# Notes on single death processes

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# Outline

1. Motivation
2. Single birth processes
3. Dual approach
4. Zero-entrance
5. Mean hitting time

# 1. Motivation

- $Q$-process: on $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$, suppose that sub-Markov transition probability matrix $P(t) = (p_{ij}(t))$ satisfies

  (1) Normal condition: $p_{ij}(t) \geq 0, \ \sum_j p_{ij}(t) \leq 1$.

  (2) Chapman-Kolmogorov eq.: $p_{ij}(t+s) = \sum_k p_{ik}(t) p_{kj}(s)$.

  (3) Jump condition: $\lim_{t\to 0} p_{ij}(t) = \delta_{ij}, \ i, j \in \mathbb{Z}_+$.

  (4) $Q$-condition: $\lim_{t\to 0}(p_{ij}(t) - \delta_{ij})/t = q_{ij}, \ i, j \in \mathbb{Z}_+$, i.e. $Q$-matrix $Q = (q_{ij})$ is the derivative matrix at time $0$ of $P(t)$,

$$0 \leqslant q_{ij} < \infty, \ i \neq j; \ \sum_{j \neq i} q_{ij} \leqslant -q_{ii} =: q_i \leqslant \infty, \ i \in \mathbb{Z}_+.$$

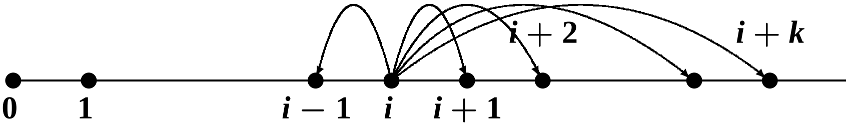
- totally stable: $q_i < \infty, \ i \in \mathbb{Z}_+$.

- conservative: $q_i = \sum_{j \neq i} q_{ij}, \ i \in \mathbb{Z}_+$, i.e. $\sum_{j \in \mathbb{Z}_+} q_{ij} = 0$.

# 1. Motivation

Single death process: $Q = (q_{ij})$ satisfies

$$q_{i,i-1} > 0, \; q_{i,i-j} = 0, \; i \geq 1, j > 1.$$




$$Q = \begin{pmatrix} - & * & * & * & \cdots \\ + & - & * & * & \cdots \\ 0 & + & - & * & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

# 1. Motivation

- branching processes: reference model

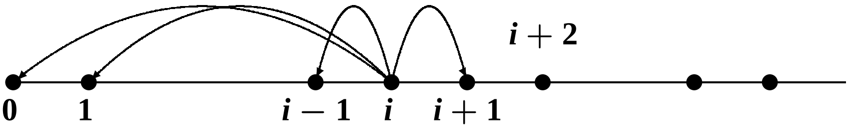
$$
\begin{aligned}
i \to i-1 \quad & \text{at rate} \quad ib_0 \\
\to i \quad & \text{at rate} \quad ib_1 = -i \\
\to i+1 \quad & \text{at rate} \quad ib_2 \\
\to i+2 \quad & \text{at rate} \quad ib_3 \\
\to \cdots &
\end{aligned}
$$

$$
b_1 = -\sum_{k \neq 1} b_k = -1
$$

# 1. Motivation

- single birth process

$$q_{i,i+1} > 0,\ q_{i,i+j} = 0,\ i \geq 0, j > 1.$$




$$Q = \begin{pmatrix} - & + & 0 & 0 & \cdots \\ * & - & + & 0 & \cdots \\ * & * & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- Uniqueness. J.K. Zhang(1984), S.J. Yan & M.F. Chen(1986).
- Recurrence. Yan & Chen(1986).
- Extinction/return probability. P.J. Brockwell(1986), W.J. Anderson(1991), Z.T. Hou & Q.F. Guo(1978, 1988). Chen & Z. (2014)
- Ergodicity. Yan & Chen(1986).
- Strong ergodicity. Z.(2001).
- Polynomial moments and stationary dist. Z.(2003, 2004, 2013)
- Exponential moments. Z.(2003)
- Laplace transform. Brockwell(1986), Anderson(1991)
- Poisson equation. Chen & Z.(2014)
- Mean occupation time. P.S. Li & Z.(2017)

