \geqslant 1, n \geqslant 1.$$

Denote the left hand side by m_i, the right hand side by h_i.

$$h_i = \frac{1}{q_{i,i-1}} \left(n M_i^{(n-1)} + \sum_{k \geqslant i+1} q_i^{(k)} h_k \right).$$

Sketch of proof

Lemma 6.

(h_i) is the minimal nonnegative solution to

$$x_i = \frac{q_i^{(i+1)}}{q_i} x_i + \sum_{k \geqslant i+1} \frac{q_i^{(k)}}{q_i} x_k + \frac{n}{q_i} M_i^{(n-1)}, \quad i \geqslant 1. \qquad (*)$$

$$m_i \geqslant h_i.$$

Sketch proof of Lemma 6:
Fix $N \geqslant 2$, define Q-matrix $Q^{(N)} = (\tilde{q}_{ij})$ on $\{0, 1, 2, \cdots, N\}$:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + nG^{(N)}a_N) & \text{if } i = N, j = N-1; \\ -(q_N \vee N)(1 + nG^{(N)}a_N) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N-1, \end{cases}$$

where $G^{(N)} = \max_{1 \leqslant i \leqslant N} G_i^{(N)}$ and

$$a_N = \begin{cases} M_N^{(n-1)} & \text{if } M_N^{(n-1)} < \infty; \\ 1 & \text{if } M_N^{(n-1)} = \infty. \end{cases}$$

Define $\tilde{q}_n^{(k)} = \sum_{j=k}^N \tilde{q}_{nj}, \ 0 \leqslant n < k \leqslant N,$

$$\widetilde{G}_i^{(i)} = 1, \qquad \widetilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \tilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \qquad 1 \leqslant n < i \leqslant N.$$

Sketch proof of Lemma 6(continued):

$$h_n^{(N)} := n\sum_{k=i}^{N}\frac{\widetilde{G}_i^{(k)}M_k^{(n-1)}}{\tilde{q}_{k,k-1}} \quad (1\leqslant i\leqslant N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i = \frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i}\cdot x_i + \sum_{k=i+1}^{N}\frac{\tilde{q}_i^{(k)}}{\tilde{q}_i}\cdot x_k + \frac{n}{\tilde{q}_i}M_i^{(n-1)}, \qquad 1\leqslant i\leqslant N.$$

$$x_N = \frac{nM_N^{(n-1)}}{(q_N\vee N)(1+nG^{(N)}a_N)},\ x_i = \frac{q_i^{(i+1)}}{q_i}\cdot x_i + \sum_{k=i+1}^{N}\frac{q_i^{(k)}}{q_i}\cdot x_k + \frac{n}{q_i}M_i^{(n-1)}.$$

$(h_n^{(N)})$ is increasing to the minimal non-negative solution of $(*)$ as $N\to\infty$.

$$h_n^{(N)} = n\sum_{k=i}^{N-1}\frac{G_i^{(k)}M_k^{(n-1)}}{q_{k,k-1}} + \frac{nG_i^{(N)}M_N^{(n-1)}}{(q_N\vee N)(1+nG^{(N)}a_N)} \to n\sum_{k=i}^{\infty}\frac{G_i^{(k)}M_k^{(n-1)}}{q_{k,k-1}} = h_i.$$

Lemma 7. Under the conditions of Theorem 1

$(\mathbb{E}_i\tau_{i_0}^n : i \geq i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0, \qquad x_i = \sum_{j\neq i, j>i_0} \frac{q_{ij}}{q_i} x_j + \frac{n}{q_i}\mathbb{E}_i\tau_{i_0}^{n-1}, \qquad i > i_0.$$

$n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)}$ satisfies the equation above.

$$\mathbb{E}_i\tau_{i_0}^n \leqslant n \sum_{i_0+1\leqslant k\leqslant i} \sum_{\ell\geqslant k} \frac{G_k^{(\ell)} M_\ell^{(n-1)}}{q_{\ell,\ell-1}} + M_{i_0+1,i}^{(n-1)} \Rightarrow m_i \leqslant h_i.$$

Sketch of proof

Proposition 5. Under the conditions of Theorem 1

$$\mathbb{E}_i \tau_{i-1}^n = n \sum_{k \geqslant i} \frac{G_i^{(k)} M_k^{(n-1)}}{q_{k,k-1}}, \ i \geqslant 1, n \geqslant 1.$$

Lemma 8. For all $1 \leqslant i \leqslant v < u$,

$$G_i^{(u)} = \sum_{i \leqslant k \leqslant v} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell=v+1}^{u} q_k^{(\ell)} G_\ell^{(u)}, \qquad 1 \leqslant i \leqslant v < u.$$

Lemma 9. Under the conditions of Theorem 1

for any nonnegative sequence $\{a_n\}$ and $n \geqslant 1$,

$$\sum_{k \geqslant i} \frac{G_i^{(k)}}{q_{k,k-1}} \sum_{\ell_1 \geqslant k+1} q_k^{(\ell_1)} \mathbb{E}_{\ell_1} \tau_{\ell_1 - 1}^n \sum_{\ell_2 = k}^{\ell_1 - 1} a_{\ell_2} = n \sum_{u \geqslant i+1} \frac{G_i^{(u)} M_u^{(n-1)}}{q_{u,u-1}} \sum_{\ell_2 = i}^{u-1} a_{\ell_2}.$$

