从单生到单死

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Outline

第一部分: 单生过程I

第二部分: 单生过程II

第三部分: 单死过程

Outline

第一部分: 单生过程I

1. Background

- Q-process: on $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t), i, j \in \mathbb{Z}_+)$ satisfies

  (1) Normal condition: $p_{ij}(t) \geq 0,\ \sum_j p_{ij}(t) \leq 1,\ i, j \in \mathbb{Z}_+, t \geq 0.$

  (2) Chapman-Kolmogorov equation: $p_{ij}(t+s) = \sum_k p_{ik}(t) p_{kj}(s).$

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij},\ i, j \in \mathbb{Z}_+.$

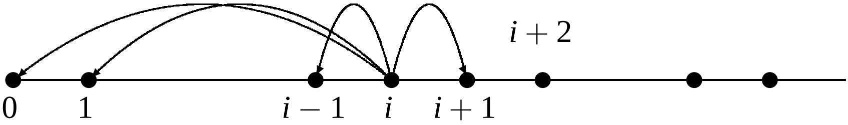
  (4) Q-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij},\ i, j \in \mathbb{Z}_+$, i.e. the Q-matrix $Q = (q_{ij})$ is the derivative matrix at time 0 of $P(t)$,

$$0 \leqslant q_{ij} < \infty,\ i \neq j;\ \sum_{j\neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty,\ i \in \mathbb{Z}_+.$$

- totally stable: $q_i < \infty,\ i \in \mathbb{Z}_+$.
- conservative: $q_i = \sum_{j\neq i} q_{ij},\ i \in \mathbb{Z}_+$, i.e. $\sum_{j\in\mathbb{Z}_+} q_{ij} = 0$.

1. Background

Single birth process: $Q = (q_{ij} : i, j \geq 0)$: $q_{i,i+1} > 0$, $q_{i,i+j} = 0$, $i \geq 0, j \geq 2$.



$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Background

1. The exit boundary of the process consists at most one single extremal point
2. Fundamental comparison tool in studying more complex processes, such as infinite-dimensional reaction-diffusion processes
3. Usually the single birth process is non-symmetric (irreversible)
4. Applications in other fields

1. Background

1. Upwardly skip-free processes (W.J. Anderson, 1991);
2. generalized birth and death processes (J.K. Zhang, 1984);
3. Population processes: birth and death processes with catastrophes; birth, death and catastrophe processes

$$\begin{aligned} i \to i+1 &\quad \text{at rate} \quad b_i \\ \to i-1 &\quad\quad a_i + c_i f_{i,i-1} \\ \to i-2 &\quad\quad c_i f_{i,i-2} \\ \to \cdots &\quad\quad \cdots \\ \to 0 &\quad\quad c_i f_{i0} \end{aligned}$$

where $\sum_{j=0}^{i-1} f_{ij} = 1$.

1. Background

Brockwell, Gani, Resnick and Pakes et al (1982-1986) for some special catastrophes f_{ij} ($b_i = b + \lambda i$, $a_i = 0$, $c_i = ci$, $i \geq 0$).

geometric: $f_{ij} = p(1-p)^{i-1-j}\,(1 \leq j < i); f_{i0} = (1-p)^{i-1}$;

uniform: $f_{ij} = 1/i\,(0 \leq j < i)$;

binomial: $f_{ij} = \binom{i-1}{j} p^j (1-p)^{i-1-j}\,(0 \leq j < i)$.

Aim: extinction times and probability of extinction.

Keys: generating function of Q-resolvent.

1. Background

B. Cairns & P. Pollett (2004).

$$
\begin{aligned}
i \to i+1 \quad & \text{at rate} \quad g_i b \\
\to i-1 \quad & \qquad\qquad g_i d_1 \\
\to \cdots \quad & \qquad\qquad \cdots \\
\to 1 \quad & \qquad\qquad g_i d_{i-1} \\
\to 0 \quad & \qquad\qquad g_i \sum_{k\geq i} d_k
\end{aligned}
$$

where $b + \sum_{k\geq 1} d_k = 1$.

1. Background

single birth process:

S.J. Yan & M.F. Chen(1986). Multidimensional Q-processes, Chinese Ann. Math., **7(B)**(1), 90-110.

M.F. Chen(2004). From Markov Chains to Non-Equilibrium Particle Systems, Second Edition, World Scientific, Singapore.

陈木法, 毛永华(2007). 随机过程导论, 高等教育出版社, 北京.

Population processes:

W.J. Anderson(1991). Continuous-Time Markov Chains, Springer-Verlag, New York.

- Uniqueness. J.K. Zhang(1984), S.J. Yan & M.F. Chen(1986).
- Recurrence. Yan & Chen(1986).
- Extinction/return probability. P.J. Brockwell(1986), W.J. Anderson(1991), Z.T. Hou & Q.F. Guo(1978, 1988). Chen & Z. (2014)
- Ergodicity. Yan & Chen(1986).
- Strong ergodicity. Z.(2001).
- Polynomial moments and stationary dist. Z.(2003, 2004, 2013)
- Exponential moments. Z.(2003)
- Laplace transform. Brockwell(1986), Anderson(1991)
- Poisson equation. Chen & Z.(2014)
- Mean occupation time. P.S. Li & Z.(2017)

2. Minimal nonnegative solution

For $a_{ij} \geq 0, b_i \geq 0$, consider the equation

$$x_i = \sum_k a_{ik} x_k + b_i, \ i \in \mathbb{Z}_+.$$

Theorem[the first successive approximation scheme]

The minimal solution (x_i^*) always exists uniquely. Furthermore, it can be obtained by the following procedure

$$x_i^{(0)} = 0, \ x_i^{(n+1)} = \sum_k a_{ik} x_k^{(n)} + b_i, \ i \in \mathbb{Z}_+, n \geq 0,$$

then $x_i^{(n)} \uparrow x_i^*$ as $n \to \infty$.

Comparison Theorem

Assume that $\tilde{a}_{ij} \geq a_{ij}, \tilde{b}_i \geq b_i$. Then for any solution $(\tilde{x}_i)$ to

$$\tilde{x}_i = \sum_k \tilde{a}_{ik}\tilde{x}_k + \tilde{b}_i, \ i \in \mathbb{Z}_+,$$

we have $\tilde{x}_i \geq x_i^*$.

Theorem[the second successive approximation scheme]

Assume that $b_i^{(n)} \geq 0$. Define

$$\tilde{x}_i^{(1)} = b_i^{(1)}, \ \tilde{x}_i^{(n+1)} = \sum_k a_{ik}\tilde{x}_k^{(n)} + b_i^{(n+1)}, \ i \in \mathbb{Z}_+, n \geq 1.$$

If $b_i^{(n)} \uparrow b_i$ (or, $\sum_{n=1}^\infty b_i^{(n)} = b_i$), then $\tilde{x}_i^{(n)} \uparrow x_i^*$ (or $x_i^* = \sum_{n=1}^\infty \tilde{x}_i^{(n)}$). In particular, if we set $\tilde{x}_i^{(1)} = b_i, \ \tilde{x}_i^{(n+1)} = \sum_k a_{ik}\tilde{x}_k^{(n)}$, then $x_i^* = \sum_{n=1}^\infty \tilde{x}_i^{(n)}$.

Localization Theorem

Let $G \subset \mathbb{Z}_+$ and $(\tilde{x}_i^* : i \in G(\text{resp.} i \in \mathbb{Z}_+))$ be the minimal solution to

$$x_i = \sum_{k \in G} a_{ik} x_k + \sum_{k \in G^c} a_{ik} x_k^* + b_i, \ i \in G(\text{resp.} i \in \mathbb{Z}_+).$$

Then we have $\tilde{x}_i^* = x_i^*, i \in G(\text{resp.} i \in \mathbb{Z}_+)$.

