非对称随机过程：从单生到单死

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Outline

单生过程:

1. Background
2. Minimal nonnegative solution
3. Classical problems
4. Criterion on uniq., recur., erg. and str. erg.
5. Poisson equation

1. Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  (1) Normal condition: $p_{ij}(t) \geq 0,\ \sum_j p_{ij}(t) \leq 1,\ i, j \in \mathbb{Z}_+, t \geq 0.$

  (2) Chapman-Kolmogorov equation: $p_{ij}(t+s) = \sum_k p_{ik}(t)p_{kj}(s).$

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij},\ i, j \in \mathbb{Z}_+.$

  (4) Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

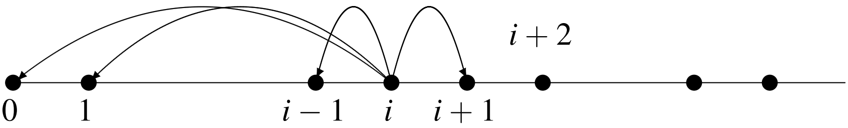
$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.

- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

1. Background

Single birth process: $Q = (q_{ij} : i,j \geq 0)$: $q_{i,i+1} > 0$, $q_{i,i+j} = 0$, $i \geq 0, j \geq 2$.



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Background

1. The exit boundary of the process consists at most one single extremal point
2. Fundamental comparison tool in studying more complex processes, such as infinite-dimensional reaction-diffusion processes
3. Usually the single birth process is non-symmetric (irreversible)
4. Applications in other fields

1. Background

1. Upwardly skip-free processes (W.J. Anderson, 1991);
2. generalized birth and death processes (J.K. Zhang, 1984);
3. Population processes: birth and death processes with catastrophes; birth, death and catastrophe processes

$$\begin{array}{lll} i \to i+1 & \text{at rate} & b_i \\ \quad \to i-1 & & a_i + c_i f_{i,i-1} \\ \quad \to i-2 & & c_i f_{i,i-2} \\ \quad \to \cdots & & \cdots \\ \quad \to 0 & & c_i f_{i0} \end{array}$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

1. Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes f_{ij} ($b_i = b + \lambda i$, $a_i = 0$, $c_i = ci$, $i \geq 0$).

geometric: $f_{ij} = p(1-p)^{i-1-j}$ $(1 \leq j < i)$; $f_{i0} = (1-p)^{i-1}$;

uniform: $f_{ij} = 1/i$ $(0 \leq j < i)$;

binomial: $f_{ij} = \binom{i-1}{j} p^j (1-p)^{i-1-j}$ $(0 \leq j < i)$.

Aim: extinction times and probability of extinction.

Keys: generating function of Q-resolvent.

B. Cairns & P. Pollett (2004).

$$
\begin{aligned}
i \to i+1 &\quad \text{at rate} \quad g_i b \\
\to i-1 &\quad g_i d_1 \\
\to \cdots &\quad \cdots \\
\to 1 &\quad g_i d_{i-1} \\
\to 0 &\quad g_i \sum_{k\geq i} d_k
\end{aligned}
$$

where $b + \sum_{k\geq 1} d_k = 1$.

1. Background

single birth process:

S.J. Yan & M.F. Chen(1986). Multidimensional Q-processes, Chinese Ann. Math., **7(B)**(1), 90-110.

M.F. Chen(2004). From Markov Chains to Non-Equilibrium Particle Systems, Second Edition, World Scientific, Singapore.

陈木法, 毛永华(2007). 随机过程导论, 高等教育出版社, 北京.

Population processes:

W.J. Anderson(1991). Continuous-Time Markov Chains, Springer-Verlag, New York.

- Uniqueness. J.K. Zhang(1984), S.J. Yan & M.F. Chen(1986).
- Recurrence. Yan & Chen(1986).
- Extinction/return probability. P.J. Brockwell(1986), W.J. Anderson(1991), Z.T. Hou & Q.F. Guo(1978, 1988). Chen & Z. (2014)
- Ergodicity. Yan & Chen(1986).
- Strong ergodicity. Z.(2001).
- Polynomial moments and stationary dist. Z.(2003, 2004, 2013)
- Exponential moments. Z.(2003)
- Laplace transform. Brockwell(1986), Anderson(1991)

- Poisson equation. Chen & Z.(2014)
- Separationn cutoff. Mao, C. Zhang & Z.(2016)
- Mean occupation time. P.S. Li & Z.(2017)

2. Minimal nonnegative solution

For $a_{ij} \geq 0, b_i \geq 0$, consider the equation

$$x_i = \sum_k a_{ik} x_k + b_i, \; i \in \mathbb{Z}_+.$$

Theorem[the first successive approximation scheme]

The minimal solution (x_i^*) always exists uniquely. Furthermore, it can be obtained by the following procedure

$$x_i^{(0)} = 0, \; x_i^{(n+1)} = \sum_k a_{ik} x_k^{(n)} + b_i, \; i \in \mathbb{Z}_+, n \geq 0,$$

then $x_i^{(n)} \uparrow x_i^*$ as $n \to \infty$.

