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单死过程和树上生灭过程

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CONTENTS

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  (1) Normal condition: $p_{ij}(t) \geqslant 0,\ \sum_j p_{ij}(t) \leqslant 1,\ \ i, j \in \mathbb{Z}_+, t \geq 0.$

  (2) Chapman-Kolmogorov equation: $p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s).$

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij},\ \ i, j \in \mathbb{Z}_+.$

  (4) Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ \ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

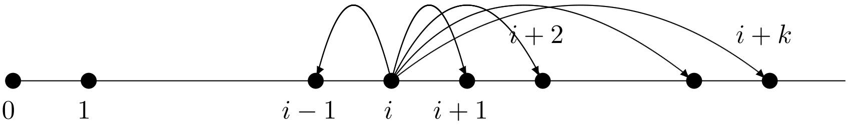
$$0 \leqslant q_{ij} < \infty,\ \ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ \ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ \ i \in \mathbb{Z}_+.$
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ \ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0.$

Background

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0,\ \ q_{i,i-j} = 0,\ \ i \geqslant 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Background

- branching processes: reference model

$$
\begin{aligned}
i &\to i-1 \quad \text{at rate} \quad \alpha i p_0 \\
&\to i \quad \text{at rate} \quad -\alpha i(1-p_1) \\
&\to i+1 \quad \text{at rate} \quad \alpha i p_2 \\
&\to i+2 \quad \text{at rate} \quad \alpha i p_3 \\
&\to \cdots
\end{aligned}
$$

Background

- single birth processes and dual approach.

Y.R. Li, A.G. Pakes, J. Li, A.H. Gu(2008). The limit behavior of dual Markov branching processes. J. Appl. Prob. 45, 176-189.

- stationary distribution. zero-entrance.
- quasi-stationary distribution: $\sup_{n\geqslant 1}\mathbf{E}_n\tau_0$.

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix. Then the limit

$$\lim_{t\to\infty} p_{ij}(t) =: \pi_j$$

exists and it is independent of i. Moreover, we have either $\sum_j \pi_j = 1$ or $\sum_j \pi_j = 0$.

- $P(t)$ is positive recurrent or ergodic if so is $P(h)$ for every $h > 0$. Equivalently, $\lim_{t\to\infty} p_{ii}(t) = \pi_i > 0$.
- $P(t)$ is strongly ergodic or uniformly ergodic if

$$\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Hitting times, criteria on ergodicity and strong ergodicity

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix and H be a non-empty finite subset of $\mathbb{Z}_+$. Define $\sigma_H = \inf\{t \geq \text{the first jump} : X(t) \in H\}$.

Theorem[Issacson & Arnold(1978)]

(1) The Q-process is ergodic iff $\mathbb{E}_i \sigma_H < \infty$ for all $i \in H$.
(2) The Q-process is strongly ergodic iff $\sup_i \mathbb{E}_i \sigma_H < \infty$.

Theorem[Tweedie(1981)]

The Q-process is ergodic (resp. strongly erergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leqslant -1, \ i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

Hitting times, criteria on ergodicity and strong ergodicity

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim\limits_{t\to\infty} |p_{ij}(t) - \pi_j| = 0.$
- Exponential ergodicity: $\lim\limits_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0.$
- Strong ergodicity: $\lim\limits_{t\to\infty} \sup\limits_i |p_{ij}(t) - \pi_j| = 0$
$$\Longleftrightarrow \lim_{t\to\infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$
- Strong ergodicity $\Rightarrow$ Exponential ergodicity $\Rightarrow$ Ordinary ergodicity.