Theorem

Assume that $\sum_j a_{ij} + b_i \leq 1$. Then the maximal solution to

$$x_i = \sum_k a_{ik} x_k + b_i, \ 0 \leq x_i \leq 1, i \in \mathbb{Z}_+,$$

can be obtained by the following procedure

$$x_i^{(0)} = 1, \ x_i^{(n+1)} = \sum_k a_{ik} x_k^{(n)} + b_i, \ i \in \mathbb{Z}_+, n \geq 0,$$

Z.T. Hou & Q.F. Guo. Homogeneous Denumerable Markov Processes. Beijing: Science Press, 1978(in Chinese); English translation, Beijing: Science Press and Springer, 1988

3. Classical problems

Q-process: $P'(t)|_{t=0} = Q$.
Backward Kolmogorov equation: $P'(t) = QP(t)$.
Forward Kolmogorov equation: $P'(t) = P(t)Q$.

$$p_{ij}(t) = \sum_{k\neq i} \int_0^t q_{ik} e^{-q_i(t-s)} p_{kj}(s) ds + \delta_{ij} e^{-q_i t},$$

$$p_{ij}(t) = \sum_{k\neq j} \int_0^t p_{ik}(s) q_{kj} e^{-q_j(t-s)} ds + \delta_{ij} e^{-q_j t},$$

- Q-process exist? unique?

Theorem

Let $Q = (q_{ij})$ be totally stable and conservative. Every Q-process satisfies the backward Kolmogorov equation.

Existence Theorem

Given a totally stable and conservative Q-matrix, there always exist a Q-process. In details, the minimal solution is a Q-process. Moreover, BE and FE have the same minimal solution.

Laplace transform: $p_{ij}(\lambda) = \int_0^\infty e^{-\lambda t} p_{ij}(t)dt$.

$$p_{ij}(\lambda) = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} p_{kj}(\lambda) + \frac{\delta_{ij}}{\lambda + q_i},$$

$$p_{ij}(\lambda) = \sum_{k \neq j} p_{ik}(\lambda) \frac{q_{kj}}{\lambda + q_j} + \frac{\delta_{ij}}{\lambda + q_i}.$$

Denote the minimal process by $P^{\min}(\lambda)$. Consider $P(\lambda) - P^{\min}(\lambda)$:

$$u_i = \sum_{k \neq i} \frac{q_{ik}}{\lambda + q_i} u_k, \; 0 \leq u_i \leq 1, i \in \mathbb{Z}_+.$$

Uniqueness Theorem[Feller(1957), Reuter(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_{ij})$, the Q-process is unique iff the equation has only the trivial solution $u_i \equiv 0$ for some (equivalently, for all) $\lambda > 0$.

- $P(t)$ is recurrent if for each $h > 0$, $P(h)$ is recurrent. Equivalently, $\int_0^\infty p_{ii}(t)dt = \infty$.

embedding chain (Π_{ij}): $\Pi_{ij} = I_{\{q_i \neq 0\}}(1 - \delta_{ij})\dfrac{q_{ij}}{q_i} + I_{\{q_i = 0\}}\delta_{ij}$.

Recurrence Theorem[Feller(1957)]

For a given totally stable and conservative Q-matrix $Q = (q_ij)$,

$$\int_0^\infty p_{ij}^{\min}(t)dt = \sum_{n=0}^{\infty} \frac{\Pi_{ij}^{(n)}}{q_j}.$$

In particular, if Q is irreducible and regular, then $P(t)$ is recurrent iff so is its embedding chain.

regular: totally stable, conservative and Q-process is unique.

Theorem

For a regular and irreducible $Q = (q_{ij})$, $P(t)$ is recurrent iff for some (equivalently, all) j_0,

$$x_i = \sum_{j \neq j_0, i} \frac{q_{ij}}{q_i} x_j, \qquad 0 \leq x_i \leq 1,\ i \geq 0$$

has only a trivial solution.

For embedding chain (Π_{ij}), $(f_{i,j_0} = \mathbb{P}_i(\sigma_{j_0} < \infty) : i \in \mathbb{Z}_+)$ is the minimal solution to the equation

$$x_i = \sum_{k \neq j_0} \Pi_{ik} x_k + \Pi_{i,j_0},\ i \in \mathbb{Z}_+.$$

(Π_{ij}) is recurrent iff $f_{i,j_0} \equiv 1$.

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix. Then the limit

$$\lim_{t\to\infty} p_{ij}(t) =: \pi_j$$

exists and it is independent of i. Moreover, we have either $\sum_j \pi_j = 1$ or $\sum_j \pi_j = 0$.

- $P(t)$ is positive recurrent or ergodic if so is $P(h)$ for every $h > 0$. Equivalently, $\lim_{t\to\infty} p_{ii}(t) = \pi_i > 0$.
- $P(t)$ is strongly ergodic or uniformly ergodic if

$$\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

Let $Q = (q_{ij})$ be a regular irreducible Q-matrix and H be a non-empty finite subset of $\mathbb{Z}_+$. Define $\sigma_H = \inf\{t \geq \text{ the first jump} : X(t) \in H\}$.

Theorem[Issacson & Arnold(1978)]

(1) The Q-process is ergodic iff $\mathbb{E}_i\sigma_H < \infty$ for all $i \in H$.
(2) The Q-process is strongly ergodic iff $\sup_i \mathbb{E}_i\sigma_H < \infty$.

Theorem[Tweedie(1981)]

The Q-process is ergodic (resp. strongly erergodic) iff the equation

$$\begin{cases} \sum_j q_{ij}y_j \leq -1, \ i \notin H \\ \sum_{i\in H} \sum_{j\neq i} q_{ij}y_j < \infty \end{cases}$$

has a finite (resp. bounded) nonnegative solution.

Theorem($y_i = 1/c(i \neq k), y_k = 0$)

Given a regular and irreducible Q-matrix $Q = (q_{ij})$. If there exists some k so that $c := \inf_{i\neq k} q_{ik} > 0$, then the Q-process is str. erg.

Assume that Q is irreducible. $\pi = (\pi_i)$ is stationary distribution.

- Ordinary ergodicity: $\lim_{t\to\infty} |p_{ij}(t) - \pi_j| = 0$.
- Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0$.
- Strong ergodicity: $\lim_{t\to\infty} \sup_i |p_{ij}(t) - \pi_j| = 0$

$$\iff \lim_{t\to\infty} e^{\beta t} \sup_i |p_{ij}(t) - \pi_j| = 0.$$

- Strong ergodicity $\Rightarrow$ Exponential ergodicity $\Rightarrow$ Ordinary ergodicity.

4. Criterion on uniqueness of SBP

For $0 \le k < n$, define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ and

$$F_i^{(i)} = 1,\ F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)},\ 0 \le i < n.$$

Define

$$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \ge 0.$$

Uniqueness[J.K. Zhang(1984); S.J. Yan & M.F. Chen(1986)]

Given a totally stable and conservative single birth Q-matrix $Q = (q_{ij})$, the process is unique (non-explosive) iff $\sum_{n=0}^{\infty} m_n = \infty$.

$$(\lambda + q_i)u_i = \sum_{k \ne i} q_{ik} u_k,\ 0 \le u_i \le 1, i \in \mathbb{Z}_+.$$

For birth-death (a_i, b_i): $a_i = q_{i,i-1} > 0, i \geq 1; b_i = q_{i,i+1} > 0, i \geq 0.$

$$m_n = \frac{1}{\mu_n b_n}\mu[0,n], \qquad n \geq 0,$$

where

$$\mu_0 = 1, \qquad \mu_i = \frac{b_0 b_1 \cdots b_{i-1}}{a_1 a_2 \cdots a_i}, \quad i \geq 1,$$

and $\mu[i,k] = \sum_{j=i}^{k} \mu_j$.

Uniqueness Criterion for BDP

Given a totally stable and conservative birth-death Q matrix (a_i, b_i). The the birth-death is unique iff $R := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n}\mu[0,n] = \infty$.

5. Criterion on recurrence of SBP

Recurrence[Yan & Chen(1986)]

Assume the single birth Q-matrix $Q = (q_{ij})$ is regular and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

$$x_i = \sum_{j\neq 0,i} \frac{q_{ij}}{q_i} x_j, \qquad 0 \le x_i \le 1,\ i \ge 0$$

For birth-death Q matrix (a_i, b_i):

$$F_n^{(0)} = \frac{b_0}{\mu_n b_n},\ n \ge 0.$$

Recurrence Criterion for BDP

Given a regular birth-death Q-matrix (a_i, b_i). The the birth-death is recurrent iff $\sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} = \infty$.

6. Criteria on ergodicity and strong ergodicity

Define

$$d_n = \sum_{k=1}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

Ergodicity[Yan & Chen(1986); Z.(2001)]

Assume the single birth Q-matrix $Q = (q_{ij})$ is regular and irreducible. Then the process is ergodic iff $d := \sup_{k\geq 0} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}} < \infty$; the process is strongly ergodic iff $S := \sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty$.