# 2. Single birth processes

For $0 \le k < n$, define $q_n^{(k)} = \sum_{j=0}^{k} q_{nj}$ and

$$F_i^{(i)} = 1, \; F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \; 0 \le i < n.$$

Define

$$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \ge 0.$$

**Uniqueness[Zhang(1984), Yan & Chen(1986)]**

Given a totally stable and conservative single birth $Q$-matrix $Q = (q_{ij})$. Then the process is unique (non-explosive) if and only if

$$\sum_{n=0}^{\infty} m_n = \infty.$$

# 2. Single birth processes

**Recurrence [Yan & Chen(1986)]**

Assume the single birth $Q$-matrix $Q = (q_{ij})$ is non-explosive and irreducible. Then the process is recurrent if and only if

$$\sum_{n=0}^{\infty} F_n^{(0)} = \infty.$$

Define

$$d_n = \sum_{k=1}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \qquad n \geq 0.$$

**Ergodicity(Yan & Chen(1986), Z.(2001))**

Assume the single birth $Q$-matrix $Q = (q_{ij})$ is regular and irreducible, and the process is recurrent. Then the process is ergodic if and only if

$$d := \varlimsup_{k\to\infty} \frac{\sum_{n=0}^{k} d_n}{\sum_{n=0}^{k} F_n^{(0)}} < \infty;$$

the process is strongly ergodic if and only if

$$\sup_{k\geq 0} \sum_{n=0}^{k} (F_n^{(0)} d - d_n) < \infty.$$

# 2. Single birth processes

**Stationary distribution[Z.(2004)]**

Assume that the process is ergodic. Define

$$c_k := \varlimsup_{i\to\infty} \frac{\sum_{j=k}^{i} m_j}{\sum_{j=k}^{i} F_j^{(k)}}.$$

Then the stationary distribution

$$\pi_k = \frac{1}{q_{k,k+1}c_k}, \quad k \geq 0.$$

# 2. Single birth processes

Define $\sigma_0 = \inf\{t \geq \text{the first jumping time: } X_t = 0\}$.

**Return/extinction probability[Brockwell(1986), Chen & Z.(2014)]**

Let the single birth $Q$-matrix $Q = (q_{ij})$ be non-explosive and irreducible. Then the return/extinction probability is as follows:

$$\mathbb{P}_0(\sigma_0 < \infty) = \frac{\sum_{k=1}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, \mathbb{P}_n(\sigma_0 < \infty) = \frac{\sum_{k=n}^{\infty} F_k^{(0)}}{\sum_{k=0}^{\infty} F_k^{(0)}}, n \geq 1.$$

**[Z.(2004)]**

$$\mathbb{E}_i \sigma_0 = \sum_{k=0}^{i-1} (F_k^{(0)} d - d_k), \quad i \geq 1.$$

# 2. Single birth processes

$$\text{Poisson}: Qg + cg = f$$

Roughly speaking, the unified treatment consists of the following three steps.

**(a)** Find out the Poisson equation corresponding to the problem .

**(b)** Apply Chen & Z.(2014) to get the solution to Poisson equation.

**(c)** Work out the answer for the problem using the solution obtained in (b).

M.F. Chen & Z.(2014): Unified representation of formulas for single birth processes, Frontiers of Mathematics in China, 9(4): 761-796.

# 2. Single birth processes

| Problem | $c_i \in \mathbb{R}$ | $f_i \in \mathbb{R}$ |
|---|---|---|
| Harmonic function | $c_i \in \mathbb{R}$ | $f_i \equiv 0$ |
| Uniqueness | $c_i \equiv -\lambda < 0$ | $f_i \equiv 0$ |
| Recurrence | $c_i \equiv 0$ | $f_i = q_{i0}(1 - \delta_{i0})$ |
| Ext./return probability | $c_i \equiv 0$ | $f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$ |
| Ergodicity | $c_i \equiv 0$ | $f_i = q_{i0}(1 - \delta_{i0})g_0 - 1$ |
| Strong ergodicity | $c_i \equiv 0$ | $f_i = q_{i0}(1 - \delta_{i0})g_0 - 1$ |
| Polynomial moment | $c_i \equiv 0$ | $f_i^{(\ell)}$ |
| Exp. moment/ergod. | $c_i \equiv \lambda > 0$ | $f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$ |
| Laplace transform | $c_i \equiv -\lambda < 0$ | $f_i = q_{i0}(1 - \delta_{i0})(g_0 - 1)$ |

where $f_i^{(\ell)} = q_{ii_0}(1 - \delta_{ii_0})g_{i_0} - \ell \mathbb{E}_i \sigma_{i_0}^{\ell-1}$.