Define $q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj},\ k > n \geqslant 0.$

$$G_i^{(i)} = 1,\ G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)},\ 1 \leqslant n < i.$$

Theorem 1

Assume that the single death Q is irreducible and regular. Then the process is ergodic if and only if

$$D := \sum_{k\geqslant 1} q_0^{(k)} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k\geqslant 1} \sum_{\ell \geqslant k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

$$G_i^{(i)} = 1, \quad G_n^{(i)} = \sum_{k=n}^{i-1} \frac{G_n^{(k)} q_k^{(i)}}{q_{k,k-1}}, \quad 1 \leqslant n \leqslant i-1.$$

- Mean hitting time of single death process: $\mathbb{E}_n \tau_{n-1} = ?$

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \ n \geqslant 1. \qquad (*)$$

Proposition 1

$$\mathbb{E}_n\tau_{n-1} = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}}, \; n \geqslant 1.$$

Denote the left hand by m_n, the right hand by h_n.

$$h_n = \frac{1}{q_{n,n-1}}\left(1 + \sum_{k\geqslant n+1} q_n^{(k)} h_k\right).$$

Lemma 1

(h_n) is the minimal nonnegative solution to

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \quad n \geqslant 1. \qquad (*)$$

$$m_n \geqslant h_n.$$

Sketch proof of Lemma 1:
Fix $N \in \mathbb{Z}_+$, define Q-matrix $Q^{(N)} = (\tilde{q}_{ij})$ on $\{0, 1, 2, \cdots, N\}$:

$$\tilde{q}_{ij} = \begin{cases} q_{ij} & \text{if } i < N, j < N; \\ q_i^{(N)} & \text{if } i < N, j = N; \\ (q_N \vee N)(1 + G^{(N)}) & \text{if } i = N, j = N-1; \\ -(q_N \vee N)(1 + G^{(N)}) & \text{if } i = N, j = N; \\ 0, & \text{if } i = N, j < N-1, \end{cases}$$

where $G^{(N)} = \max_{1 \leqslant n \leqslant N} G_n^{(N)}$.
Define

$$\tilde{q}_n^{(k)} = \sum_{j=k}^{N} \tilde{q}_{nj}, \ \ 0 \leqslant n < k \leqslant N$$

and

$$\widetilde{G}_i^{(i)} = 1, \ \ \widetilde{G}_n^{(i)} = \frac{1}{\tilde{q}_{n,n-1}} \sum_{k=n+1}^{i} \tilde{q}_n^{(k)} \widetilde{G}_k^{(i)}, \ \ 1 \leqslant n < i \leqslant N.$$

Sketch proof of Lemma 1(continued):

$$h_n^{(N)} := \sum_{k=n}^{N} \frac{\widetilde{G}_n^{(k)}}{\tilde{q}_{k,k-1}} \quad (1 \leqslant n \leqslant N)$$

is a unique solution (the minimal non-negative solution) to the following equations

$$x_i = \frac{\tilde{q}_i^{(i+1)}}{\tilde{q}_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{\tilde{q}_i^{(\ell)}}{\tilde{q}_i} \cdot x_\ell + \frac{1}{\tilde{q}_i}, \ 1 \leqslant i \leqslant N.$$

$$x_N = \frac{1}{(q_N \vee N)(1+G^{(N)})}, \ x_i = \frac{q_i^{(i+1)}}{q_i} \cdot x_i + \sum_{\ell=i+1}^{N} \frac{q_i^{(\ell)}}{q_i} \cdot x_\ell + \frac{1}{q_i}, \ 1 \leqslant i < N.$$

$(h_n^{(N)})$ is increasing to the minimal non-negative solution of $(*)$ as $N \to \infty$.

$$h_n^{(N)} = \sum_{k=n}^{N-1} \frac{G_n^{(k)}}{q_{k,k-1}} + \frac{G_n^{(N)}}{(q_N \vee N)(1+G^{(N)})} \to \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} = h_n \text{ as } N \to \infty.$$

Lemma 2

$(\mathbf{E}_i \tau_{i_0} : i \geq i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0,\ x_i = \sum_{j \neq i, j > i_0} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i},\ i > i_0.$$

$\sum_{i_0+1 \leqslant n \leqslant i} h_n$ satisfies the equation above.

$$\mathbf{E}_i \tau_{i_0} \leqslant \sum_{i_0+1 \leqslant n \leqslant i} h_n \Rightarrow m_n \leqslant h_n.$$