$H = \{0\}$

$$\sum_{j\neq i} q_{ij}(y_j - y_i) + 1 \leq 0, \; i \geq 1.$$

For birth-death Q matrix (a_i, b_i):

$$d_n = \frac{1}{\mu_n b_n}\mu[1, n], \qquad n \geq 0.$$

Ergodicity Criteria for BDP

Given a regular birth-death Q-matrix (a_i, b_i). The the birth-death chain is ergodic iff $\mu[0, \infty) < \infty$; it is strongly ergodic iff $S := \sum_{n=0}^{\infty} \frac{1}{\mu_n b_n} \sum_{k=n+1}^{\infty} \mu_k < \infty$.

H.J. Zhang, X. Lin & Z.T. Hou, 2000.

7. Stationary distribution

Stationary distribution(Y.H. Zhang, 2004)

Assume that the process is ergodic. Define

$$c_k := \varlimsup_{i\to\infty} \frac{\sum_{j=k}^{i} m_j}{\sum_{j=k}^{i} F_j^{(k)}}. \quad \text{Then} \quad \pi_k = \frac{1}{q_{k,k+1}c_k}, \quad k \geq 0.$$

Example 1

$b_i = q_{i,i+1} > 0\,(i \geq 0)$, $a_i = q_{i,i-1} > 0\,(i \geq 1)$. Define $\mu_0 = 1$, $\mu_i = b_0 \cdots b_{i-1}/a_1 \cdots a_i\,(i \geq 1)$ and $\mu = \sum_{i=0}^{\infty} \mu_i$. Then $c_i = \mu/(\mu_i b_i)$, $\pi_i = \mu_i/\mu\,(i \geq 0)$.

Example 2

$q_{i,i+1} = 1\,(i \geq 0)$ and $q_{i0} = 1\,(i \geq 1)$. Then $c_k = 2^{k+1}$, $\pi_k = 2^{-k-1}\,(k \geq 0)$.

Example 3

$q_{i,i+1} = 1\,(i \geq 0)$, $q_{10} = 1$ and $q_{i,i-2} = 1\,(i \geq 2)$. Then $\{F_j^{(k)}\}_{j=k}^{\infty}$ is a Fibonacci sequence: $F_{k+n}^{(k)} = \frac{1}{\sqrt{5}}[A^{n+1} - (-B)^{n+1}]$, $n \geq 0, k \geq 0$, where $A = (\sqrt{5}+1)/2, B = (\sqrt{5}-1)/2$. Then $c_k = A^{k+2}$, $\pi_k = B^{k+2}\,(k \geq 0)$.

Example 4(uniform catastrophes)

$q_{i,i+1} = \lambda_i := a + \lambda i\,(i \geq 0)$, $q_{ij} = \beta\,(0 \leq j < i)$, where $a > 0$, $\lambda > 0$, $\beta > 0$. Then

$$\pi_0 = \frac{\beta}{a+\beta}, \quad \pi_k = \frac{(k+1)\beta}{a+\beta} \prod_{j=0}^{k-1} \frac{\lambda_j}{\lambda_{j+1} + (j+2)\beta}, \quad k \geq 1.$$

In particular, when $a = \lambda$, then $\pi_k = \beta a^k/(a+\beta)^{k+1}\,(k \geq 0)$. Furthermore, when $a = \lambda = \beta = 1$, then $\pi_k = 2^{-k-1}\,(k \geq 0)$.

Define $\sigma_j = \inf\{t \geq \text{the first jumping time: } X(t) = j\}$ and

$$\tau_j = \inf\{t \geq 0 : X(t) = j\}.$$

Theorem[Z.(2004)]

Fix $i_0 \geq 0$. For recurrent single process, we have

$$\mathbb{E}_i \tau_{i_0} = \sum_{j=i}^{i_0-1} m_j, \ i < i_0;$$

$$\mathbb{E}_i \tau_{i_0} = \sum_{j=i_0}^{i-1} (F_j^{(i_0)} c_{i_0} - m_j), \ i \geq i_0 + 1.$$

For recurrent Q-process, $(\mathbb{E}_i\tau_{i_0} : i \in \mathbb{Z}_+)$ is the minimal solution to the equation

$$x_{i_0} = 0,\ x_i = \sum_{j\neq i,i_0} \frac{q_{ij}}{q_i} x_j + \frac{1}{q_i},\ i \neq i_0.$$

$$\mathbb{E}_i\sigma_i = \frac{q_{i,i+1}c_i}{q_i},\ i \geq 0.$$

$$\pi_i q_i \mathbb{E}_i\sigma_i = 1 \Rightarrow \pi_i = \frac{1}{q_{i,i+1}c_i}.$$

$$\mathbb{E}_1\tau_0 = d,\ \mathbb{E}_i\tau_0 = \sum_{j=0}^{i-1}(F_j^{(0)}d - d_j),\ i \geq 1.$$

$$\mathbb{E}_0\sigma_0 = q_{01}^{-1} + \mathbb{E}_1\tau_0 = q_{01}^{-1} + d.$$

Probabilistic meaning:

- Uniqueness. $R = \mathbb{E}_0 \sigma_\infty = \infty$.
- Recurrence. $\mathbb{P}_0(\sigma_0 < \infty) = 1 - (\sum_{n=0}^{\infty} F_n^{(0)})^{-1} = 1$.
- Ergodicity. $d = \mathbb{E}_1 \tau_0 < \infty \Leftrightarrow \mathbb{E}_0 \sigma_0 < \infty$.
- Strong ergodicity. $S = \sup_{i \geq 1} \mathbb{E}_i \tau_0 < \infty$.

8. Criterion on ℓ-ergodicity

For a positive integer ℓ, the recurrent chain $P(t)$ is said to be ℓ-ergodic if $\mathbb{E}_j\sigma_j^\ell < \infty$ for some (and hence for all) $j \in E$.

1-ergodic = positive recurrent (ergodic), 0-ergodic = null recurrent.

Discrete time:

- J.G. Kemeny, J.L. Snell and A.W. Knapp, 1976
- Y.H. Mao, 2003
- Z.T. Hou and Y.Y. Liu, 2003. Queue Theory

Continuous time:

- P. Coolen-Schrijner and E.A. van Doorn, 2002
- Y.H. Mao, 2004

- ℓ-ergodicity provides an algebraic convergence rate: $p_{ij}(t) - \pi_j = o(t^{-(\ell-1)})$ as $t \to \infty$.
- Given a regular birth-death (a_i, b_i). Assume that $P(t)$ is recurrent. Then $P(t)$ is ℓ-ergodic iff $\sum_{i=1}^{\infty} \mu_i \mathbb{E}_i \sigma_0^{\ell-1} < \infty$,

$$\mathbb{E}_i \sigma_0^n = n \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=j+1}^{\infty} \mu_k \mathbb{E}_k \sigma_0^{n-1}, \; i \geq 1, \, n \geq 1.$$

Q-process is ℓ-ergodic iff $\sum_{i=1}^{\infty} \pi_i \mathbb{E}_i \sigma_0^{\ell-1} < \infty$.

Q-process is ℓ-regodic if and only if for some (and then for all) $j \in E$, the following system has a nonnegative finite solution

$$\begin{cases} \sum_k q_{ik} y_k \leq -\ell \mathbb{E}_i \sigma_j^{\ell-1}, & i \neq j, \\ \sum_{k \neq j} q_{jk} y_k < \infty. \end{cases}$$

Define

$$d_n^{(\ell)} = \sum_{k=1}^{n} \frac{F_n^{(k)} \mathbb{E}_k \sigma_0^{\ell-1}}{q_{k,k+1}}, \; n \geq 0; \, d^{(\ell)} := \varlimsup_{i\to\infty} \frac{\sum_{j=0}^{i} d_j^{(\ell)}}{\sum_{j=0}^{i} F_j^{(0)}}.$$

Theorem

Given a regular irreducible single birth Q-matrix $Q = (q_{ij})$. Assume that the Q-process is $(\ell - 1)$-ergodic for some $\ell \geq 1$. Then the single birth process is ℓ-ergodic if and only if $d^{(\ell)} < \infty$.