Sketch of proof

Lemma 10. Under the conditions of Theorem 1

for any $1 \leqslant i \leqslant k < \ell_1$ and $n \geqslant 2$,

$$M_{k,\ell_1-1}^{(n-1)} + \sum_{s=1}^{n-1} \binom{n}{s} \mathbb{E}_{\ell_1-1}\tau_{k-1}^{n-s}\mathbb{E}_{k-1}\tau_{i-1}^{s} = \sum_{\ell_2=k}^{\ell_1-1}\sum_{s=1}^{n-1}\binom{n}{s}\mathbb{E}_{\ell_2}\tau_{\ell_2-1}^{n-s}\mathbb{E}_{\ell_2-1}\tau_{i-1}^{s}.$$

Lemma 11. Under the conditions of Theorem 1

for all $1 \leqslant s_1 < n$,

$$\sum_{k\geqslant i}\frac{G_i^{(k)}}{q_{k,k-1}}\sum_{\ell_1\geqslant k+1} q_k^{(\ell_1)}\mathbb{E}_{\ell_1}\tau_{\ell_1-1}^{n-s_1}\left(M_{k,\ell_1-1}^{(s_1-1)} + \sum_{s_2=1}^{s_1-1}\binom{s_1}{s_2}\mathbb{E}_{\ell_1-1}\tau_{k-1}^{s_1-s_2}\mathbb{E}_{k-1}\tau_{i-1}^{s_2}\right)$$

$$= (n-s_1)\sum_{u\geqslant i+1}\frac{G_i^{(u)}}{q_{u,u-1}}M_u^{(n-s_1-1)}M_{i,u-1}^{(s_1-1)}.$$

Sketch of proof

Sketch of proof of Theorem 1.

$$\mathbb{E}_i \tau_{i-1}^n = h_i = n \sum_{k \geqslant i} \frac{G_i^{(k)} \mathbb{E}_k \tau_{i-1}^{n-1}}{q_{k,k-1}}, \qquad i \geqslant 1, n \geqslant 1.$$

$$\mathbb{E}_i \tau_k^n = n \sum_{k+1 \leqslant j \leqslant i} \sum_{\ell \geqslant j} \frac{G_j^{(\ell)} \mathbb{E}_\ell \tau_k^{n-1}}{q_{\ell,\ell-1}}, \qquad 0 \leqslant k < i, n \geqslant 1.$$

Future work

Define a positive matrix $H = (h_{ij})_{i,j \geqslant 1}$:

$$h_{ij} = \frac{1}{q_{j,j-1}} \sum_{k=1}^{i \wedge j} G_k^{(j)}.$$

$$\mathbb{E}_k e^{\alpha \tau_0} = \sum_{n=0}^{\infty} ((\alpha H)^n \mathbb{1})(k).$$

$$\frac{1}{\alpha_{\max}} = \inf_{y \in \mathscr{F}} \sup_{k \geqslant 1} \frac{(Hy + H\mathbb{1})(k)}{y_k} < \infty \Leftrightarrow \text{exp. erg.} \Leftrightarrow \lambda_{\max} < \infty.$$

where

$$\mathscr{F} = \{y : 0 < y \upuparrows, \sum_{k \geqslant 1} q_{0k} y_k < \infty\}.$$

Future work

Theorem [Tweedie(1981)]

Given a regular irreducible Q-matrix $Q = (q_{ij})$. Then the process $P(t)$ is exponentially ergodicity if and only if for some $\alpha > 0$ with $\alpha < q_i$ for all i,

$$\begin{cases} y_i \geqslant 1, \quad i \in E \\ \sum_j q_{ij} y_j \leqslant -\alpha y_i, \quad i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite solution (y_i).

$$H = ((-Q)|_{i \geqslant 1})^{-1} = \left(\int_0^\infty p_{ij}^{\min}(t) \mathrm{d}t \right).$$

Future work

Example 12.

Given a constant $b > 2$ (for regularity only needed that $b > 1$). Let

$$q_{ij} = \frac{b-1}{b^{j-i+2}}, \quad j \geqslant i+1; \quad q_{i,i-1} = \frac{b-1}{b}, \quad q_i = -q_{ii} = \frac{b^2-b+1}{b^2}, \quad i \geqslant 1;$$

$$q_{0j} = \frac{b-1}{b^{j+1}}, \qquad j \geqslant 1; \qquad q_0 = -q_{00} = \frac{1}{b}.$$

Then $q_n^{(k)} = 1/b^{k-n+1}(1 \leqslant n < k)$, $q_0^{(k)} = 1/b^k(k \geqslant 1)$, $q = 1/b$ and

$$G_n^{(i)} = \frac{1}{b(b-1)^{i-n}}, \qquad 1 \leqslant n < i.$$

We know that the process is exp. erg. but not strongly ergodic. Now

$$\delta = \frac{b-1}{(b-2) \wedge (b-2)^2}, \quad \sum_{k \geqslant 1} \frac{q_0^{(k)}}{G_1^{(k)}} = \frac{b^2-b+1}{b} \quad \text{and} \quad M = \frac{b(b-1)}{(b-2)^2}.$$

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Thank you for your attention!

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