Comparison Theorem

Assume that $\tilde{a}_{ij} \geq a_{ij}, \tilde{b}_i \geq b_i$. Then for any solution $(\tilde{x}_i)$ to

$$\tilde{x}_i = \sum_k \tilde{a}_{ik}\tilde{x}_k + \tilde{b}_i, \ i \in \mathbb{Z}_+,$$

we have $\tilde{x}_i \geq x_i^*$.

Theorem[the second successive approximation scheme]

Assume that $b_i^{(n)} \geq 0$. Define

$$\tilde{x}_i^{(1)} = b_i^{(1)}, \ \tilde{x}_i^{(n+1)} = \sum_k a_{ik}\tilde{x}_k^{(n)} + b_i^{(n+1)}, \ i \in \mathbb{Z}_+, n \geq 1.$$

If $b_i^{(n)} \uparrow b_i$ (or, $\sum_{n=1}^{\infty} b_i^{(n)} = b_i$), then $\tilde{x}_i^{(n)} \uparrow x_i^*$ (or $x_i^* = \sum_{n=1}^{\infty} \tilde{x}_i^{(n)}$). In particular, if we set $\tilde{x}_i^{(1)} = b_i, \ \tilde{x}_i^{(n+1)} = \sum_k a_{ik}\tilde{x}_k^{(n)}$, then $x_i^* = \sum_{n=1}^{\infty} \tilde{x}_i^{(n)}$.

Localization Theorem

Let $G \subset \mathbb{Z}_+$ and $(\tilde{x}_i^* : i \in G(\text{resp}.i \in \mathbb{Z}_+))$ be the minimal solution to

$$x_i = \sum_{k \in G} a_{ik} x_k + \sum_{k \in G^c} a_{ik} x_k^* + b_i, \ i \in G(\text{resp}.i \in \mathbb{Z}_+).$$

Then we have $\tilde{x}_i^* = x_i^*, i \in G(\text{resp}.i \in \mathbb{Z}_+)$.

Theorem

Assume that $\sum_j a_{ij} + b_i \leq 1$. Then the maximal solution to

$$x_i = \sum_k a_{ik} x_k + b_i, \ 0 \leq x_i \leq 1, i \in \mathbb{Z}_+,$$

can be obtained by the following procedure

$$x_i^{(0)} = 1, \ x_i^{(n+1)} = \sum_k a_{ik} x_k^{(n)} + b_i, \ i \in \mathbb{Z}_+, n \geq 0,$$

Z.T. Hou & Q.F. Guo. Homogeneous Denumerable Markov Processes. Beijing: Science Press, 1978(in Chinese); English translation, Beijing: Science Press and Springer, 1988

侯振挺(2017)获得中国数学会第十三届华罗庚数学奖.

3. Classical problems

Q-process: $P'(t)|_{t=0} = Q$.
Backward Kolmogorov equation: $P'(t) = QP(t)$.
Forward Kolmogorov equation: $P'(t) = P(t)Q$.

$$p_{ij}(t) = \sum_{k \neq i} \int_0^t q_{ik} e^{-q_i(t-s)} p_{kj}(s) ds + \delta_{ij} e^{-q_i t},$$

$$p_{ij}(t) = \sum_{k \neq j} \int_0^t p_{ik}(s) q_{kj} e^{-q_j(t-s)} ds + \delta_{ij} e^{-q_j t},$$

- Q-process exist? unique?

Theorem

Let $Q = (q_{ij})$ be totally stable and conservative. Every Q-process satisfies the backward Kolmogorov equation.

3. Classical problems

Existence Theorem

Given a totally stable and conservative Q-matrix, there always exist a Q-process. In details, the minimal solution is a Q-process. Moreover, BE and FE have the same minimal solution.