Given a regular irreducible Q-matrix $Q = (q_{ij})$. Assume that the Q-process is $(n-1)$-ergodic Then $(\mathbb{E}_i \sigma_0^n)$ is the minimal solution to the equation:

$$x_0 = 0, \; x_i = \sum_{j \neq i} \frac{q_{ij}}{q_i} x_j + \frac{n}{q_i} \mathbb{E}_i \sigma_0^{n-1}, \; i \geq 1.$$

第二部分: 单生过程II

1. Poisson equation
2. Uniqueness
3. Recurrence, extinction/return probability
4. Ergodicity, strong ergodicity, and the first moment of return time
5. Polynomial moments of hitting time and life time
6. Exponential ergodicity and Laplace transform of return time
7. Examples
8. Related topics and applications

Given a function c, define

$$\Omega g = Qg + cg, \quad \text{where} \quad (Qg)_i = \sum_j q_{ij}(g_j - g_i).$$

Clearly, if $c \le 0$, then Ω is an operator corresponding to a single birth process with killing rates $(-c_i)$. Define

$$\widetilde{F}_i^{(i)} = 1, \qquad \widetilde{F}_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \qquad n > i \ge 0,$$

$$\tilde{q}_n^{(k)} = q_n^{(k)} - c_n := \sum_{j=0}^{k} q_{nj} - c_n, \qquad 0 \le k < n.$$

Theorem 1

Given a single-birth Q-matrix $Q = (q_{ij})$ and functions c and f, the solution g to the Poisson equation

$$\Omega g = f$$

has the following representation:

$$g_n = g_0 + \sum_{0 \le k \le n-1} \sum_{0 \le j \le k} \frac{\widetilde{F}_k^{(j)} (f_j - c_j g_0)}{q_{j,j+1}}, \quad n \ge 0.$$

In particular, the harmonic function of Ω can be represented as

$$g_n = g_0 \left(1 - \sum_{0 \le k \le n-1} \sum_{0 \le j \le k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j,j+1}} \right), \quad n \ge 0.$$

Proof For each $i \geq 0$, we have

$$
\begin{aligned}
(\Omega g)_i &= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} q_{ij} \sum_{k=j}^{i-1} (g_{k+1} - g_k) + c_i g_i \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} \sum_{j=0}^{k} q_{ij} (g_{k+1} - g_k) + c_i g_i \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} \left(\sum_{j=0}^{k} q_{ij} - c_i \right) (g_{k+1} - g_k) + c_i g_0 \\
&= q_{i,i+1}(g_{i+1} - g_i) - \sum_{0 \leq j \leq i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + c_i g_0.
\end{aligned}
$$

$$
g_{i+1} - g_i = \frac{1}{q_{i,i+1}} \left(\sum_{0 \leq k \leq i-1} \tilde{q}_i^{(k)} (g_{k+1} - g_k) + f_i - c_i g_0 \right).
$$

$$g_{i+1} - g_i = \sum_{j=0}^{i} \frac{\widetilde{F}_i^{(j)}(f_j - c_j g_0)}{q_{j,j+1}}, \qquad i \geq 0.$$

Proposition1

For given f, the sequence (h_n) defined successively by

$$h_n = \frac{1}{q_{n,n+1}}\left(f_n + \sum_{i \leq k \leq n-1} \tilde{q}_n^{(k)} h_k\right), \qquad n \geq i$$

has an unified expression as follows

$$h_n = \sum_{k=i}^{n} \frac{\widetilde{F}_n^{(k)} f_k}{q_{k,k+1}}, \qquad n \geq i.$$

1. Poisson equation

$$\widetilde{F}_i^{(i)} = 1, \quad \widetilde{F}_n^{(i)} = \sum_{k=i+1}^{n} \frac{\widetilde{F}_n^{(k)} \tilde{q}_k^{(i)}}{q_{k,k+1}}, \quad n \geq i+1.$$

Roughly speaking, the unified treatment consists of the following three steps.

(a) Find out the Poisson equation corresponding to the problem .

(b) Apply Theorem 1 to get the solution to Poisson equation.

(c) Work out a criterion for the problem using the solution obtained in (b).

M.F. Chen & Z.(2014): Unified representation of formulas for single birth processes, Frontiers of Mathematics in China, 9(4): 761-796.

Problem	$c_i \in \mathbb{R}$	$f_i \in \mathbb{R}$
Harmonic function	$c_i \in \mathbb{R}$	$f_i \equiv 0$
Uniqueness	$c_i \equiv -\lambda < 0$	$f_i \equiv 0$
Recurrence	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})$
Ext./return probability	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0 - 1)$
Ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Strong ergodicity	$c_i \equiv 0$	$f_i = q_{i0}(1-\delta_{i0})g_0 - 1$
Polynomial moment	$c_i \equiv 0$	$f_i^{(\ell)}$
Exp. moment/ergodicity	$c_i \equiv \lambda > 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0 - 1)$
Laplace transform	$c_i \equiv -\lambda < 0$	$f_i = q_{i0}(1-\delta_{i0})(g_0 - 1)$

where $f_i^{(\ell)} = q_{ii_0}(1-\delta_{ii_0})g_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}$.

2. Uniqueness

Define

$$\widetilde{m}_n = \sum_{k=0}^{n} \frac{\widetilde{F}_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

Proposition

Given a totally stable and single birth conservative Q-matrix $Q = (q_{ij})$, the process is unique (non-explosive) iff $\sum_{n=0}^{\infty} m_n = \infty$.

Proof. unique iff the solution (u_i) to the equation

$$(\lambda + q_i)u_i = \sum_{j\neq i} q_{ij}u_j, \qquad i \geq 0; \qquad u_0 = 1$$

is unbounded for some (equivalently for all) $\lambda > 0$. Rewrite it as

$$\Omega u = Qu - \lambda u = 0; \qquad u_0 = 1.$$

Applying Theorem 1 to $c_i \equiv -\lambda$ and $f_i \equiv 0$, we obtain:

$$u_n = 1 + \lambda \sum_{0\leq k\leq n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)}}{q_{j,j+1}} = 1 + \lambda \sum_{0\leq k\leq n-1} \widetilde{m}_k, \qquad n \geq 0.$$

Clearly, u_n is increasing in n and then is unbounded iff $\sum_n \widetilde{m}_n = \infty$. Thus, it remains to show that $\sum_n \widetilde{m}_n = \infty$ iff $\sum_n m_n = \infty$.

3. Recurrence, extinction/return probability

Proposition

Assume the single birth Q-matrix $Q = (q_{ij})$ is non-explosive and irreducible. Then the process is recurrent iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Proof. recurrent iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad 0 \leq x_i \leq 1, \qquad i \geq 0$$

has only zero solution, where $\Pi_{ik} = (1 - \delta_{ik}) q_{ik}/q_i$. It is easily seen that the equation has a non-trivial solution iff the equation

$$x_i = \sum_{k \neq 0} \Pi_{ik} x_k, \qquad i \geq 0; \qquad x_0 = 1$$

has a nonnegative bounded solution.

Rewrite the previous equation as

$$(Qx)_0 = 0, \qquad (Qx)_i = q_{i0}, \qquad i \geq 1; \qquad x_0 = 1.$$

Applying Theorem 1 to $c_i \equiv 0$ and $f_i = q_{i0}(1 - \delta_{i0})$, we obtain the unique solution as follows

$$x_0 = 1, \quad x_n = 1 + \sum_{k=1}^{n-1} F_k^{(0)} = \sum_{k=0}^{n-1} F_k^{(0)}, \quad n \geq 1.$$

Clearly, (x_n) is bounded iff $\sum_{k=0}^{\infty} F_k^{(0)} < \infty$. In other words, the equation has only a trivial solution iff $\sum_{k=0}^{\infty} F_k^{(0)} = \infty$.

P. J. Brockwell(1986), W. J. Anderson(1991). Hou & Guo(1978, 1988)

Proposition

Let the single birth Q-matrix $Q = (q_{ij})$ be non-explosive and irreducible. Then the return/extinction probability is as follows:

$$\mathbb{P}_0(\sigma_0 < \infty) = \frac{\sum_{k=1}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad \mathbb{P}_n(\sigma_0 < \infty) = \frac{\sum_{k=n}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \quad n \geq 1.$$

Furthermore, $\mathbb{P}_n(\sigma_0 < \infty) = 1$ for all $n \geq 0$ iff $\mathbb{P}_0(\sigma_0 < \infty) = 1$, equivalently iff $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$.