Hitting times, criteria on ergodicity and strong ergodicity

Proposition 1

$$m_n = h_n := \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}}, \ n \geqslant 1.$$

$$\begin{aligned}
\mathbf{E}_0 \sigma_0 &= \frac{1}{q_0} + \sum_{j \geqslant 1} \frac{q_{0j}}{q_0} \mathbf{E}_j \tau_0 = \frac{1}{q_0} + \sum_{j \geqslant 1} \frac{q_{0j}}{q_0} \sum_{1 \leqslant k \leqslant j} m_k \\
&= \frac{1}{q_0} + \frac{1}{q_0} \sum_{k \geqslant 1} q_0^{(k)} h_k \\
&=: \frac{1}{q_0} + \frac{1}{q_0} D. (\Leftrightarrow \text{ergodicity iff } D < \infty)
\end{aligned}$$

$$\sup_{i \geqslant 1} \mathbb{E}_i \tau_0 = \sup_{i \geqslant 1} \sum_{1 \leqslant k \leqslant i} m_k = \sum_{k \geqslant 1} h_k =: S, \ \ D \leqslant q_0 S$$

$\Leftrightarrow$ strong ergodicity iff $S < \infty$.

S. Martinez, J. S. Martin , D. Villemonais(2014). Existence and uniqueness of a quasi- stationary distribution for Markov Processes with fast return from infinity. Journal of Applied Probability, 51(3): 756-768.

Theorem 2

Assume that the single death Q is 'irreducible' and regular such that 0 is an absorbing state and absorption occurs almost surely. If

$$\sum_{n=1}^{\infty}\sum_{k=n}^{\infty}\frac{G_n^{(k)}}{q_{k,k-1}}<\infty,$$

then there exists a quasi-stationary distribution uniquely.

Three examples

Example 1

For birth-death chain (a_i, b_i), define

$$\mu_0 = 1, \quad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geqslant 1; \quad \mu[i, +\infty) = \sum_{k \geqslant i} \mu_k, \quad i \geqslant 0.$$

Then

$$G_n^{(i)} = \frac{\mu_i a_i}{\mu_n a_n}, \quad 1 \leqslant n \leqslant i; \qquad m_n = \frac{\mu[n, +\infty)}{\mu_n a_n}, \quad n \geqslant 1.$$

Therefore, the birth-death process is ergodic if and only if $D = \mu[1, +\infty) < \infty$, equivalently, $\mu := \mu[0, +\infty) < \infty$; the birth-death process is strongly ergodic if and only if

$$S = \sum_{n \geqslant 1} \frac{\mu[n, +\infty)}{\mu_n a_n} = \sum_{n \geqslant 0} \frac{\mu[n+1, +\infty)}{\mu_n b_n} < \infty.$$

Example 2

Given a constant $b > 2$. Let

$$q_{ij} = \frac{b-1}{b^{j-i+2}}, \quad j \geqslant i+1; \quad q_{i,i-1} = \frac{b-1}{b}, \quad q_i = -q_{ii} = \frac{b^2-b+1}{b^2}, \quad i \geqslant 1;$$

$$q_{0j} = \frac{b-1}{b^{j+1}}, \quad j \geqslant 1; \quad q_0 = -q_{00} = \frac{1}{b}.$$

Then $q_n^{(k)} = 1/b^{k-n+1}(1 \leqslant n < k)$, $q_0^{(k)} = 1/b^k(k \geqslant 1)$,

$$G_n^{(i)} = \frac{1}{b(b-1)^{i-n}}, \quad 1 \leqslant n < i; \quad m_n = \frac{b-1}{b-2}, \quad n \geqslant 1.$$

Therefore, by $D = 1/(b-2)$ and $S = +\infty$, we know that the process is ergodic but not strongly ergodic.

Three examples

The original branching process can be described as follows. Let $\alpha > 0$ and $(p_j : j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate $\alpha i p_0 : i \to i-1\,(i \geq 1)$ and growth rate $\alpha i p_{k+1} : i \to i+k\,(k \geq 1, i \in \mathbb{Z}_+)$. An extended class of branching processes:

Example 3

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i-1,\ i \geq 1; \\ r_i p_{k+1}, & j = i+k,\ i, k \geq 1; \\ -r_i(1-p_1), & j = i \geq 1; \\ 0, & \text{else}, \quad i, j \in \mathbb{Z}_+. \end{cases}$$

where $r_i > 0$ for all $i \geq 1$ and $0 < q_0 := \sum_{j \geq 1} q_{0j} < \infty$.