Laplace transform: $p_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt$.

$$p_{ij}(\lambda) = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} p_{kj}(\lambda) + \frac{\delta_{ij}}{\lambda + q_i},$$

$$p_{ij}(\lambda) = \sum_{k \neq j} p_{ik}(\lambda) \frac{q_{kj}}{\lambda + q_j} + \frac{\delta_{ij}}{\lambda + q_i}.$$

Denote the minimal process by $P^{\min}(\lambda)$. Consider $P(\lambda) - P^{\min}(\lambda)$:

$$u_i = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} u_k, \ 0 \leq u_i \leq 1, i \in \mathbb{Z}_+.$$

Uniqueness Theorem[Feller(1957), Reuter(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$, the Q-process is unique iff the equation has only the trivial solution $u_i \equiv 0$ for some (equivalently, for all) $\lambda > 0$.

- $P(t)$ is recurrent if for each $h > 0$, $P(h)$ is recurrent. Equivalently, $\int_0^\infty p_{ii}(t)dt = \infty$.

embedding chain (Π_{ij}): $\Pi_{ij} = I_{\{q_i \neq 0\}}(1 - \delta_{ij})\dfrac{q_{ij}}{q_i} + I_{\{q_i = 0\}}\delta_{ij}$.

Recurrence Theorem[Feller(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_ij)$,

$$\int_0^\infty p_{ij}^{\min}(t)dt = \sum_{n=0}^\infty \frac{\Pi_{ij}^{(n)}}{q_j}.$$

In particular, if Q is irreducible and regular, then $P(t)$ is recurrent iff so is its embedding chain.

regular: totally stable, conservative and Q-process is unique.

Theorem

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ is recurrent iff for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} \frac{q_{ij}}{q_i} x_j, \qquad 0 \leq x_i \leq 1,\ i \geq 0$$

has only a trivial solution.

For embedding chain (Π_{ij}), $(f_{i,j_0} = \mathbb{P}_i(\sigma_{j_0} < \infty) : i \in \mathbb{Z}_+)$ is the minimal solution to the equation

$$x_i = \sum_{k \neq j_0} \Pi_{ik} x_k + \Pi_{i,j_0},\ i \in \mathbb{Z}_+.$$

(Π_{ij}) is recurrent iff $f_{i,j_0} \equiv 1$.

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix. Then the limit

$$\lim_{t\to\infty} p_{ij}(t) =: \pi_j$$

exists and it is independent of i. Moreover, we have either $\sum_j \pi_j = 1$ or $\sum_j \pi_j = 0$.

- $P(t)$ is positive recurrent or ergodic if so is $P(h)$ for every $h > 0$. Equivalently, $\lim_{t\to\infty} p_{ii}(t) = \pi_i > 0$.
- $P(t)$ is strongly ergodic or uniformly ergodic if

$$\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix and H be a non-empty finite subset of $\mathbb{Z}_+$. Define $\sigma_H = \inf\{t \geq \text{ the first jump} : X(t) \in H\}$.

Theorem[Issacson & Arnold(1978)]

(1) The Q-process is ergodic iff $\mathbb{E}_i \sigma_H < \infty$ for all $i \in H$.
(2) The Q-process is strongly ergodic iff $\sup_i \mathbb{E}_i \sigma_H < \infty$.

Theorem[Tweedie(1981)]

The Q-process is ergodic (resp. strongly erergodic) iff the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t \to \infty} |p_{ij}(t) - \pi_j| = 0$.
- Exponential ergodicity: $\lim_{t \to \infty} e^{\alpha t} |p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim_{t \to \infty} \sup_i |p_{ij}(t) - \pi_j| = 0$

$$\Longleftrightarrow \lim_{t \to \infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

- Strong ergodicity $\Rightarrow$ Exponential ergodicity $\Rightarrow$ Ordinary ergodicity.

4.1 Criterion on uniqueness of SBP

For $0 \le k < n$, define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ and

$$F_i^{(i)} = 1, \; F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \; 0 \le i < n.$$

Define

$$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \ge 0.$$

Uniqueness[J.K. Zhang(1984); S.J. Yan & M.F. Chen(1986)]

Given a totally stable and conservative single birth Q-matrix $Q = (q_{ij})$, the process is unique (non-explosive) iff $\sum_{n=0}^{\infty} m_n = \infty$.

$$(\lambda + q_i)u_i = \sum_{k \ne i} q_{ik} u_k, \; 0 \le u_i \le 1, i \in \mathbb{Z}_+.$$

4.2 Criterion on recurrence of SBP

Recurrence[Yan & Chen(1986)]

Assume the single birth Q-matrix $Q = (q_{ij})$ is regular and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

$$x_i = \sum_{j\neq 0,i} \frac{q_{ij}}{q_i} x_j, \qquad 0 \le x_i \le 1,\ i \ge 0$$

4.3 Criteria on ergodicity and strong ergodicity

Define

$$d_n = \sum_{k=1}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

Ergodicity[Yan & Chen(1986); Z.(2001)]

Assume the single birth Q-matrix $Q = (q_{ij})$ is regular and irreducible. Then the process is ergodic iff $d := \sup_{k\geq 0} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}} < \infty$; the process is strongly ergodic iff $S := \sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty$.