Proof. $(\mathbb{P}_i(\sigma_0 < \infty) : i \in E)$ is the minimal nonnegative solution to the equation

$$x_i = \sum_{k \neq 0, i} \frac{q_{ik}}{q_i} x_k + \frac{q_{i0}}{q_i}(1 - \delta_{i0}), \qquad i \in E.$$

The last equation is equivalent to

$$(Qx)_i = q_{i0}(1 - \delta_{i0})(x_0 - 1), \qquad i \geq 0.$$

Applying Theorem 1 to $c_i \equiv 0$ and $f_i = q_{i0}(1 - \delta_{i0})(x_0 - 1)$, we obtain the solution to the last equation:

$$x_n = x_0 \sum_{0 \leq k \leq n-1} F_k^{(0)} - \sum_{1 \leq k \leq n-1} F_k^{(0)}, \qquad n \geq 0.$$

$$x_0^* = 1 - \frac{1}{\sum_{k=0}^{\infty} F_k^{(0)}}, \qquad x_n^* = 1 - \frac{\sum_{k=0}^{n-1} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \qquad n \geq 1.$$

Consider the occupation time of the state j:

$$T_j := \int_0^\infty \mathbf{1}_{\{X(t)=j\}} \mathrm{d}\, t.$$

By the return probability and the embedding chain, we get

Proposition[P.S. Li & Z.(2017)]

Let the single birth Q-matrix $Q = (q_{ij})$ be regular and irreducible. Then the mean occupation time

$$\mathbf{E}_i T_j = \frac{1}{q_{j,j+1}} \sum_{n=i\vee j}^{\infty} F_n^{(j)}.$$

Define

$$\tilde{d}_n = \sum_{1\le j\le n} \frac{\widetilde{F}_n^{(j)}}{q_{j,j+1}}, \quad n \ge 0.$$

Proposition[Yan & Chen(1986); Z.(2001, 2003)]

Assume that the single birth Q-matrix $Q = (q_{ij})$ is irreducible and corresponding process is recurrent. Then

$$\mathbb{E}_0\sigma_0 = \frac{1}{q_{01}} + d, \qquad \mathbb{E}_n\sigma_0 = \sum_{k=0}^{n-1} \left(F_k^{(0)} d - d_k\right), \quad n \ge 1,$$

Proposition(contin.)

where

$$d = \varlimsup_{k\to\infty} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}} = \lim_{n\to\infty} \frac{d_n}{F_n^{(0)}} \quad \text{if the limit exists.}$$

Furthermore, the process is ergodic (i.e. positive recurrent) iff $d < \infty$; and it is strongly ergodic iff $\sup_{k\in E} \sum_{n=0}^{k} \left(F_n^{(0)} d - d_n\right) < \infty$. Actually, for the last conclusion, the recurrence assumption can be replaced by the uniqueness one.

Proof. $(\mathbb{E}_i\sigma_0 : i \in E)$ is the minimal nonnegative solution (x_i^*) to the equation

$$x_i = \frac{1}{q_i}\sum_{k\notin\{0,i\}} q_{ik}x_k + \frac{1}{q_i}, \qquad i \in E.$$

Then, we have

$$(Qx)_i = q_{i0}x_0 - 1, \qquad i \geq 1; \qquad (Qx)_0 = -1.$$

Applying Theorem 1 to $c = 0$ and $f_i = q_{i0}(1-\delta_{i0})x_0 - 1$ $(i \geq 0)$, we obtain the solution to the last equation:

$$x_n = \sum_{k=0}^{n-1}\left[F_k^{(0)}\left(x_0 - \frac{1}{q_{01}}\right) - d_k\right], \quad n \geq 1.$$

$$x_0^* = \frac{1}{q_{01}} + \sup_{n\geq 1} \frac{\sum_{k=0}^{n-1} d_k}{\sum_{k=0}^{n-1} F_k^{(0)}}$$

and

$$x_n^* = \sum_{k=0}^{n-1} \left(F_k^{(0)} \sup_{n\geq 1} \frac{\sum_{j=0}^{n-1} d_j}{\sum_{j=0}^{n-1} F_j^{(0)}} - d_k \right), \quad n \geq 1.$$

Proposition[Z.(2004, 2013); Chen & Z.(2014)]

Assume that the single birth Q-matrix $Q = (q_{ij})$ is irreducible and the corresponding process is $(\ell - 1)$-ergodic ($\ell \geq 1$), i.e. $\mathbb{E}_i \sigma_{i_0}^{\ell-1} < \infty$ for every $i \geq 0$. When $\ell = 1$, assume additionally that the process is unique. Then we have

$$\mathbb{E}_n \sigma_{i_0}^{\ell} = \begin{cases} \ell \sum_{n \leq k \leq i_0 - 1} v_k^{(\ell)} + \left[1 - \sum_{n \leq k \leq i_0 - 1} u_k\right] \mathbb{E}_{i_0} \sigma_{i_0}^{\ell} & 0 \leq n \leq i_0; \\ -\ell \sum_{i_0 \leq k \leq n-1} v_k^{(\ell)} + \left[1 + \sum_{i_0 \leq k \leq n-1} u_k\right] \mathbb{E}_{i_0} \sigma_{i_0}^{\ell}, & n > i_0; \end{cases}$$

Proposition(contin.)

$$u_k = \begin{cases} \sum_{j=i_0-1}^{k} q_{j,j+1}{}^{-1} F_k^{(j)} q_{ji_0}(1-\delta_{ji_0}), & k \geq i_0, \\ 1, & k = i_0 - 1, \\ 0, & 0 \leq k \leq i_0 - 2 \end{cases}$$

$$v_k^{(\ell)} = \sum_{j=0}^{k} \frac{F_k^{(j)}}{q_{j,j+1}} \mathbb{E}_j \sigma_{i_0}^{\ell-1}, \qquad k \geq 0,$$

$$\mathbb{E}_{i_0} \sigma_{i_0}^{\ell} = \ell \varlimsup_{n\to\infty} \left(\sum_{i_0 \leq k \leq n} v_k^{(\ell)} \right) \left[1 + \sum_{i_0 \leq k \leq n} u_k \right]^{-1} = \ell \lim_{n\to\infty} \frac{v_n^{(\ell)}}{u_n} \text{ if } \exists.$$

Proposition[Z.(2013); Chen & Z.(2014)]

Let the single birth Q-matrix $Q = (q_{ij})$ be irreducible and explosive (i.e. $\sum_n m_n < \infty$). Assume that the minimal process has finite $(\ell - 1)$-th moments of τ_∞ for some integer $\ell \geq 1$ (i.e. $E_i \tau_\infty^{\ell-1} < \infty$ for all $i \geq 0$). Then

$$\mathbb{E}_n \tau_\infty^\ell = \ell \sum_{k \geq n} \overline{m}_k^{(\ell)}, \qquad n \geq 0,$$

where

$$\overline{m}_n^{(\ell)} = \frac{1}{q_{n,n+1}} \left[\mathbb{E}_n \tau_\infty^{\ell-1} + \sum_{0 \leq k \leq n-1} q_n^{(k)} \overline{m}_k^{(\ell)} \right] = \sum_{j=0}^{n} \frac{F_n^{(j)} \mathbb{E}_j \tau_\infty^{\ell-1}}{q_{j,j+1}}, \qquad n \geq 0.$$

Proposition[Z.(2003); Chen & Z.(2014)]

Let the single birth Q-matrix (q_{ij}) be irreducible. Assume that its process is ergodic. Define $\big(\widetilde{F}_k^{(i)}\big)$ and $\big(\tilde{d}_k\big)$ by setting $c_i \equiv \lambda > 0$. Then for small λ,

$$\mathbb{E}_0 e^{\lambda\sigma_0} = \frac{q_{01}(1+\lambda\tilde{d})}{q_{01}-\lambda} < \infty, \quad \mathbb{E}_n e^{\lambda\sigma_0} = 1+\lambda\sum_{k=0}^{n-1}\Big(\widetilde{F}_k^{(0)}\tilde{d}-\tilde{d}_k\Big) < \infty, \quad n \geq 1$$

iff

$$\tilde{d} := \varlimsup_{n\to\infty} \mathbb{1}_{\{\sum_{k=0}^n \widetilde{F}_k^{(0)}>0\}} \frac{\sum_{k=0}^n \tilde{d}_k}{\sum_{k=0}^n \widetilde{F}_k^{(0)}} < \infty$$

and

$$\tilde{d}\sum_{k=0}^{n-1}\widetilde{F}_k^{(0)} > \sum_{k=0}^{n-1}\tilde{d}_k \quad \text{whenever} \quad \sum_{k=0}^{n-1}\widetilde{F}_k^{(0)} \leq 0 \quad \text{for } n \geq 2. \qquad (1)$$

Proposition(contin.)