# 3. Dual approach

Given a $Q$-matrix $Q$, denote by $P(t) = (P_{ij}(t))$ the corresponding minimal $Q$-process. The $Q$-process $P(t)$ is said to be stochastically monotone if

$$\sum_{j \geq k} P_{ij}(t) \leq \sum_{j \geq k} P_{mj}(t), \quad i \leq m, k = 0, 1, 2, \cdots .$$

$\Rightarrow$

$$\sum_{j \geq k} q_{ij} \leq \sum_{j \geq k} q_{mj}, \quad i \leq m, k \in \{0, \cdots, i\} \cup \{m+1, m+2, \cdots\}.$$

We also say a $Q$-matrix $Q = (q_{ij})$ is monotone if the inequality above holds. If a $Q$-matrix is regular and monotone, then the minimal $Q$-process is also stochastically monotone.

# 3. Dual approach

**Propositioin[Siegmund(1976)]**

A transition function $P(t)$ is stochastically monotone if and only if there exists a dual process $\widetilde{P}(t)$ of $P(t)$ such that

$$\widetilde{P}(t)\Lambda = \Lambda P(t)^{\mathsf{T}},$$

where

$$\Lambda(i,j) = \mathbf{1}_{\{i\le j\}}.$$

D. Siegmund(1976): The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov processes. Ann. Probab. 41, 914-924.

# 3. Dual approach

**Theorem[M.F. Chen, P.S. Li & Z.(2017+)]**

Let $Q = (q_{ij})$ be a regular stochastically monotone single birth $Q$-matrix and let $\tilde{Q}$ be the dual of $Q$: $\tilde{Q} = \Lambda Q^{\mathsf{T}} \Lambda^{-1}$. Denote the single birth process by $(X_t)_{t\geq 0}$ and denote the (dual) single death process by $(\widetilde{X}_t)_{t\geq 0}$. Then we have:

(1) The (dual) single death matrix $\widetilde{Q}$ is regular if and only if the single birth process $(X_t)_{t\geq 0}$ is not strongly ergodic, i.e.,

$$\sup_{i\geq 1} \sum_{j=0}^{i-1} (F_j^{(0)} d - d_j) = \infty.$$

# 3. Dual approach

**Theorem(continued)**

(2) The single birth process is ergodic if and only if for any initial state $i > 0$, the extinction probability of the (dual) single death process $\mathbf{P}_i(\tilde{\sigma}_0 < \infty) < 1$. Indeed the extinction probability is given by

$$\mathbf{P}_i(\tilde{\sigma}_0 < \infty) = 1 - \sum_{j=0}^{i-1} \pi_j, \quad i = 1, 2, \cdots .$$

**Theorem(continued)**

(3) Denote the occupation time of the dual single birth process by $\widetilde{T}_j = \int_0^\infty \mathbf{1}_{\{\widetilde{X}_t = j\}} dt \; j = 1, 2, \ldots$. Then

$$\mathrm{E}_i(\tilde{T}_j) = \int_0^\infty \mathrm{P}_i(\tilde{X}_t = j)dt = \sum_{k=0}^{i-1}\left(\mathbf{1}_{\{k\le j-1\}}\frac{F_{j-1}^{(k)}}{q_{k,k+1}} - m_{j-1}\pi_k\right).$$

Moreover, when the extinction probability of the (dual) single death process is $1$, then

$$\mathrm{E}_i\tilde{T}_j = \sum_{k=0}^{i-1}\mathbf{1}_{\{k\le j-1\}}\frac{F_{j-1}^{(k)}}{q_{k,k+1}} = \sum_{k=0}^{i\wedge j-1}\frac{F_{j-1}^{(k)}}{q_{k,k+1}}.$$