R.R. Chen(1997). An extended class of time-continuous branching processes. J. Appl. Prob., 34(1), 14-23.

Three examples

Theorem 3

Assume that the extended branching Q is irreducible and regular. Then the process is ergodic if and only if

$$\sum_{\ell\geqslant 1}\frac{1}{r_\ell}\left(q_0^{(\ell)}+\sum_{1\leqslant k\leqslant \ell-1}\frac{(\mathbf{f}^{*k}*\mathbf{q})_{\ell-k+1}}{p_0^k}\right)<\infty;$$

it is strongly ergodic if and only if

$$\sum_{\ell\geqslant 1}\frac{1}{r_\ell}\left(1+\sum_{1\leqslant k\leqslant \ell-1}\frac{(\mathbf{f}^{*k}*\mathbf{1})_{\ell-k+1}}{p_0^k}\right)<\infty.$$

where $\mathbf{f}=(\sum_{k\geqslant n}p_k, n\geqslant 2)$, $\mathbf{1}=(1,1,\cdots)$ and $\mathbf{q}=(q_0^{(n-1)}, n\geqslant 2)$ and

$$(\mathbf{a}*\mathbf{b})_n=\sum_{2\leqslant m\leqslant n}a_{n+2-m}b_m,\ \ n\geqslant 2.$$

Three examples

Come back to Example 2. Fix a positive constant a such that $a < 1 - 1/(b^2 - b + 1)$. Then Example 2 is the special case of Example 3:

$$r_i = (b-1)/(ab),\ p_0 = a,\ p_j = a/b^j\,(j \geqslant 2),\ p_1 = 1 - a - a/(b^2 - b).$$

$$f_n = \frac{a}{(b-1)b^{n-1}}, \quad n \geqslant 2; \quad q_0^{(\ell)} = \frac{1}{b^\ell}, \quad \ell \geqslant 1.$$

$$(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1} = \frac{a^k}{(b-1)^k b^\ell} C_{\ell-1}^k, \qquad 1 \leqslant k \leqslant \ell - 1.$$

$$\sum_{1 \leqslant k \leqslant \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} = \frac{1}{b(b-1)^{\ell-1}} - \frac{1}{b^\ell}.$$

$$\sum_{\ell \geqslant 1} \frac{1}{r_\ell} \left(q_0^{(\ell)} + \sum_{1 \leqslant k \leqslant \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{q})_{\ell-k+1}}{p_0^k} \right) = \frac{a}{b-2} < \infty.$$

Three examples

Note that

$$(\mathbf{f} * \mathbf{1})_\ell = \frac{a}{(b-1)^2}\left(1 - \frac{1}{b^{\ell-1}}\right).$$

From the equality above, it follows that

$$\sum_{\ell \geqslant 1} \frac{1}{r_\ell}\left(1 + \sum_{1 \leqslant k \leqslant \ell-1} \frac{(\mathbf{f}^{*k} * \mathbf{1})_{\ell-k+1}}{p_0^k}\right) \geqslant \sum_{\ell \geqslant 2} \frac{1}{r_\ell} \cdot \frac{(\mathbf{f} * \mathbf{1})_\ell}{p_0} = \infty.$$

So the process in Example 2 is ergodic but not strong ergodic. This is the case of $b > 2$. Note that

$$M_1 := \sum_{k=1}^{\infty} k p_k = \frac{a}{(b-1)^2} + 1 - a.$$

So $M_1 \leqslant 1$ if and only if $b \geqslant 2$. Hence, it is easy to be known that the process is unique and null recurrent when $b = 2$; the process is unique and transient when $1 < b < 2$.