Furthermore, once $\widetilde{F}_n^{(0)} > 0$ for large enough n and $\sum_n \widetilde{F}_n^{(0)} = \infty$, we have

$$\tilde{d} = \lim_{n\to\infty} \frac{\tilde{d}_n}{\widetilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

Finally, the process is exponentially ergodic iff both $\tilde{d} < \infty$ and (1) holds.

Proposition[Brockwell(1986), Anderson(1991), Chen & Z.(2014)]

Define $\left(\widetilde{F}_k^{(i)}\right)$ and $\left(\tilde{d}_k\right)$ with $c_i \equiv -\lambda < 0$. Let the single birth process be recurrent. Then the Laplace transform of σ_0 is given by

$$\mathbb{E}_0 e^{-\lambda\sigma_0} = \frac{q_{01}(1-\lambda\tilde{d})}{q_{01}+\lambda}, \ \mathbb{E}_n e^{-\lambda\sigma_0} = 1 - \lambda\sum_{k=0}^{n-1}\left(\widetilde{F}_k^{(0)}\tilde{d} - \tilde{d}_k\right), \ n \geq 1,$$

where

$$\tilde{d} = \lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\tilde{d}_k}{\sum_{k=0}^{n-1}\widetilde{F}_k^{(0)}} = \lim_{n\to\infty}\frac{\tilde{d}_n}{\widetilde{F}_n^{(0)}} \quad \text{if the limit exists.}$$

Proposition[Chen & Z.(2014)]

Assume that the single birth Q-matrix $Q = (q_{ij})$ is explosive and irreducible. Define $(\widetilde{m}_k)$ with $c_i \equiv \lambda$. For the corresponding minimal process,

(i) if $\exists \lambda > 0$ such that $\lambda \sum_{k=0}^{n-1} \widetilde{m}_k < 1$ for every $n > 1$, then

$$\mathbb{E}_n e^{\lambda \tau_\infty} = 1 + \lambda \left[\bar{c} \left(1 - \lambda \sum_{k=0}^{n-1} \widetilde{m}_k \right) - \sum_{k=0}^{n-1} \widetilde{m}_k \right], \quad n \geq 0,$$

$$\bar{c} = \varlimsup_{n \to \infty} \frac{\sum_{k=0}^{n} \widetilde{m}_k}{1 - \lambda \sum_{k=0}^{n} \widetilde{m}_k}.$$

Furthermore, the process decays exponentially fast provided $\bar{c} < \infty$.

(ii) For $\lambda > 0$, the Laplace transform of τ_∞ is given by

$$\mathbb{E}_n e^{-\lambda \tau_\infty} = \frac{1 + \lambda \sum_{0 \leq k \leq n-1} \widetilde{m}_k}{1 + \lambda \sum_{k \geq 0} \widetilde{m}_k}, \quad n \geq 0.$$

Example 1[uniform catastrophes]

Let

$$q_{i,i+1} = b\,i, \qquad i \geq 0; \qquad q_{ij} = a, \qquad j = 0, 1, \ldots, i-1;$$

and $q_{ij} = 0$ for other $j > i+1$, where a and b are positive constants. Then the extinction of the process has an exponential distribution

$$\mathbb{E}_n e^{-\lambda\tau_0} = \frac{a}{a+\lambda}, \qquad \lambda > 0,\ n \geq 1.$$

It is surprising that the distribution is independent of b and the starting point n. Redefine $q_{01} = 1$. Then the irreducible process is indeed strongly ergodic.

P. J. Brockwell et al(1982)

Example 2[Chen & Z.(2014)]

Consider the single birth Q-matrix (q_{ij}) with

$$q_{i0} > 0, \quad q_{i,i+1} > 0, \quad q_{ij} = 0 \ \text{ for all other } j \neq i.$$

Assume that $q_{n0} - c_n \equiv q_{10} - c_1$ for every $n \geq 1$. Then the process is explosive if

$$\kappa' := \lim_{n\to\infty} \frac{n(q_{n+1,n+2} - q_{n,n+1} - q_{10})}{q_{n,n+1} + q_{10}} > 1$$

($q_{n,n+1} = (n+1)^\gamma$ for $\gamma > 1$ for example). Otherwise, if $\kappa' < 1$ ($q_{n,n+1} = (n+1)^\gamma$ for some $\gamma \leq 1$ for instance), then the process is unique. If so, the process is indeed strongly ergodic.

- Exponential ergodicity: $\lim_{t\to\infty} e^{\alpha t}|p_{ij}(t) - \pi_j| = 0.$

Theorem[Mao & Z.(2004)]

Given a regular irreducible single birth Q-matrix $Q = (q_{ij})$. If

$$\inf_i q_i > 0 \text{ and } M := \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} \frac{1}{q_{j,j+1}F_j^{(0)}} < \infty,$$

then the Q-process is exponential ergodicity.

Example

Let $q_{n,n+1} = 1$ for all $n \geq 0$, $q_{10} = 1$, $q_{n,n-2} = 1$ for all $n \geq 2$ and $q_{ij} = 0$ for other $i \neq j$. Then the single birth process is exponentially ergodic and not strongly ergodic.

Theorem([Chen(2000)])

Given a regular birth-death Q-matrix (a_i, b_i). Then the process is exponentially ergodic iff

$$\delta := \sup_{i>0} \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{j=i}^{\infty} \mu_j < \infty.$$

$$\frac{1}{4\delta} \le \lambda_1 \le \frac{1}{\delta}, \quad \lambda_1 > 0 \iff \delta < \infty.$$

$$\lambda_1 = \text{ exponential convergence rate.}$$

Key: birth-death processes are reversible!

Theorem

Given a regular irreducible Q-matrix $Q = (q_{ij})$. Then the process $P(t)$ is exponentially ergodicity if and only if for some $\lambda > 0$ with $\lambda < q_i$ for all i,

$$\begin{cases} \sum_j q_{ij} y_j \leq -\lambda y_i - 1, & i \notin H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty \end{cases}$$

has a nonnegative finite solution (y_i).

"$\inf_i q_i > 0$" is necessary essentially for exponential ergodicity.

Difficulty: two parameters!

Mao and Zhang (2004) gives another proof.

Keys:

1. Construct a test function with double summations for sufficiency. The construction comes from the study of spectral gap essentially.

2. Necessity. Note that

$$\mathbb{E}_i \sigma_0^n \geq n\left(\sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k\right) \mathbb{E}_i \sigma_0^{n-1}$$

$$\geq \cdots \geq n!\left(\sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k\right)^n.$$

$$\mathbb{E}_i e^{\lambda \sigma_0} < \infty \Rightarrow \sum_{n=1}^{\infty}\left(\lambda \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k\right)^n < \infty$$

$$\Rightarrow \lambda \sum_{j=0}^{i-1} \frac{1}{\mu_j b_j} \sum_{k=i}^{\infty} \mu_k < 1 \Rightarrow \delta \leq \lambda^{-1} < \infty.$$

Unfortunately, "$M < \infty$" is not necessary for exponential ergodicity.

Example

Given $q_{01} > 0$ arbitrarily. Let $q_{i,i+1} = i^\gamma$, $q_{i0} = i^{\gamma-1}$ for all $i \geq 1$ where the constant $\gamma \in (1, 2)$, and $q_{ij} = 0$ for other $i \neq j$. Obviously, the Q-matrix is irreducible. It is easily computed that $F_i^{(0)} = 1$ for all $i \geq 0$. So the single birth process is recurrent and furthermore regular. Note that $\inf_{i\geq 1} q_{i0} = 1 > 0$. It follows that the single birth process is strongly ergodic and furthermore exponentially ergodic immediately. But $M = \infty$.