# 3. Dual approach

**Theorem(continued)**

(4) Define the explosion time of the (dual) single death process $\tilde{\zeta} = \lim_{n\to\infty} \tilde{\zeta}_n$, where $\tilde{\zeta}_n = \inf\{t \geq 0 : \widetilde{X}_t \geq n\}$. The conditional expectation of the explosion time and the extinction time are given by

$$\mathbf{E}_i(\tilde{\zeta}|\tilde{\sigma}_0 = \infty) = \sum_{j=1}^{\infty} \frac{\sum_{k=0}^{j-1} \pi_k}{\sum_{k=0}^{i-1} \pi_k} \sum_{k=0}^{i-1} \left( \mathbf{1}_{\{k\leq j-1\}} \frac{F_{j-1}^{(k)}}{q_{k,k+1}} - m_{j-1}\pi_k \right),$$

$$\mathbf{E}_i(\tilde{\sigma}_0|\tilde{\sigma}_0 < \infty) = \sum_{j=1}^{\infty} \frac{1 - \sum_{k=0}^{j-1} \pi_k}{1 - \sum_{k=0}^{i-1} \pi_k} \sum_{k=0}^{i-1} \left( \mathbf{1}_{\{k\leq j-1\}} \frac{F_{j-1}^{(k)}}{q_{k,k+1}} - m_{j-1}\pi_k \right).$$

# 3. Dual approach

**Theorem(continued)**

If the extinction probability of the (dual) single death process is $1$, then the mean extinction time of the (dual) single death process is given by

$$\mathrm{E}_i \tilde{\sigma}_0 = \sum_{j=1}^{\infty} \sum_{k=0}^{i \wedge j-1} \frac{F_{j-1}^{(k)}}{q_{k,k+1}}, \qquad i = 1, 2, \cdots .$$

H.J. Zhang, A.Y. Chen, X. Lin, Z.T. Hou(2001): Strong ergodicity of monotone transition functions, Statist. Probab. Lett. 55: 63-69.

# 4. Zero-entrance

Define $c_i := q_i - \sum_{j\neq i} q_{ij}$ (non-conservative quantity).

**[L.D. Wang &Z.(2014)]**

Given a totally stable single birth $Q$-matrix $Q = (q_{ij})$. Then the single birth $Q$-matrix is zero-exit(equivalently, $BQ$-process is unique) if and only if

$$R := \sum_{n=0}^{\infty} m_n = \infty,$$

$$m_0 = \frac{1+c_0}{q_{01}},\ m_n = \frac{1}{q_{n,n+1}}\left(1 + c_n + \sum_{k=0}^{n-1}(c_n + q_n^{(k)})m_k\right),\ n \geq 1.$$

$$\lambda x = Qx,\ 0 \leq x \leq 1.$$

# 4. Zero-entrance

**[L.D. Wang & Z.(2014)]**

Given a totally stable single death $Q$-matrix $Q = (q_{ij})$. Define $\bar{q}_n^{(k)} = \sum_{j=k}^{\infty} q_{nj}$ $(k > n)$. Then the single death $Q$-matrix is zero-entrance if and only if $\overline{R} := \sum_{n=0}^{\infty} \overline{m}_n = \infty$, where

$$\overline{m}_0 = 0,\ \overline{m}_n = \frac{1}{q_{n,n-1}}\left(1 + \sum_{k=0}^{n-1}(c_k + \bar{q}_n^{(k)})\overline{m}_k\right),\ n \geq 1.$$

If the single death $Q$-matrix is non-conservative, then $FQ$-process is unique if and only if $\overline{R} = \infty$.

$$\lambda y = yQ,\ y \geq 0, ||y||_1 < \infty.$$

# 4. Zero-entrance

**[L.D. Wang & Z.(2014)]**

Given a totally stable single death $Q$-matrix $Q = (q_{ij})$. Then there exists a unique invariant measure as follows:

$$\mu_0 = 1,\ \mu_n = \frac{1}{q_{n,n-1}} \sum_{k=0}^{n-1} (c_k + \bar{q}_n^{(k)}) \mu_k,\ n \geq 1.$$