Exponential ergodicity

Define $G_n = \underline{\lim}_{m\to\infty} G_n^{(m)}/G_1^{(m)}$, $n \geqslant 1$. Then

$$G_n \geqslant \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{\infty} q_n^{(k)} G_k, \quad n \geqslant 1.$$

Theorem 4

Let the single death Q-matrix be regular and irreducible. Assume that

$$\sum_{k\geqslant 1} q_0^{(k)} G_k < \infty.$$

If

$$q := \inf_{n\geqslant 0} q_n > 0 \quad \text{and} \quad M := \sup_{n\geqslant 1} \left[\sum_{k=1}^{n} G_k\right]\left[\sum_{j=n}^{\infty} \frac{1}{q_{j,j-1} G_j}\right] < \infty,$$

then the process is exponentially ergodic.

Exponential ergodicity

Theorem [Tweedie(1981)]

Given a regular irreducible Q-matrix $Q = (q_{ij})$. Then the process $P(t)$ is exponentially ergodicity if and only if for some $\lambda > 0$ with $\lambda < q_i$ for all i,

$$\begin{cases} y_i \geqslant 1, \quad i \in E \\ \sum_j q_{ij} y_j \leqslant -\lambda y_i, \quad i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite solution (y_i).

Sketch of proof: Define an operator

$$II_i(f) = \frac{1}{f_i} \sum_{j=1}^{i} G_j \sum_{k=j}^{\infty} \frac{f_k}{q_{k,k-1} G_k}, \quad i \geqslant 1$$

and

$$\varphi_i = \sum_{j=1}^{i} G_j, \qquad i \geqslant 1.$$

Then φ is increasing in i and $\varphi_1 = G_1$. Let $f = cq_{10}\sqrt{\varphi}/\sqrt{G_1}$ for some $c > 1$. Then f is increasing and $f_1 = cq_{10}$. Finally, define $g = f\,II(f)$ and $g_0 = 1$. Then g is increasing, $1 \leqslant g_i < \infty$ for all $i \geqslant 0$ and $\sum_{j \geqslant 1} q_{0j} g_j < \infty$. Then we can take any λ:

$$0 < \lambda < \left(\frac{c-1}{c} \, II_1(f)^{-1} \right) \bigwedge \left(\inf_{i \geqslant 2} II_i(f)^{-1} \right) \bigwedge q.$$

provided the right-hand side is positive or, equivalently, $\sup_{i \geqslant 1} II_i(f) < \infty$.

Exponential ergodicity

To prove the last property, define another operator

$$I_i(f) = \frac{G_{i+1}}{f_{i+1} - f_i} \sum_{k=i+1}^{\infty} \frac{f_k}{q_{k,k-1} G_k}. \quad i \geqslant 1,$$

By the proportion property, we get

$$\sup_{i \geqslant 1} II_i(f) \leqslant \sup_{i \geqslant 1} I_i(f).$$

By the condition $M < \infty$, it follows that

$$I_i(f) = \frac{G_{i+1}}{\sqrt{\varphi_{i+1}} - \sqrt{\varphi_i}} \sum_{k=i+1}^{\infty} \frac{\sqrt{\varphi_k}}{q_{k,k-1} G_k} \leqslant \frac{2MG_{i+1}}{(\sqrt{\varphi_{i+1}} - \sqrt{\varphi_i})\sqrt{\varphi_{i+1}}} \leqslant 4M$$

for all $i \geqslant 1$. Therefore, $\sup_{i \geqslant 1} II_i(f) \leqslant 4M < \infty$ as required. We have thus constructed a solution (g_i) to the equation with $1 \leqslant g_i < \infty$ for all $i \geqslant 0$ and $\sum_{j \geqslant 1} q_{0j} g_j < \infty$. This implies the exponential ergodicity of the single death process.

Exponential ergodicity

For the birth-death process (a_i, b_i), $G_n^{(m)}/G_1^{(m)} = \mu_1 a_1/(\mu_n a_n)$. Thus

$$G_n = \lim_{m\to\infty} \frac{G_n^{(m)}}{G_1^{(m)}} = \frac{\mu_1 a_1}{\mu_n a_n}, \quad n \geqslant 1$$

and $\sum_{k\geqslant 1} q_0^{(k)} G_k = b_0 < \infty$. The equality holds. Now

$$M = \sup_{n\geqslant 1} \left[\sum_{k=1}^{n} \frac{1}{\mu_k a_k}\right] \left[\sum_{j=n}^{\infty} \mu_j\right] = \sup_{n\geqslant 1} \left[\sum_{k=0}^{n-1} \frac{1}{\mu_k b_k}\right] \left[\sum_{j=n+1}^{\infty} \mu_j\right].$$

The birth-death process is exponentially ergodic if and only if $M < \infty$.