$H = \{0\}$

$$\sum_{j\neq i} q_{ij}(y_j - y_i) + 1 \leq 0, \; i \geq 1.$$

Define $\tau_j = \inf\{t \geq 0 : X(t) = j\}$.

Theorem[Z.(2004)]

Fix $i_0 \geq 0$. For recurrent single process, we have

$$\mathbb{E}_i\tau_{i_0} = \sum_{j=i}^{i_0-1} m_j,\ i < i_0;\ \mathbb{E}_i\tau_{i_0} = \sum_{j=i_0}^{i-1} (F_j^{(i_0)} c_{i_0} - m_j),\ i \geq i_0 + 1.$$

For recurrent Q-process, $(\mathbb{E}_i\tau_{i_0} : i \in \mathbb{Z}_+)$ is the minimal solution to the equation

$$x_{i_0} = 0,\ x_i = \sum_{j \neq i, i_0} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i},\ i \neq i_0.$$

5. Poisson equation

Given a function c, define

$$\Omega g = Qg + cg, \quad \text{where} \quad (Qg)_i = \sum_j q_{ij}(g_j - g_i).$$

Clearly, if $c \le 0$, then Ω is an operator corresponding to a single birth process with killing rates $(-c_i)$. Define

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \qquad n > i \ge 0,$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^{k} q_{nj} - c_n, \qquad 0 \le k < n.$$

Theorem 1

Given a single-birth Q-matrix $Q = (q_{ij})$ and functions c and f, the solution g to the Poisson equation

$$\Omega g = f$$

has the following representation:

$$g_n = g_0 + \sum_{0\le k\le n-1} \sum_{0\le j\le k} \frac{\widetilde{F}_k^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \quad n \ge 0.$$

In particular, the harmonic function of Ω can be represented as

$$g_n = g_0 \left(1 - \sum_{0\le k\le n-1} \sum_{0\le j\le k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad n \ge 0.$$

7. Poisson equation

$$\widetilde{F}_i^{(i)} = 1, \quad \widetilde{F}_n^{(i)} = \sum_{k=i+1}^{n} \frac{\widetilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, \quad n \geq i+1.$$

Roughly speaking, the unified treatment consists of the following three steps.

(a) Find out the Poisson equation corresponding to the problem .

(b) Apply Theorem 1 to get the solution to Poisson equation.

(c) Work out a criterion for the problem using the solution obtained in (b).

M.F. Chen & Z.(2014): Unified representation of formulas for single birth processes, Frontiers of Mathematics in China, 9(4): 761-796.

Problem	$c_i \in \mathbb{R}$	$f_i \in \mathbb{R}$
Harmonic function	$c_i \in \mathbb{R}$	$f_i \equiv 0$
Uniqueness	$c_i \equiv -\lambda < 0$	$f_i \equiv 0$
Recurrence	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})$
Ext./return probability	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$
Ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Strong ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Polynomial moment	$c_i \equiv 0$	$f_i^{(\ell)}$
Exp. moment/ergodicity	$c_i \equiv \lambda > 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$
Laplace transform	$c_i \equiv -\lambda < 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0-1)$

where $f_i^{(\ell)} = q_{ii_0}(1-\delta_{ii_0})g_{i_0} - \ell\mathbb{E}_i\sigma_{i_0}^{\ell-1}$.

Define

$$\widetilde{m}_n = \sum_{k=0}^{n} \frac{\widetilde{F}_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

Proposition

Given a totally stable and single birth conservative Q-matrix $Q = (q_{ij})$, the process is unique (non-explosive) iff $\sum_{n=0}^{\infty} m_n = \infty$.