The stationary distribution of the single death process is represented as follows: $\pi_n = \mu_n / \mu,\ n \geq 0$, where

$$\mu_0 = 1,\ \mu_n = \frac{1}{q_{n,n-1}} \sum_{k=0}^{n-1} \bar{q}_n^{(k)} \mu_k,\ n \geq 1;$$

$$\mu = \sum_{n=0}^{\infty} \mu_n.$$

# 4. Zero-entrance

$$\pi Q = 0.$$

Transpose the matrix equation $\pi Q = 0$ to

$$Q^{\mathsf{T}} \pi^{\mathsf{T}} = 0^{\mathsf{T}}.$$

Then

$$Q^{\mathsf{T}} g =: \Omega g = Q^* g + c g = 0^{\mathsf{T}},$$

where $Q^*$ is a totally stable and conservative single birth $Q$-matrix and

$$c_i = \sum_{j=0}^{i-1} q_{ji} + q_{i+1,i} - \sum_{j=i+1}^{\infty} q_{ij} - q_{i,i-1}, \ i \geq 0;$$

$$f_i \equiv 0.$$

# 4. Zero-entrance

**Proposition[Z.(2016)]**

The stationary distribution of the single death process is represented as follows: $\pi_n = \mu_n/\mu, \ n \geq 0$, where

$$\mu_0 = 1, \ \mu_n = 1 - \sum_{k=0}^{n-1} \sum_{j=0}^{k} \frac{\widetilde{F}_k^{(j)} c_j}{q_{j+1,j}}, n \geq 1; \ \mu := \sum_{n=0}^{\infty} \mu_n;$$

$$\widetilde{F}_i^{(i)} = 1, \quad \widetilde{F}_n^{(i)} = \frac{1}{q_{n+1,n}} \sum_{k=i}^{n-1} \tilde{q}_n^{(k)} \widetilde{F}_k^{(i)}, \quad n > i \geq 0,$$

$$\tilde{q}_n^{(k)} = \sum_{j=0}^{k} q_{jn} - c_n, \quad n > k \geq 0.$$

# 5. Mean hitting time

- single birth process

$$m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \; n \geq 0.$$

$$\mathbb{E}_n \tau_{n+1} = m_n = \sum_{k=0}^{n} \frac{F_n^{(k)}}{q_{k,k+1}}, \; n \geq 0.$$

$$x_n = \frac{1}{q_n}\left(1 + \sum_{k=0}^{n-1} q_n^{(k)} x_k + q_n^{(n-1)} x_n\right), \; n \geq 0.$$

$$F_i^{(i)} = 1, \; F_n^{(i)} = \frac{1}{q_{n,n+1}} \sum_{k=i}^{n-1} q_n^{(k)} F_k^{(i)}, \; 0 \leq i < n.$$

# 5. Mean hitting time

- single death process: $\mathbb{E}_n \tau_{n-1} = ?$

$$x_n = \frac{1}{q_n}\left(1 + \sum_{k=n+1}^{\infty} \bar{q}_n^{(k)} x_k + \bar{q}_n^{(n+1)} x_n\right), \ n \geq 1.$$

$$\bar{q}_n^{(k)} := \sum_{j=k}^{\infty} q_{nj}, \ k > n \geq 0$$

$$G_i^{(i)} = 1, \ G_n^{(i)} = \frac{1}{q_{n,n-1}} \sum_{k=n+1}^{i} q_n^{(k)} G_k^{(i)}, \ 0 \leq n < i.$$

# 5. Mean hitting time

$$x_n = \frac{1}{q_n}\left(1 + \sum_{k=n+1}^{\infty} \bar{q}_n^{(k)} x_k + \bar{q}_n^{(n+1)} x_n\right), \; n \geq 1.$$

$$\mathbb{E}_n \tau_{n-1} = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} (n \geq 1)?$$

**The minimal nonnegative solution**

$$h_n = \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} (n \geq 1).$$

$$\sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \frac{G_n^{(k)}}{q_{k,k-1}} < \infty \Rightarrow \exists \text{ a QSD uniquely.}$$

***Thank you for your attention!***

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