Now, for the single birth Q-matrix (q_{ij}), define a conservative birth-death Q-matrix (a_i, b_i) as follows.

$$a_i = \frac{q_{i,i+1}F_i^{(0)}}{F_{i-1}^{(0)}}, \quad i \geq 1; \qquad b_i = q_{i,i+1}, \quad i \geq 0.$$

Then $\mathbb{E}_i\tau_0^n \leq \mathbb{E}_i\overline{\tau}_0^n\,(i \geq 1, n \geq 0)$, furthermore $\mathbb{E}_i e^{\lambda\tau_0} \leq \mathbb{E}_i e^{\lambda\overline{\tau}_0}\,(i \geq 1, \lambda > 0)$.

- Single birth Q-matrix with absorbing boundary: $1 \leq N := \max\{i+1 : q_{i,i+1} = 0\} < \infty$. When $N = \max\emptyset = 0$, it is single birth.

  M.F. Chen(1999), Single birth processes, Chin. Ann. Math., **20B**, 77-82. Uniqueness.

- Single birth Q-matrix with immigration: $q_{0j} > 0$ for $j \geq 2$.

  Y.H. Zhang, Q.Q. Zhao, 2010.

- Cutoff

  Mao & Z.(2014). Explicit criteria on separation cutoff for birth and death chains, Frontiers of Mathematics in China, 9(4): 881-898.

  Mao, C. Zhang & Z.(2016). Separation cutoff for upward skip-free chains, J. Applied Prob., 53(1): 299-306.

- Applications

Yan & Chen(1986). Multidimensional Q-processes. Chinese Ann. Math. 7B: 90-110.

To study sufficient conditions for these problems of multidimensional Q-processes, we use the comparison method, i.e., compare them with a single birth Q-processes. The method reduces the multidimensional problems to one-dimensional ones.

The typical example is Schögl's model: a model of chemical reaction with diffusion in a container.

The method is efficient for necessary condition.

The Q-process corresponding to Brusselator model is not strongly ergodic.

Let E be a countable set. $Q = (q(x,y) : x,y \in E)$ be a conservative Q-matrix. Suppose that there exists a partition $\{E_k\}$ of E such that $\sum_{k=0}^{\infty} E_k = E$ and

(i) If $q(x,y) > 0$ and $x \in E_k$, then $y \in \sum_{j=0}^{k+1} E_j$ for all $k \geq 0$.

(ii) $\sum_{y \in E_{k+1}} q(x,y) > 0$ for all $x \in E_k$ and all $k \geq 0$.

(iii) $C_k := \sup\{q(x) : x \in E_k\} < \infty$ for all $k \geq 0$.

Define a conservative single birth Q-matrix $Q = (q_{ij})$:

$$q_{ij} = \begin{cases} \sup\{\sum_{y \in E_j} q(x,y) : x \in E_i\}, & \text{if } j = i+1; \\ \inf\{\sum_{y \in E_j} q(x,y) : x \in E_i\}, & \text{if } j < i; \\ 0, & \text{other cases of } j \neq i. \end{cases}$$

(0) If the (q_{ij})-process is unique (i.e. $R = \infty$), then so is the $(q(x, y))$-process.

Now suppose that $E_0 = \{\theta\}$ where $\theta \in E$ is a reference point, and that both $(q(x, y))$ and (q_{ij}) are irreducible and (q_{ij}) is regular.

(1) Moreover, assume that E_k is finite for all $k \geq 1$. If the (q_{ij})-process is recurrent (i.e. $\sum_{n=0}^{\infty} F_n^{(0)} = \infty$), then so is the $(q(x, y))$-process.
(2) If $\widehat{d} := \sup_{k\geq 0} d_k / F_k^{(0)} < \infty$, then both processes are ergodic.
(3) If $\inf_i q_i > 0$ and $M = \sup_{i>0} \sum_{j=0}^{i-1} F_j^{(0)} \sum_{j=i}^{\infty} (q_{j,j+1} F_j^{(0)})^{-1} < \infty$, then both processes are exponentially ergodic.
(4) If $\sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} \widehat{d} - d_n) < \infty$, then both processes are strongly ergodic.

Keys: increasing solution of equations.

Let S be a finite set and $E=\mathbb{Z}_+^S$. The model is defined by the following Q-matrix $Q=(q(x,y):x,y\in E)$:

$$q(x,y)=\begin{cases}\lambda_1\binom{x(u)}{2}+\lambda_4, & \text{if } y=x+e_u,\\ \lambda_2\binom{x(u)}{3}+\lambda_3 x(u), & \text{if } y=x-e_u,\\ x(u)p(u,v), & \text{if } y=x-e_u+e_v,\\ 0, & \text{other } y\neq x,\end{cases}$$

$q(x)=-q(x,x)=\sum_{y\neq x}q(x,y)$, where $x=(x(u):u\in S)$, $\binom{n}{k}$ is the usual combination, $\lambda_1,\cdots,\lambda_4$ are positive constants. $(p(u,v):u,v\in S)$ is a transition probability matrix on S and e_u is the element in E having value 1 at u and 0 elsewhere.

Conclusion

The Q-process corresponding to finite dimensional Schlögl model is strongly ergodic.

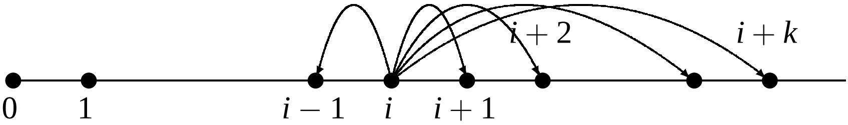
第三部分: 单死过程

1. Motivation
2. Dual approach
3. Zero-entrance
4. Mean hitting time

1. Motivation

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0,\ q_{i,i-j} = 0,\ i \geq 1, j > 1.$$



$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

1. Motivation

- branching processes: reference model

$$\begin{aligned} i &\to i-1 \quad \text{at rate} \quad ib_0 \\ &\to i \qquad\quad \text{at rate} \quad ib_1 = -i \\ &\to i+1 \quad \text{at rate} \quad ib_2 \\ &\to i+2 \quad \text{at rate} \quad ib_3 \\ &\to \cdots \end{aligned}$$

$$b_1 = -\sum_{k\neq 1} b_k = -1.$$

- single birth processes

Y.R. Li, A.G. Pakes, J. Li, A.H. Gu(2008). The limit behavior of dual Markov branching processes. J. Appl. Prob. 45, 176-189.

2. Dual approach

Given a Q-matrix Q, denote by $P(t) = (P_{ij}(t))$ the corresponding minimal Q-process. The Q-process $P(t)$ is said to be stochastically monotone if

$$\sum_{j \geq k} P_{ij}(t) \leq \sum_{j \geq k} P_{mj}(t), \quad i \leq m, k = 0, 1, 2, \cdots .$$

$\Rightarrow$

$$\sum_{j \geq k} q_{ij} \leq \sum_{j \geq k} q_{mj}, \quad i \leq m, k \in \{0, \cdots, i\} \cup \{m+1, m+2, \cdots\}.$$

We also say a Q-matrix $Q = (q_{ij})$ is monotone if the inequality above holds. If a Q-matrix is regular and monotone, then the minimal Q-process is also stochastically monotone.

2. Dual approach

Propositioin[Siegmund(1976)]

A transition function $P(t)$ is stochastically monotone if and only if there exists a dual process $\widetilde{P}(t)$ of $P(t)$ such that

$$\widetilde{P}(t)\Lambda = \Lambda P(t)^{\mathrm{T}},$$

where

$$\Lambda(i,j) = 1_{\{i \le j\}}.$$

D. Siegmund(1976): The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes. Ann. Probab. 41, 914-924.

Theorem[M.F. Chen, P.S. Li & Z.(2017+)]

Let $Q = (q_{ij})$ be a regular stochastically monotone single birth Q-matrix and let $\tilde{Q}$ be the dual of Q: $\tilde{Q} = \Lambda Q^{\mathrm{T}} \Lambda^{-1}$. Denote the single birth process by $(X_t)_{t\geq 0}$ and denote the (dual) single death process by $(\widetilde{X}_t)_{t\geq 0}$. Then we have:

(1) The (dual) single death matrix $\widetilde{Q}$ is regular if and only if the single birth process $(X_t)_{t\geq 0}$ is not strongly ergodic, i.e.,

$$\sup_{i\geq 1} \sum_{j=0}^{i-1} (F_j^{(0)} d - d_j) = \infty.$$

H.J. Zhang, A.Y. Chen, X. Lin, Z.T. Hou(2001): Strong ergodicity of monotone transition functions, Statist. Probab. Lett. 55: 63-69.