Exponential ergodicity

Come back to Example 2 for $b > 2$. We see that $G_n^{(m)}/G_1^{(m)} = (b-1)^{n-1}$. Then

$$G_n = \lim_{m\to\infty} \frac{G_n^{(m)}}{G_1^{(m)}} = (b-1)^{n-1}, \qquad n \geqslant 1,$$

and $\sum_{k\geqslant 1} q_0^{(k)} G_k = 1 < \infty$. The equality holds. Note that $q = 1/b > 0$ and

$$M = \frac{b(b-1)}{(b-2)^2} < \infty.$$

Hence, the process is exponentially ergodic.

Exponential ergodicity

Define

$$a_i = q_{i,i-1},\ b_i = \frac{q_{i,i-1}G_i}{G_{i+1}},\ i \geqslant 1;\ b_0 = \frac{q_{10}}{G_1}.$$

Then $\mathbb{E}_i\tau_0^n \leqslant \mathbb{E}_i\bar{\tau}_0^n,\ n \geqslant 0, i \geqslant 1.$

Theorem 5

Let the single death Q-matrix be regular and irreducible. Assume that $\sum_{k\geqslant 1} G_k = \infty$. If the birth-death process is ergodic (or exponentially ergodic, or strongly ergodic respectively), then so is the single death process.

$$\sum_{i\geqslant 1}\frac{1}{q_{i,i-1}G_i} < \infty \Rightarrow \text{ergodic}$$

$$M < \infty \Rightarrow \text{exponentially ergodic}$$

$$\sum_{n\geqslant 1} G_n \sum_{k\geqslant n}\frac{1}{q_{k,k-1}G_k} < \infty \Rightarrow \text{strongly ergodic}$$

Recurrence and return probability

Define

$$\widetilde{G}_n = \overline{\lim_{m\to\infty}} \frac{G_n^{(m)}}{G_1^{(m)}}, \ n \geqslant 1.$$

Assume that

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} < \infty, \ n \geqslant 1. \qquad (**)$$

Then

$$\widetilde{G}_n \leqslant \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{\infty} q_n^{(k)} \widetilde{G}_k, \qquad n \geqslant 1.$$

Theorem 6

Let the single death Q-matrix be regular and irreducible. If $\sum_{n\geqslant 1} G_n = \infty$, then the process is recurrent. Conversely, under the assumption $(**)$, if the process is recurrent, then $\sum_{n\geqslant 1} \widetilde{G}_n = \infty$.

Recurrence and return probability

- For the birth-death process (a_i, b_i), we have

$$\sum_{n\geqslant 1} G_n = \sum_{n\geqslant 1} \widetilde{G}_n = \sum_{n\geqslant 1} \frac{\mu_1 a_1}{\mu_n a_n} = \sum_{n\geqslant 0} \frac{b_0}{\mu_n b_n}$$

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} = \frac{b_0}{\mu_n} < \infty, \ n \geqslant 1.$$

  Hence the birth-death process is recurrent if and only if $\sum_{n\geqslant 0} 1/(\mu_n b_n) = \infty$.

- For Example 2 when $b > 1$. Then

$$\sum_{k\geqslant n+1} \frac{q_n^{(k)}}{G_1^{(k)}} = (b-1)^n < \infty, \ n \geqslant 1$$

$$\sum_{n\geqslant 1} G_n = \sum_{n\geqslant 1} \widetilde{G}_n = \sum_{n\geqslant 1} (b-1)^{n-1},$$

  which is infinite when $b \geqslant 2$ and finite when $1 < b < 2$. So the process is recurrent if and only if $b \geqslant 2$.