Proof. unique iff the solution (u_i) to the equation

$$(\lambda + q_i)u_i = \sum_{j\neq i} q_{ij}u_j, \qquad i \geq 0; \qquad u_0 = 1$$

is unbounded for some (equivalently for all) $\lambda > 0$. Rewrite it as

$$\Omega u = Qu - \lambda u = 0; \qquad u_0 = 1.$$

Applying Theorem 1 to $c_i \equiv -\lambda$ and $f_i \equiv 0$, we obtain:

$$u_n = 1 + \lambda \sum_{0\leq k\leq n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)}}{q_{j,j+1}} = 1 + \lambda \sum_{0\leq k\leq n-1} \widetilde{m}_k, \qquad n \geq 0.$$

Clearly, u_n is increasing in n and then is unbounded iff $\sum_n \widetilde{m}_n = \infty$. Thus, it remains to show that $\sum_n \widetilde{m}_n = \infty$ iff $\sum_n m_n = \infty$.

Proposition

Assume the single birth Q-matrix $Q = (q_{ij})$ is non-explosive and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Proof. recurrent iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad 0 \leq x_i \leq 1, \qquad i \geq 0$$

has only zero solution, where $\Pi_{ik} = (1 - \delta_{ik}) q_{ik} / q_i$. It is easily seen that the equation has a non-trivial solution iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad i \geq 0; \qquad x_0 = 1$$

has a nonnegative bounded solution.

Rewrite the previous equation as

$$(Qx)_0 = 0, \qquad (Qx)_i = q_{i0}, \qquad i \geq 1; \qquad x_0 = 1.$$

Applying Theorem 1 to $c_i \equiv 0$ and $f_i = q_{i0}(1 - \delta_{i0})$, we obtain the unique solution as follows

$$x_0 = 1, \quad x_n = 1 + \sum_{k=1}^{n-1} F_k^{(0)} = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1.$$

Clearly, (x_n) is bounded iff $\sum_{k=0}^{\infty} F_k^{(0)} < \infty$. In other words, the equation has only a trivial solution iff $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$.

7.3 One example

Example: uniform catastrophes

Let $q_{i,i+1} = b\,i,\ i \geq 0;\ q_{ij} = a,\ j = 0, 1, \ldots, i-1$; and $q_{ij} = 0$ for other $j > i+1$, where a and b are positive constants. Then the extinction of the process has an exponential distribution

$$\mathbb{E}_n \mathrm{e}^{-\lambda \tau_0} = \frac{a}{a+\lambda}, \qquad \lambda > 0,\ n \geq 1.$$

It is surprising that the distribution is independent of b and the starting point n. Redefine $q_{01} = 1$. Then the irreducible process is indeed strongly ergodic.

P. J. Brockwell et al(1982)

- Single birth Q-matrix with absorbing boundary: $1 \leq N := \max\{i+1 : q_{i,i+1} = 0\} < \infty$. When $N = \max \emptyset = 0$, it is single birth.

  M.F. Chen(1999), Single birth processes, Chin. Ann. Math., **20B**, 77-82. Uniqueness.

- Single birth Q-matrix with immigration: $q_{0j} > 0$ for $j \geq 2$.

  Z. & Q.Q. Zhao(2010).

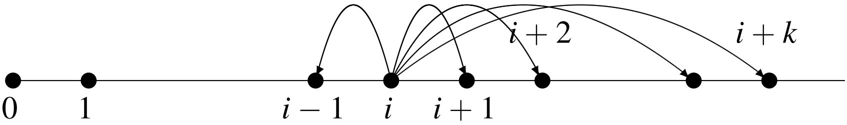
单死过程:

1. Motivation
2. Mean hitting time
3. Ergodicity and strong ergodicity

1. Motivation

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0,\ q_{i,i-j} = 0,\ i \geq 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Motivation

- branching processes: reference model

$$
\begin{aligned}
i \to i-1 &\quad \text{at rate} \quad ib_0 \\
\to i &\quad \text{at rate} \quad ib_1 \\
\to i+1 &\quad \text{at rate} \quad ib_2 \\
\to i+2 &\quad \text{at rate} \quad ib_3 \\
\to \cdots &
\end{aligned}
$$

$$b_1 = -\sum_{k\neq 1} b_k \geq -1.$$

1. Motivation

- single birth processes and dual approach.

Y.R. Li, A.G. Pakes, J. Li, A.H. Gu(2008). The limit behavior of dual Markov branching processes. J. Appl. Prob. 45, 176-189.

- stationary distribution. zero-entrance.
- quasi-stationary distribution: $\sup_{n\geq 1} \mathbf{E}_n \tau_0$.