2. Dual approach

Theorem(continued)

(2) The single birth process is ergodic if and only if for any initial state $i > 0$, the extinction probability of the (dual) single death process $\mathbf{P}_i(\tilde{\sigma}_0 < \infty) < 1$. Indeed the extinction probability is given by

$$\mathbf{P}_i(\tilde{\sigma}_0 < \infty) = 1 - \sum_{j=0}^{i-1} \pi_j, \quad i = 1, 2, \cdots .$$

2. Dual approach

Theorem(continued)

(3) Denote the occupation time of the dual single birth process by $\widetilde{T}_j = \int_0^\infty 1_{\{\widetilde{X}_t = j\}} dt \; j = 1, 2, \ldots$. Then

$$\mathbf{E}_i(\tilde{T}_j) = \int_0^\infty \mathbf{P}_i(\tilde{X}_t = j)dt = \sum_{k=0}^{i-1} \left(1_{\{k \le j-1\}} \frac{F_{j-1}^{(k)}}{q_{k,k+1}} - m_{j-1}\pi_k \right).$$

Moreover, when the extinction probability of the (dual) single death process is 1, then

$$\mathbf{E}_i \tilde{T}_j = \sum_{k=0}^{i-1} 1_{\{k \le j-1\}} \frac{F_{j-1}^{(k)}}{q_{k,k+1}} = \sum_{k=0}^{i \wedge j-1} \frac{F_{j-1}^{(k)}}{q_{k,k+1}}.$$

Theorem(continued)

(4) Define the explosion time of the (dual) single death process $\tilde{\zeta} = \lim_{n\to\infty} \tilde{\zeta}_n$, where $\tilde{\zeta}_n = \inf\{t \geq 0 : \widetilde{X}_t \geq n\}$. The conditional expectation of the explosion time and the extinction time are given by

$$\mathbf{E}_i(\tilde{\zeta}|\tilde{\sigma}_0 = \infty) = \sum_{j=1}^{\infty} \frac{\sum_{k=0}^{j-1} \pi_k}{\sum_{k=0}^{i-1} \pi_k} \sum_{k=0}^{i-1} \left(1_{\{k \leq j-1\}} \frac{F_{j-1}^{(k)}}{q_{k,k+1}} - m_{j-1}\pi_k \right),$$

$$\mathbf{E}_i(\tilde{\sigma}_0|\tilde{\sigma}_0 < \infty) = \sum_{j=1}^{\infty} \frac{1 - \sum_{k=0}^{j-1} \pi_k}{1 - \sum_{k=0}^{i-1} \pi_k} \sum_{k=0}^{i-1} \left(1_{\{k \leq j-1\}} \frac{F_{j-1}^{(k)}}{q_{k,k+1}} - m_{j-1}\pi_k \right).$$

If the extinction probability of the (dual) single death process is 1, then the mean extinction time of the (dual) single death process is given by

$$\mathbf{E}_i\tilde{\sigma}_0 = \sum_{j=1}^{\infty} \sum_{k=0}^{i \wedge j-1} \frac{F_{j-1}^{(k)}}{q_{k,k+1}}, \qquad i = 1, 2, \cdots.$$

3. Zero-entrance

Define $c_i := q_i - \sum_{j\neq i} q_{ij}$ (non-conservative quantity).

[L.D. Wang &Z.(2014)]

Given a totally stable single birth Q-matrix $Q = (q_{ij})$. Then the single birth Q-matrix is zero-exit(equivalently, BQ-process is unique) if and only if

$$R := \sum_{n=0}^{\infty} m_n = \infty,$$

$$m_0 = \frac{1+c_0}{q_{01}}, \; m_n = \frac{1}{q_{n,n+1}}\left(1 + c_n + \sum_{k=0}^{n-1}(c_n + q_n^{(k)})m_k\right), \; n \geq 1.$$

$$\lambda x = Qx, \; 0 \leq x \leq 1.$$

3. Zero-entrance

Theorem[L.D. Wang & Z.(2014)]

Given a totally stable single death Q-matrix $Q = (q_{ij})$. Define $\bar{q}_n^{(k)} = \sum_{j=k}^{\infty} q_{nj}$ $(k > n)$. Then the single death Q-matrix is zero-entrance if and only if $\overline{R} := \sum_{n=0}^{\infty} \overline{m}_n = \infty$, where

$$\overline{m}_0 = 0,\ \overline{m}_n = \frac{1}{q_{n,n-1}} \left(1 + \sum_{k=0}^{n-1} (c_k + \bar{q}_n^{(k)}) \overline{m}_k \right),\ n \geq 1.$$

If the single death Q-matrix is non-conservative, then FQ-process is unique if and only if $\overline{R} = \infty$.

$$\lambda y = yQ,\ y \geq 0, ||y||_1 < \infty.$$

3. Zero-entrance

Theorem[L.D. Wang & Z.(2014)]

Given a totally stable single death Q-matrix $Q = (q_{ij})$. Then there exists a unique invariant measure as follows:

$$\mu_0 = 1,\ \mu_n = \frac{1}{q_{n,n-1}} \sum_{k=0}^{n-1} (c_k + \bar{q}_n^{(k)}) \mu_k,\ n \geq 1.$$

The stationary distribution of the single death process is represented as follows: $\pi_n = \mu_n/\mu,\ n \geq 0$, where

$$\mu_0 = 1,\ \mu_n = \frac{1}{q_{n,n-1}} \sum_{k=0}^{n-1} \bar{q}_n^{(k)} \mu_k,\ n \geq 1;$$

$$\mu = \sum_{n=0}^{\infty} \mu_n.$$

3. Zero-entrance

$$\pi Q = \mathbf{0}.$$

Transpose the matrix equation $\pi Q = \mathbf{0}$ to

$$Q^{\mathrm{T}} \pi^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}.$$

Then

$$Q^{\mathrm{T}} g =: \Omega g = Q^* g + c g = \mathbf{0}^{\mathrm{T}},$$

where Q^* is a totally stable and conservative single birth Q-matrix and

$$c_i = \sum_{j=0}^{i-1} q_{ji} + q_{i+1,i} - \sum_{j=i+1}^{\infty} q_{ij} - q_{i,i-1}, \; i \geq 0;$$

$$f_i \equiv 0.$$

3. Zero-entrance

Proposition[Z.(2016)]

The stationary distribution of the single death process is represented as follows: $\pi_n = \mu_n/\mu, \; n \geq 0$, where

$$\mu_0 = 1, \; \mu_n = 1 - \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j+1,j}}, n \geq 1; \; \mu := \sum_{n=0}^{\infty} \mu_n;$$

$$\widetilde{F}_i^{(i)} = 1, \quad \widetilde{F}_n^{(i)} = \frac{1}{q_{n+1,n}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \quad n > i \geq 0,$$

$$\tilde{q}_n^{(k)} = \sum_{j=0}^{k} q_{jn} - c_n, \quad n > k \geq 0.$$

4. Mean hitting time

- single birth process

$$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \ n \geq 0.$$

$$\mathbb{E}_n \tau_{n+1} = m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \ n \geq 0.$$

$$x_n = \frac{1}{q_n}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} x_k + q_n^{(n-1)} x_n\right), \ n \geq 0.$$

$$F_i^{(i)} = 1, \ F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \ 0 \leq i < n.$$

4. Mean hitting time

- single death process: $\mathbb{E}_n \tau_{n-1} = ?$

$$x_n = \frac{1}{q_n}\left(1 + \sum_{k=n+1}^{\infty} \bar{q}_n^{(k)} x_k + \bar{q}_n^{(n+1)} x_n\right), \ n \geq 1.$$

$$\bar{q}_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}, \ k > n \geq 0$$

$$G_i^{(i)} = 1, \ G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \ 0 \leq n < i.$$

4. Mean hitting time

$$x_n = \frac{1}{q_n}\left(1 + \sum_{k=n+1}^{\infty} \bar{q}_n^{(k)} x_k + \bar{q}_n^{(n+1)} x_n\right),\ n \geq 1.$$

$$\mathbb{E}_n \tau_{n-1} = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} (n \geq 1)?$$

The minimal nonnegative solution

$$h_n = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} (n \geq 1).$$

$$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} < \infty \Rightarrow \exists \text{ a QSD uniquely.}$$

Thank you for your attention!

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