Recurrence and return probability

Define

$$G_0 = \frac{1}{q_0} \sum_{n\geqslant 1} q_0^{(n)} G_n,\ G = \sum_{n\geqslant 1} G_n,\ \widetilde{G}_0 = \frac{1}{q_0} \sum_{n\geqslant 1} q_0^{(n)} \widetilde{G}_n,\ \widetilde{G} = \sum_{n\geqslant 1} \widetilde{G}_n.$$

Theorem 7

Let the single death Q-matrix be regular and irreducible. Assume that $(**)$ holds. Then

$$\mathbb{P}_0(\sigma_0 < \infty) \leqslant 1 - \frac{\widetilde{G}_0}{\widetilde{G}},\ \mathbb{P}_n(\sigma_0 < \infty) \leqslant 1 - \frac{1}{\widetilde{G}} \sum_{1\leqslant k\leqslant n} \widetilde{G}_k,\ n \geqslant 1.$$

In addition, assume that $\displaystyle \lim_{m\to\infty} \frac{1}{G_1^{(m)}} \sum_{1\leqslant j\leqslant k} G_j^{(m)} = \sum_{1\leqslant j\leqslant k} G_j,\ k \geqslant 1.$ $\qquad (***)$

Then

$$\mathbb{P}_0(\sigma_0 < \infty) \geqslant 1 - \frac{G_0}{G},\ \mathbb{P}_n(\sigma_0 < \infty) \geqslant 1 - \frac{1}{G} \sum_{1\leqslant k\leqslant n} G_k,\ n \geqslant 1.$$

Recurrence and return probability

Lemma 3

$$x_0 = 1 - \frac{\widetilde{G}_0}{\widetilde{G}}, \; x_k = 1 - \frac{1}{\widetilde{G}} \sum_{1 \leqslant n \leqslant k} \widetilde{G}_n$$

satisfies the inequality $x_i \geqslant \sum_{k \neq 0, i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i}(1 - \delta_{i0}), \; i \geqslant 0.$

$(\mathbb{P}_i(\sigma_0 < \infty))$ is the minimal nonnegative solution to the equation above. Thus

$$\mathbb{P}_i(\sigma_0 < \infty) \leqslant x_i, \; i \geqslant 0.$$

Lemma 4

$$\mathbb{P}_k(\tau_0 < \tau_m) = \frac{\sum_{k+1 \leqslant n \leqslant m} G_n^{(m)}}{\sum_{1 \leqslant n \leqslant m} G_n^{(m)}}, \; k \geqslant 0.$$

$$\mathbb{P}_k(\sigma_0 < \infty) \geqslant 1 - \frac{1}{G} \sum_{1 \leqslant n \leqslant k} G_n, \; k \geqslant 1.$$

Remark

If there exist the limits

$$\lim_{m\to\infty}\frac{G_n^{(m)}}{G_1^{(m)}}$$

and the assumption $(**)$ holds, then $G_n=\widetilde{G}_n\,(n\geqslant 0)$ and $(***)$ holds. Hence

$$\mathbb{P}_0(\sigma_0<\infty)=1-\frac{G_0}{G},\ \mathbb{P}_n(\sigma_0<\infty)=1-\frac{1}{G}\sum_{1\leqslant n\leqslant k}G_n,\ \ n\geqslant 1.$$

Thus, $\mathbb{P}_i(\sigma_0<\infty)=1$ for all $i\geqslant 1$ if and only if $\mathbb{P}_0(\sigma_0<\infty)=1$, equivalently, if and only if $G=\infty\,(G_0<\infty)$. See Example 1 and 2.

Birth-death processes on trees

The tree T is a connected graph without cycles. We fix a point on T as the root, denoted by o. For any vertex $i \in T \setminus \{o\}$, there is a unique simple path from i to the root o.

- $\mathscr{P}(i)$: the set of all the vertices on this path (the root o is excluded).
- $|i|$: the number of segments of this path is the length of i.
- $i \sim j$: two vertices i and j are called adjacent if they are joined by a segment.
- When $|j| = |i| + 1$ and $i \sim j$, j is called one offspring of i and the set of all the offsprings of i is denoted by $J(i)$.
- When $|j| = |i| - 1$ and $i \sim j$, j is called the father of i and denoted by i^*.
- Denote T_i the subtree with i as its root, including all the descendants of i.
- Assume that any vertex has finite offsprings.