2. Mean hitting time

- single birth process

$$\mathbb{E}_n \tau_{n+1} = m_n := \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \; n \geq 0.$$

$$x_n = \sum_{k=0}^{n-1} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n-1)}}{q_n} x_n + \frac{1}{q_n}, \; n \geq 0.$$

$$F_i^{(i)} = 1, \; F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \; 0 \leq i < n.$$

2. Mean hitting time

- single death process: $\mathbb{E}_n \tau_{n-1} = ?$

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \ n \geq 1.$$

$$q_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}, \ k > n \geq 0$$

$$G_i^{(i)} = 1, \ G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \ 0 \leq n < i.$$

2. Mean hitting time

$$\mathbb{E}_n \tau_{n-1} = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} =: m_n (n \geq 1)?$$

Lemma 1

(m_n) is the minimal nonnegative solution to

$$x_n = \sum_{k=n+1}^{\infty} \frac{q_n^{(k)}}{q_n} x_k + \frac{q_n^{(n+1)}}{q_n} x_n + \frac{1}{q_n}, \ n \geq 1.$$

$$\mathbb{E}_n \tau_{n-1} \geq m_n.$$

2. Mean hitting time

Lemma 2

$(\mathbf{E}_i\tau_{i_0} : i \geq i_0)$ is the minimal nonnegative solution to

$$x_{i_0} = 0,\ x_i = \sum_{j\neq i, j>i_0} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i},\ i > i_0.$$

$\sum_{i_0+1\leq n\leq i} m_n$ satisfies the equation above.

$$\mathbf{E}_i\tau_{i_0} \leq \sum_{i_0+1\leq n\leq i} m_n \Rightarrow \mathbb{E}_n\tau_{n-1} \leq m_n.$$

Theorem 2

$$\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} < \infty \Rightarrow \exists \text{ a QSD uniquely.}$$

3. Ergodicity and strong ergodicity

Theorem 3

Assume that the single death Q is irreducible and regular. Then the process is ergodic if and only if

$$D := \sum_{k\geq 1} q_0^{(k)} \sum_{\ell\geq k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty;$$

the process is strongly ergodic if and only

$$S := \sum_{k\geq 1} \sum_{\ell\geq k} \frac{G_k^{(\ell)}}{q_{\ell,\ell-1}} < \infty.$$

3. Ergodicity and strong ergodicity

The original branching process can be described as follows. Let $\alpha > 0$ and $(p_j : j \in \mathbb{Z}_+)$ be a probability distribution. Then the process has death rate $\alpha i p_0 : i \to i-1\,(i \geq 1)$ and growth rate $\alpha i p_{k+1} : i \to i+k\,(k \geq 1, i \in \mathbb{Z}_+)$. An extended class of branching processes:

$$q_{ij} = \begin{cases} q_{0j}, & j > i = 0; \\ -q_0, & j = i = 0; \\ r_i p_0, & j = i-1,\ i \geq 1; \\ r_i p_{k+1}, & j = i+k,\ i,k \geq 1; \\ -r_i(1-p_1), & j = i \geq 1; \\ 0, & \text{else}, \quad i,j \in \mathbb{Z}_+. \end{cases}$$

where $r_i > 0$ for all $i \geq 1$ and $0 < q_0 := \sum_{j\geq 1} q_{0j} < \infty$.

R.R. Chen(1997). An extended class of time-continuous branching processes. J. Appl. Prob., 34(1), 14-23.

3. Ergodicity and strong ergodicity

Theorem 4

Assume that the extended branching Q is irreducible and regular. Then the extended branching process is ergodic if and only if

$$\sum_{\ell\geq 1}\frac{1}{r_\ell}\left(q_0^{(\ell)}+\sum_{1\leq k\leq \ell-1}\frac{(\mathbf{f}^{*k}*\mathbf{q})_{\ell-k+1}}{p_0^k}\right)<\infty;$$

it is strongly ergodic if and only if

$$\sum_{\ell\geq 1}\frac{1}{r_\ell}\left(1+\sum_{1\leq k\leq \ell-1}\frac{(\mathbf{f}^{*k}*\mathbf{1})_{\ell-k+1}}{p_0^k}\right)<\infty.$$

where $\mathbf{f}=(\sum_{k\geq n}p_k, n\geq 2)$, $\mathbf{1}=(1,1,\cdots)$ and $\mathbf{q}=(q_0^{(n-1)}, n\geq 2)$.

$(\mathbf{a}*\mathbf{b})_n=\sum_{2\leq m\leq n}a_{n+2-m}b_m,\ n\geq 2.$

Thank you for your attention!

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