We consider a birth-death process on this tree which Q-matrix satisfies: $q_{ij} > 0$ if and only if $i \sim j$, i.e., $j = i^*$, or $j \in J(i)$.
$q_i := -q_{ii} = q_{ii^*} + \sum_{j \in J(i)} q_{ij} < \infty$ for all $i \in T$.
Define a measure μ on T as follows.

$$\mu_o = 1, \qquad \mu_i = \prod_{j \in \mathscr{P}(i)} \frac{q_{j^*j}}{q_{jj^*}}, \qquad i \in T \setminus \{0\},$$

which is invariant with respect to Q. In fact, μ satisfies the so-called detailed balance equation:

$$\mu_i q_{ij} = \mu_j q_{ji}, \qquad i \sim j.$$

Define $q_i^{(+)} = \sum_{j \in J(i)} q_{ij}$ for all $i \in T$ and

$$h_i = \frac{1}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k, \qquad i \in T \setminus \{o\}.$$

Lemma 5

$(h_i, i \in T \setminus \{o\})$ is the minimal nonnegative solution of the following equations

$$y_i = \frac{q_i^{(+)}}{q_i} y_i + \sum_{j \in J(i)} \frac{q_{ij}}{q_i} y_j + \frac{1}{q_i}, \qquad i \in T \setminus \{o\}.$$

Birth-death processes on trees

Lemma 6

Assume that the Q-matrix on the tree T is regular and the process is recurrent. Given $i_0 \in T$ arbitrarily. Then

$$\mathbb{E}_i \tau_{i_0} \leqslant \sum_{\ell \in \mathscr{P}(i) \setminus \mathscr{P}(i_0)} \frac{1}{\mu_\ell q_{\ell\ell^*}} \sum_{k \in T_\ell} \mu_k, \qquad i \in T_{i_0}.$$

In particular,

$$\mathbb{E}_i \tau_{i^*} \leqslant \frac{1}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k, \qquad i \in T \setminus \{o\}.$$

Theorem 8

Assume that the Q-matrix on the tree T is regular and the process is recurrent. Then

$$\mathbb{E}_i \tau_{i^*} = \frac{1}{\mu_i q_{ii^*}} \sum_{k \in T_i} \mu_k, \qquad i \in T \setminus \{o\}.$$

Corollary 1

Assume that the Q-matrix on the tree T is regular. Then the process is ergodic if and only if $\mu < \infty$; the process is strongly ergodic if and only if

$$\sup_{i \in T \setminus \{o\}} \sum_{j \in \mathscr{P}(i)} \frac{1}{\mu_j q_{jj^*}} \sum_{\ell \in T_j} \mu_\ell < \infty.$$

S. Martinez, B. Ycart(2001). Decay rates and cutoff for convergence and hitting times of Markov chains with countably infinite state space. Adv. Appl. Prob., 33(1): 188-205

Birth-death processes on trees

Conjecture

Assume that the process is $(n-1)$-ergodic. Then

$$\mathbb{E}_i\tau_o^n = n\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_j}\mu_\ell\mathbb{E}_\ell\tau_o^{n-1}, \qquad i\in T, n\geqslant 1.$$

If the conjecture holds, then by $\mathbb{E}_i\tau_o^{n-1}\leqslant\mathbb{E}_\ell\tau_o^{n-1}$ for all $\ell\in T_i$,

$$\mathbb{E}_i\tau_o^n \geqslant n\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\mathbb{E}_\ell\tau_o^{n-1}\geqslant n\left(\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\right)\mathbb{E}_i\tau_o^{n-1}$$
$$\geqslant\cdots\geqslant n!\left(\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell\right)^n, \qquad n\geqslant 1.$$

From the exponential ergodicity, it follows that

$$\delta := \sup_{i\in T\setminus\{o\}}\sum_{j\in\mathscr{P}(i)}\frac{1}{\mu_j q_{jj^*}}\sum_{\ell\in T_i}\mu_\ell<\infty.$$

Thank you for your attention!

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